Computation of operators in wavelet coordinates

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Overview

- Linear operator equation $Lu = g$ with $L : \mathcal{H} \to \mathcal{H}'$
- Riesz basis $\Psi = \{\psi_\lambda\}$ of $\mathcal{H}$, e.g. $u = \sum_\lambda \mathbf{u}_\lambda \psi_\lambda$
- Infinite dimensional matrix-vector system $\mathbf{Lu} = \mathbf{g}$, with $\mathbf{u} = (\mathbf{u}_\lambda)_\lambda$ and $\mathbf{L} : \ell_2 \to \ell_2$
- Convergent iterations such as $\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \alpha[\mathbf{g} - \mathbf{Lu}^{(i)}]$
- We can approximate $\mathbf{Lu}^{(i)}$ by a finitely supported vector
- How cheap can we compute this approximation?
- The answer will depend on L and Ψ

Linear operator equations

- Let Ω be an n-dimensional domain or smooth manifold
- $H^t \subset H^t(\Omega)$ be a subspace, and H^{-t} be its dual space
- Consider the problem of finding u from

$$Lu = g$$

- where $L : H^t \to H^{-t}$ is a self-adjoint elliptic operator of order $2t$
- and $g \in H^{-t}$ is a linear functional

Differential operators

- Partial differential operators of order $2t$

$$\langle v, Lu\rangle = \sum_{|\alpha|,|\beta|\leq t} \langle \partial^\alpha v, a_{\alpha\beta}\partial^\beta u\rangle,$$

- Example: The reaction-diffusion equation ($t = 1$)

$$\langle v, Lu\rangle = \int_\Omega \nabla v \cdot \nabla u + \kappa^2 vu,$$

Singular integral operators

- Boundary integral operators

$$[Lu](x) = \int_{\Omega} K(x,y)u(y)d\Omega_y$$

with the kernel $K(x,y)$ singular at $x = y$

- Example: The single layer operator for the Laplace BVP in 3-d domain ($t = -\frac{1}{2}$)

$$K(x,y) = \frac{1}{4\pi|x-y|}$$

Multiresolution analysis

- $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset H^t$ and $\tilde{\mathcal{S}}_0 \subset \tilde{\mathcal{S}}_1 \subset \ldots \subset H^{-t}$
- $\dim \mathcal{S}_j \,, \dim \tilde{\mathcal{S}}_j = \mathcal{O}(2^{jn})$ (dyadic refinements)
- $\mathcal{S}_j$ contains all piecewise pols of degree $\boxed{d-1}$
- $\tilde{\mathcal{S}}_j$ contains all piecewise pols of degree $\tilde{d}-1$
- $\mathcal{S}_j$ is globally C^r-smooth

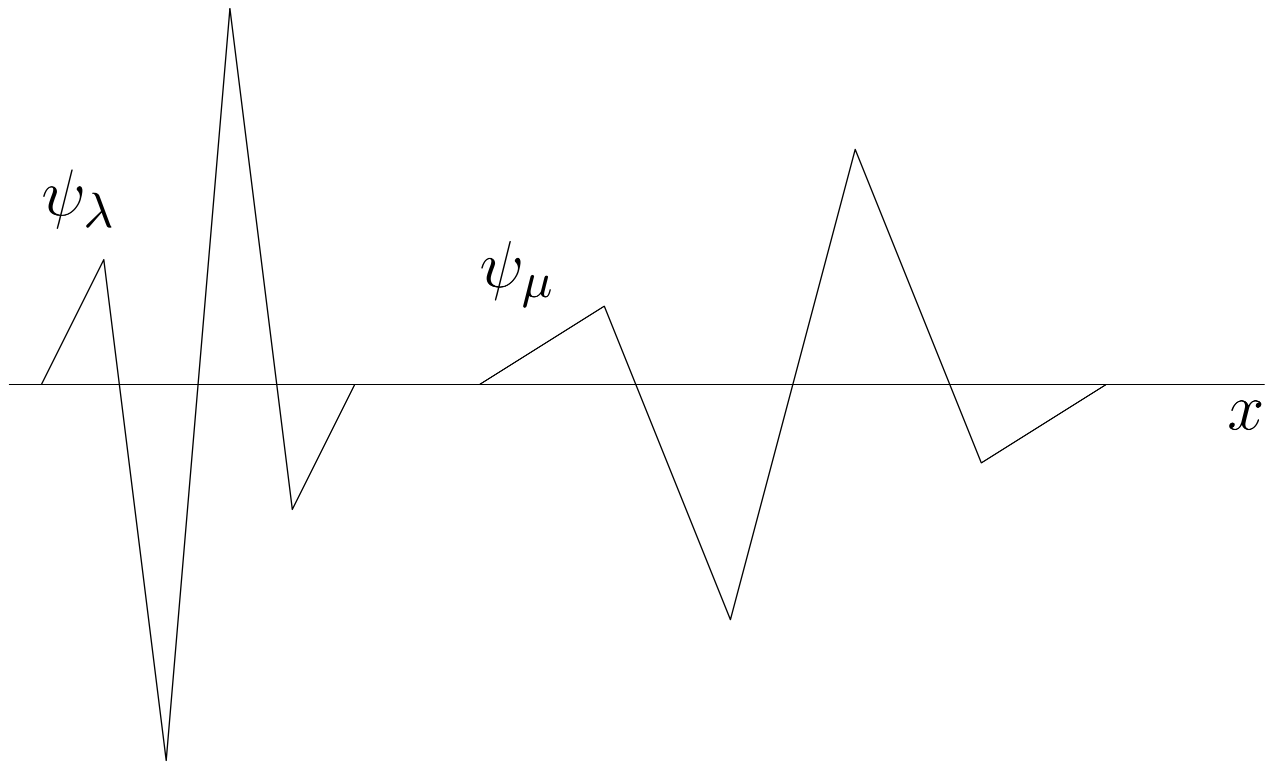
Wavelet bases

- $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ is a Riesz basis for H^t
  – each $v \in H^t$ has a unique expansion

$$v = \sum_{\lambda \in \Lambda} \mathbf{v}_\lambda \psi_\lambda \quad \text{s.t.} \quad c\|\mathbf{v}\| \leq \|v\|_{H^t} \leq C\|\mathbf{v}\|$$

- For every index $\lambda \in \Lambda$, there is a number $|\lambda| \in I\!N_0$ called the level of ψ_λ
- $\mathrm{span}\{\psi_\lambda : |\lambda| \leq j\} = \mathcal{S}_j$
- $\langle \psi_\lambda, v \rangle = 0$ for any $v \in \tilde{\mathcal{S}}_{|\lambda|-1}$
- $\mathrm{diam}(\mathrm{supp}\,\psi_\lambda) = \mathcal{O}(2^{-|\lambda|})$

Typical wavelets



- ψ_λ is a piecewise polynomial of degree $d-1$
- $\int x^k \psi_\lambda(x)dx = 0$ for $k < \tilde{d}$ ($\tilde{d}$ vanishing moments)

Galerkin methods

- Wavelet basis $\Psi_j := \{\psi_\lambda : |\lambda| \leq j\}$ of $\mathcal{S}_j$
- Stiffness $\mathbf{L}_{(j)} = \langle L\psi_\lambda, \psi_\mu \rangle_{|\lambda|,|\mu|\leq j}$
- load $\mathbf{g}_{(j)} = \langle g, \psi_\lambda \rangle_{|\lambda|\leq j}$
- Linear equation in $\mathbb{R}^{N_j}$ $\quad (N_j := \dim \mathcal{S}_j)$

$$\mathbf{L}_{(j)}\mathbf{u}_{(j)} = \mathbf{g}_{(j)}$$

- $\mathbf{L}_{(j)} : \mathbb{R}^{N_j} \to \mathbb{R}^{N_j}$ SPD and $\mathbf{g}_{(j)} \in \mathbb{R}^{N_j}$
- $u_{(j)} = \sum_\lambda [\mathbf{u}_{(j)}]_\lambda \psi_\lambda$ approximates the solution of $Lu = g$

Galerkin approximation

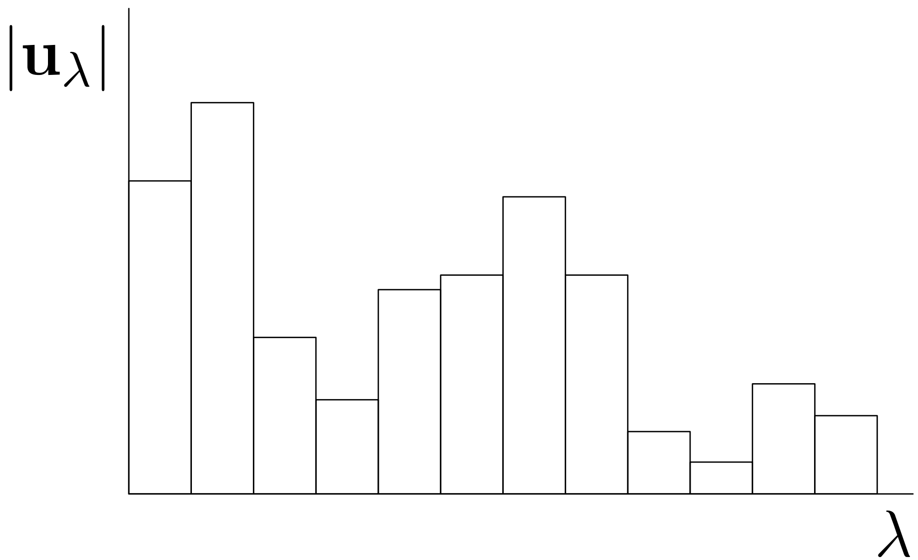
- If $u \in H^s$ for some $s \in [t, d]$

$$\varepsilon_{(j)} := \|u_{(j)} - u\|_{H^t} \leq \mathcal{O}(2^{-j(s-t)})$$

- $N_j = \dim \mathcal{S}_j = \mathcal{O}(2^{jn})$
- $\varepsilon_{(j)} \leq \mathcal{O}(N_j^{-\frac{s-t}{n}})$
- Solve $\mathbf{L}_{(j)}\mathbf{u}_{(j)} = \mathbf{g}_{(j)}$ with CG $\rightsquigarrow$ complexity $\mathcal{O}(N_j)$
- Similar estimates for FEM
- Better convergence? Adaptive methods?

Nonlinear approximation

- Given $\mathbf{u} = (\mathbf{u}_\lambda)_\lambda \in \ell_2$
- Approximate $\mathbf{u}$ using N coeffs



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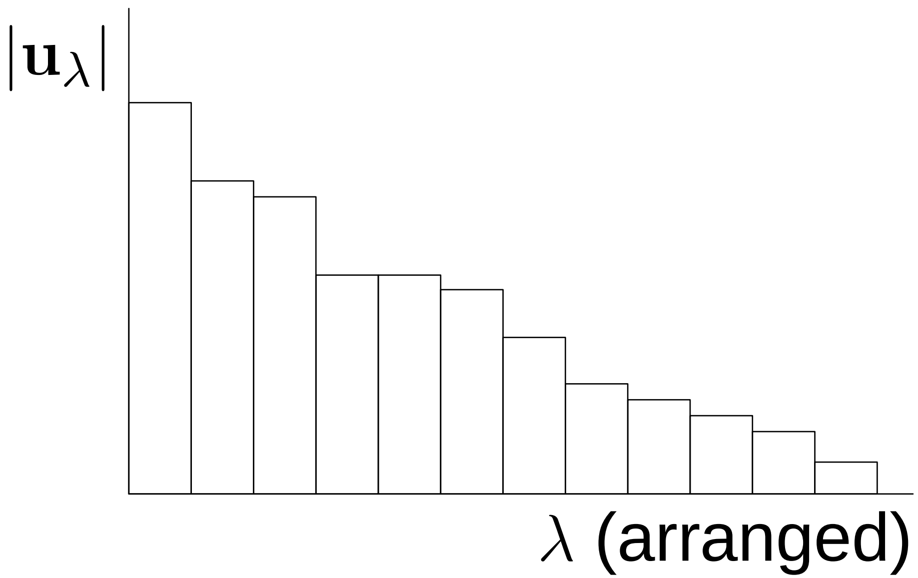

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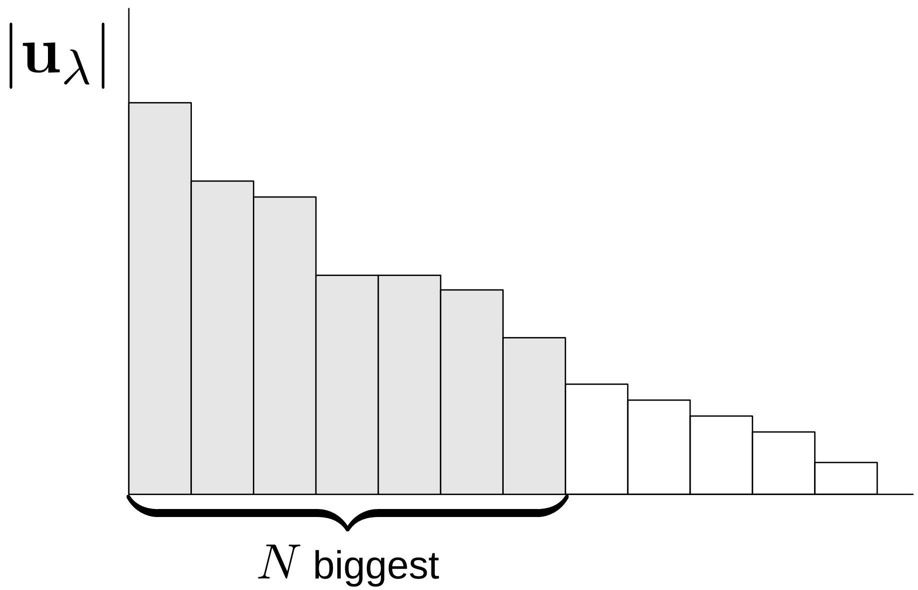

- $\mathbf{u}_N$ best approximation of $\mathbf{u}$ with $\#\mathrm{supp}\,\mathbf{u}_N \leq N$

Nonlinear approximation

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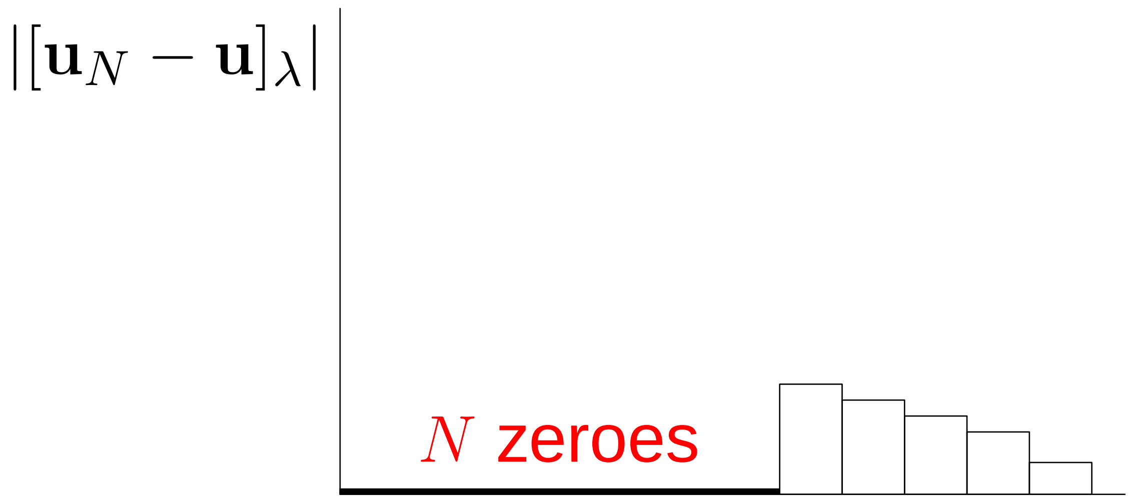
- Approximate $\mathbf{u}$ using N coeffs



- $\mathbf{u}_N$ best approximation of $\mathbf{u}$ with $\#\mathrm{supp}\,\mathbf{u}_N \le N$

Nonlinear vs. linear approximation

- If $u \in B^s_{\tau,\tau}$ with $\frac{1}{\tau} = \frac{1}{2} + \frac{s-t}{n}$ for some $s < d$

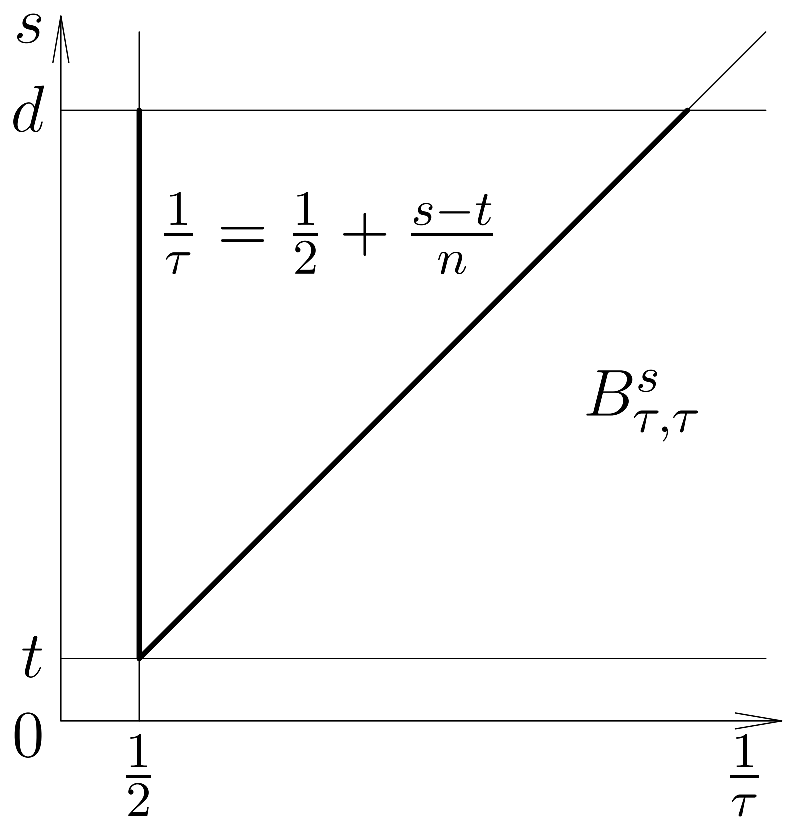
$$\varepsilon_N = \|\mathbf{u}_N - \mathbf{u}\| \leq \mathcal{O}(N^{-\frac{s-t}{n}})$$

- If $u \in H^s$ for some $s \leq d$, uniform refinement

$$\varepsilon_{(j)} = \|\mathbf{u}_{(j)} - \mathbf{u}\| \leq \mathcal{O}(N_j^{-\frac{s-t}{n}})$$

- $B^s_{\tau,\tau}$ is bigger than H^s

Besov vs. Sobolev regularity



- [Dahlke, DeVore]: $u \in B^d_{\tau,\tau}$ with $\frac{1}{\tau} = \frac{1}{2} + \frac{d-t}{n}$ "often"

Equivalent problem in ℓ_2

[Cohen, Dahmen, DeVore '02]

- Wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$
- Stiffness $\mathbf{L} = \langle L\psi_\lambda, \psi_\mu \rangle_{\lambda,\mu}$ and load $\mathbf{g} = \langle g, \psi_\lambda \rangle_\lambda$
- Linear equation in $\ell_2(\Lambda)$

$$\mathbf{Lu} = \mathbf{g}$$

- $\mathbf{L} : \ell_2(\Lambda) \to \ell_2(\Lambda)$ SPD and $\mathbf{g} \in \ell_2(\Lambda)$
- $u = \sum_\lambda \mathbf{u}_\lambda \psi_\lambda$ is the solution of $Lu = g$

Richardson iterations in ℓ_2

- $\mathbf{u}^{(0)} = \mathbf{0}$
- $\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \alpha[\mathbf{g} - \mathbf{L}\mathbf{u}^{(i)}] \qquad i = 0, 1, \ldots$
- $\mathbf{g}$ and $\mathbf{L}\mathbf{u}^{(i)}$ are infinitely supported
- Approximate them by finitely supported sequences
- Algorithm $\mathbf{SOLVE}[N, \mathbf{L}, \mathbf{g}] \to \mathbf{u}_{[N]}$ (N operations)
- $\#\mathrm{supp}\,\mathbf{u}_{[N]} \leq \mathcal{O}(N)$ and

$$\varepsilon_{[N]} = \|\mathbf{u}_{[N]} - \mathbf{u}\| \to 0 \quad \text{as } N \to \infty$$

- $\varepsilon_{[N]}$ speed of convergence?

Complexity of SOLVE

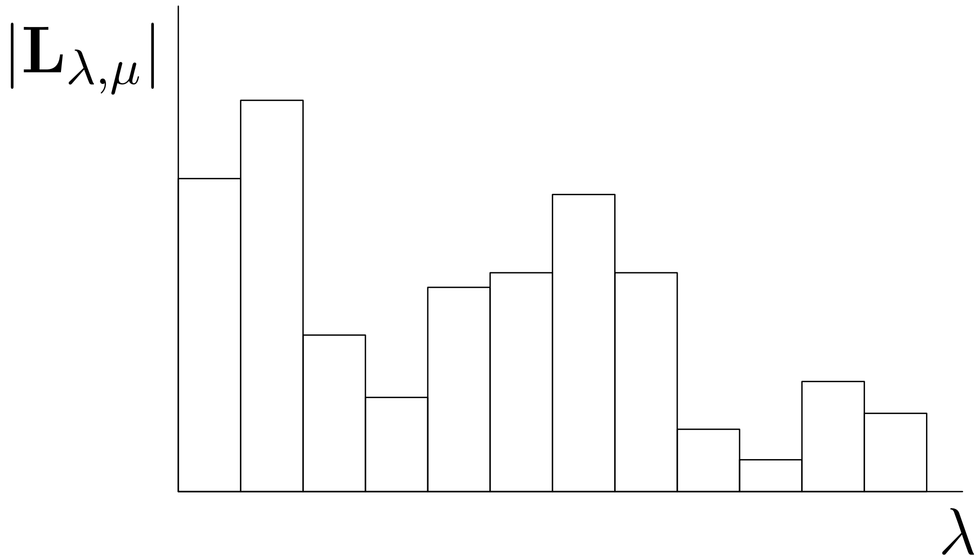
- Matrix $\mathbf{L}$ is called q^*-computable, when for each N one can construct an infinite matrix $\mathbf{L}_N$ s.t.
  - for any $q < q^*$, $\|\mathbf{L}_N - \mathbf{L}\| \leq \mathcal{O}(N^{-q})$
  - having in each column $\mathcal{O}(N)$ non-zero entries
  - whose computation takes $\mathcal{O}(N)$ operations
- [CDD'02]: Suppose that
  - $\|\mathbf{u}_N - \mathbf{u}\| \leq \mathcal{O}(N^{-s}) \qquad [s < \frac{d-t}{n}]$
  - $\mathbf{L}$ is q^*-computable with $q^* > s$

  then for suitable $\mathbf{g}$, $\mathbf{u}_{[N]} = \mathbf{SOLVE}[N, \mathbf{L}, \mathbf{g}]$ satisfies

$$\|\mathbf{u}_{[N]} - \mathbf{u}\| \leq \mathcal{O}(N^{-s})$$

Computability

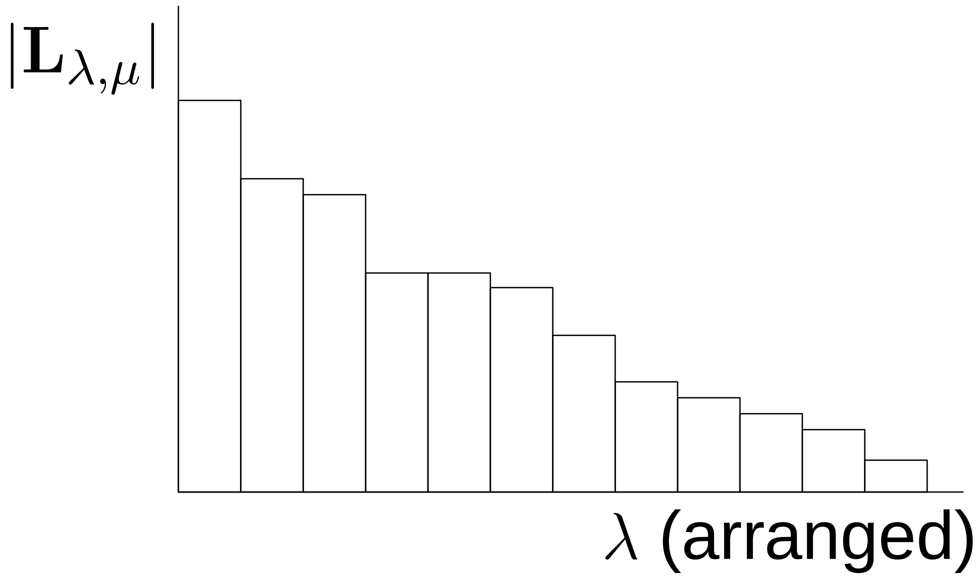
- $[\mathbf{L}_{\lambda,\mu}]_{\lambda\in\Lambda}$ – μ-th column



- Approximate by N entries?

Computability

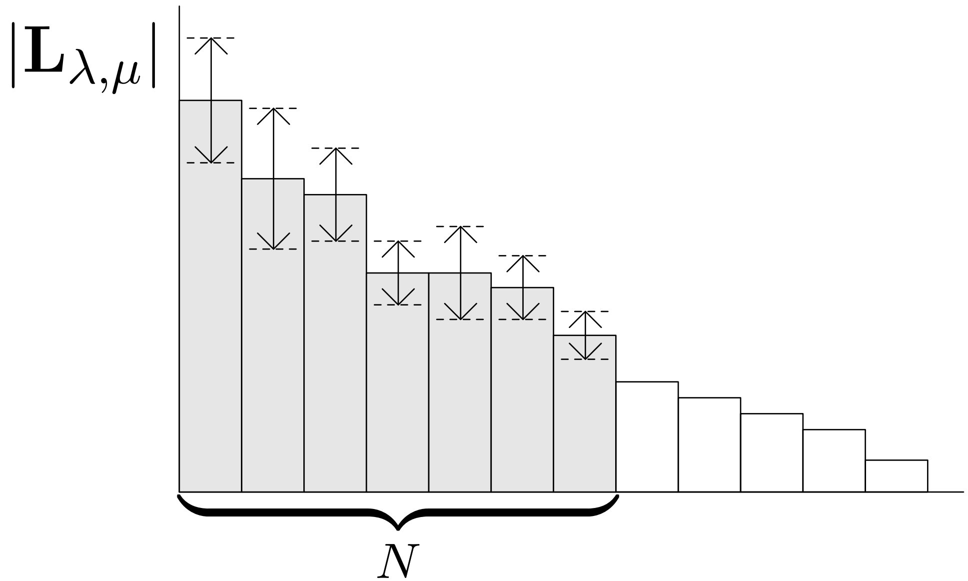
- $[\mathbf{L}_{\lambda,\mu}]_{\lambda\in\Lambda}$ – μ-th column arranged by modulus



- N biggest entries?

Computability

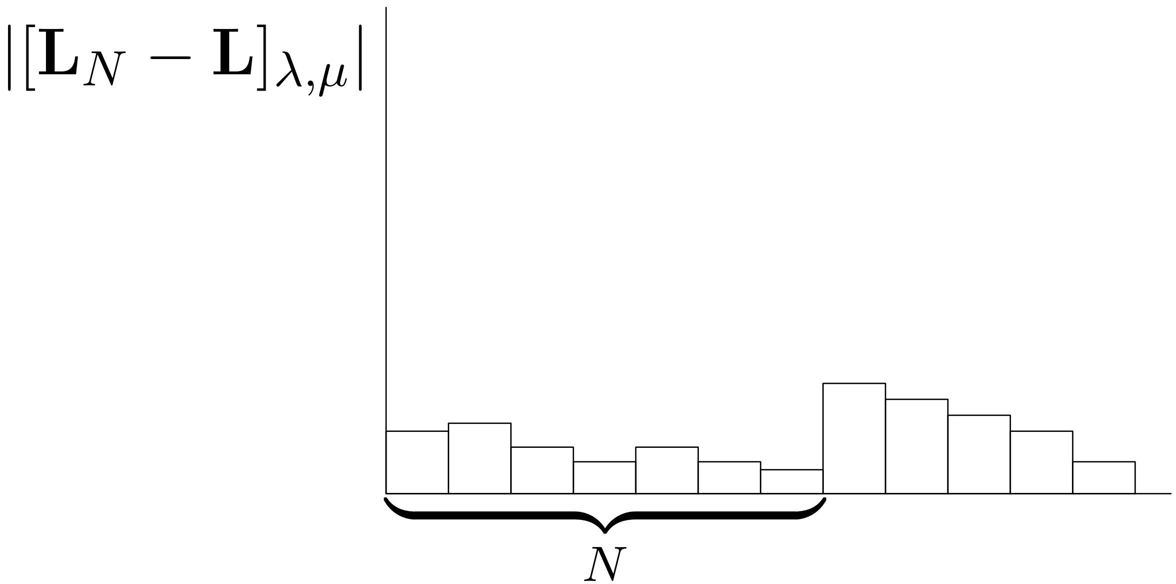
- $[\mathbf{L}_{\lambda,\mu}]_{\lambda\in\Lambda}$ – μ-th column



- Compute the N biggest entries

Computability

- The μ-th column of the difference



- Need to locate the biggest entries a priori

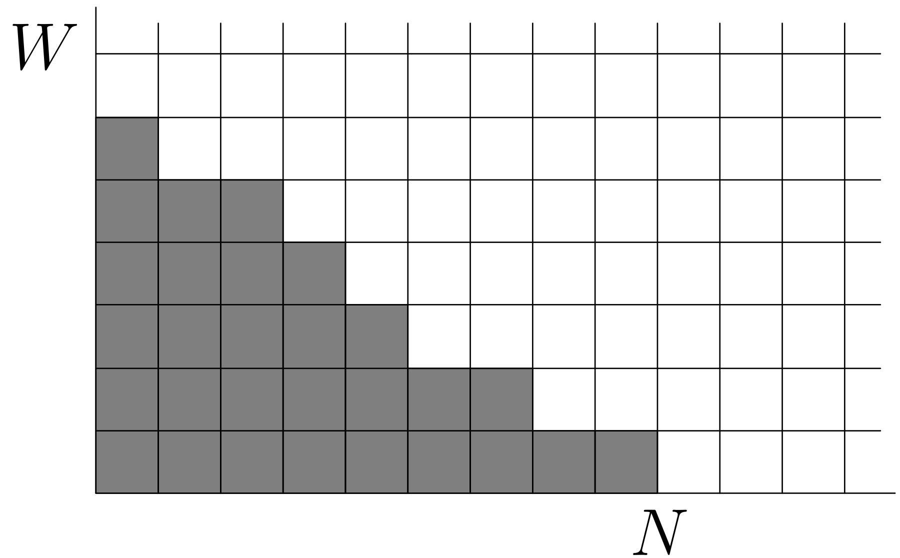
Compressibility

- $\mathbf{L}$ is called q^*-compressible, when $\mathbf{L}$ is q^*-computable assuming each entry of $\mathbf{L}$ is available at unit cost
- [CDD'01], [Stevenson '04]: Suppose
  - $\{\psi_\lambda\}$ are piecewise polynomial wavelets that
    - are sufficiently smooth and
    - have sufficiently many vanishing moments
  - $\mathbf{L}$ is either differential or singular integral operator

  then $\mathbf{L}$ is q^*-compressible for some $q^* \geq \frac{d-t}{n} \quad (> s)$

Computability

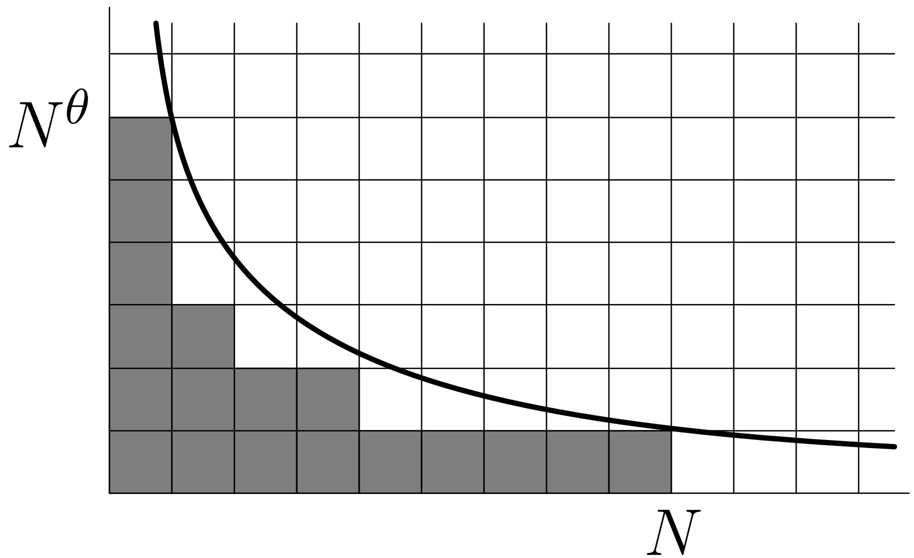
- Distribute computational works over the entries



- Require: shaded area $= \mathcal{O}(N)$

Computability

- Distribute computational works over the entries



- $p(x) = N^\theta x^{-\varrho} \rightsquigarrow$ when $\theta \le \varrho < 1$

$$\int_0^N N^\theta x^{-\varrho} dx = \tfrac{1}{1-\varrho} N^{1+\theta-\varrho} = \mathcal{O}(N)$$

Computability

[T.G., Stevenson '04], [T.G.'04]: Suppose

- $\{\psi_\lambda\}$ are piecewise polynomial wavelets that
  - are sufficiently smooth and
  - have sufficiently many vanishing moments
- $\mathbf{L}$ is either differential or singular integral operator

then $\mathbf{L}$ is q^*-computable for some $q^* \geq \frac{d-t}{n} \quad (> s)$

So the adaptive wavelet method has the optimal convergence rate and optimal computational complexity

References

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