Nonconstant mean curvature solutions of the Einstein constraint equations

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Outline

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Einstein constraint equations	$\mathrm{Ric}(N, \cdot) = 0$
Conformal parameterization	$g = u^4 g_0$
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Lichnerowicz equation	$-8\Delta u + Ru + \alpha u^5 = \beta u^{-7}$
Coupling	$\beta = \lvert\sigma + Lw\rvert^2$ and $\omega = u^6 \mathrm{d}\tau$
Near-CMC solutions	$\mathrm{d}\tau$ small
Small-TT solutions	σ small
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Einstein constraint equations

Let M be a closed 3-manifold. The initial data for the Einstein evolution equation on M consist of

- Riemannian metric h on M
- Symmetric 2-tensor K (extrinsic curvature of M inside the space-time that is to be "grown")

They must satisfy the constraint equations

$$\mathrm{scal} - |K|^2 + (\underbrace{\mathrm{tr}K}_{\mathrm{MC}})^2 = \rho, \qquad \mathrm{div}K - \mathrm{d}\,\underbrace{\mathrm{tr}K}_{\mathrm{MC}} = j,$$

where ρ and j are related to energy-momentum density.

The system is highly underdetermined.

Conformal parameterization

This is a proposed way to parameterize the constraint solution set. The free (or conformal) data consist of

- Riemannian metric g on M, representing the conformal class $[g] = \{e^\alpha g : \alpha \in C^\infty(M)\}$,
- Symmetric transverse traceless 2-tensor σ (TT-tensor),
- Scalar function τ, specifying the mean curvature,

and the determined data consist of

- Positive scalar function $u \in C^\infty(M, \mathbb{R}_+)$,
- 1-form $w \in \Omega^1(M)$.

We assume $\qquad h = u^4 g, \qquad K = u^{-2}(\sigma + Lw) + \frac{1}{3}\tau g,$

where

$$(Lw)_{ab} = \nabla_a w_b + \nabla_b w_a - \frac{2}{3} g_{ab} \operatorname{div} w.$$

Then the (vacuum) constraint equations are equivalent to

$$-8\Delta u + Ru + \frac{2}{3}\tau^2 u^5 = |\sigma + Lw|^2 u^{-7}, \qquad \operatorname{div} Lw = \frac{2}{3} u^6 \mathrm{d}\tau.$$

Momentum constraint

Note that if $\mathrm{d}\tau = 0$ (CMC case), then the constraint system

$$-8\Delta u + Ru + \frac{2}{3}\tau^2 u^5 = |\sigma + Lw|^2 u^{-7}, \qquad \mathrm{div} Lw = \frac{2}{3} u^6 \mathrm{d}\tau,$$

decouples. The equation

$$Aw \equiv \mathrm{div} Lw = \omega,$$

is called the momentum (or vector) constraint equation. Note that in the constraint system, the momentum constraint appears with $\omega = \frac{2}{3} u^6 \mathrm{d}\tau$.

The operator A is self-adjoint and strongly elliptic, and its kernel is given by the conformal Killing fields.

$$\ker A = \ker L$$

In particular, we have the elliptic estimate

$$\|w\|_{W^{2,p}} \lesssim \|Aw\|_{L^p} + \|w\|_{L^p}$$

Lichnerowicz equation

With $\alpha \geq 0$ and $\beta \geq 0$, the equation

$$-8\Delta u + Ru + \alpha u^5 = \beta u^{-7}$$

is called the Lichnerowicz equation. Note that in the constraint system, the Lichnerowicz equation appears with $\alpha = \frac{2}{3}\tau^2$ and $\beta = |\sigma + Lw|^2$.

Suppose $\alpha + \beta \not\equiv 0$ and $\alpha, \beta \in L^p$ with $p > 3$. Then there exists a positive solution $u \in W^{2,p}$ if and only if one of the following conditions holds:

- g is Yamabe positive, and $\beta \not\equiv 0$
- g is Yamabe null, $\alpha \not\equiv 0$, and $\beta \not\equiv 0$
- g is Yamabe negative, and there is a metric in $[g]$ with scalar curvature equal to $-\alpha$

Moreover, in each case the solution is unique.

This settles the CMC case. There one even has $\alpha = \mathrm{const}$.

Contributors: York, Choquet-Bruhat, Isenberg, Ó Murchadha, Maxwell, ...

Conformal invariance

Let $\bar{g} = \theta^4 g$. Then

$$-8\Delta u + Ru + \alpha u^5 \gtreqless \beta u^{-7} \qquad \Longleftrightarrow \qquad -8\bar{\Delta} v + \bar{R} v + \alpha v^5 \gtreqless \bar{\beta} v^{-7}$$

with $v = \theta^{-1} u$, and $\bar{\beta} = \theta^{-12}\beta$.

Example usage: Isenberg and Ó Murchadha proved that the constraint system has no solution if $R \geq 0$, $\sigma \equiv 0$, and $\frac{\|\mathrm{d}\tau\|_\infty}{\min|\tau|} \sim 0$. Let us extend it to the full nonnegative Yamabe class.

Let $\theta > 0$ be such that $\bar{g} = \theta^4 g$ has $\bar{R} \geq 0$. Suppose that the constraint system has a solution u. Then $v = \theta^{-1} u$ is the solution of the θ-scaled Lichnerowicz equation with $\bar{\beta} = \theta^{-12}|Lw|^2$. At maximum of v

$$\alpha v^5 \leq \bar{\beta} v^{-7} = \theta^{-12}|Lw|^2 v^{-7} \lesssim \theta^{-12}\|\mathrm{d}\tau\|_{L^p}^2 \|u\|_{L^\infty}^{12} v^{-7} \leq \theta^{-12}\|\mathrm{d}\tau\|_{L^p}^2 \|\theta\|_{L^\infty}^{12} v^5$$

which gives a contradiction if $\frac{\|\mathrm{d}\tau\|_{L^p}^2}{\alpha} \equiv \frac{\|\mathrm{d}\tau\|_{L^p}^2}{\tau^2}$ is small enough.

Sub- and supersolutions

Positive function u is called a supersolution if

$$-8\Delta u + Ru + \alpha u^5 \geq \beta u^{-7}.$$

Subsolutions are defined analogously.

If $u_- > 0$ is a subsolution and $u_+ \geq u_-$ is a supersolution, then there exists a solution u to the Lichnerowicz equation, satisfying $u_- \leq u \leq u_+$. If uniqueness available, this implies pointwise bounds. Note also that subsolutions (supersolutions) can always be scaled down (up).

Example subsolution: Let $\theta > 0$ be such that $\bar{g} = \theta^4 g$ has $\bar{R} \geq 0$. Then solve

$$-8\bar{\Delta} v + (\bar{R} + \alpha) v = \bar{\beta} \equiv \theta^{-12} \beta,$$

which has a unique positive solution if $\bar{R} + \alpha \not\equiv 0$ and $\beta \equiv 0$. From

$$-8\bar{\Delta} cv + \bar{R} cv + \alpha (cv)^5 - \bar{\beta} (cv)^{-7} = \alpha \left((cv)^5 - cv \right) + \bar{\beta} \left(c - (cv)^{-7} \right),$$

we see that cv is a subsolution of the θ-scaled Lichnerowicz equation if $c > 0$ is sufficiently small. Hence $\theta^{-1} cv$ is a subsolution of the original Lichnerowicz equation.

Uniqueness

Suppose $u>0$ and $\theta>0$ are two solutions of

$$-8\Delta u+Ru+\alpha u^5=\beta u^{-7}.$$

Let $\bar{g}=\theta^4 g$. Then

$$\bar{R}=\theta^{-5}(-8\Delta\theta+R\theta)=\theta^{-5}(\beta\theta^{-7}-\alpha\theta^5)=\bar{\beta}-\alpha,$$

and so $v=\theta^{-1}u$ solves

$$0=-8\bar{\Delta}v+\bar{R}v+\alpha v^5-\bar{\beta}v^{-7}=-8\bar{\Delta}v+\alpha(v^5-1)+\bar{\beta}(1-v^{-7}).$$

Hence

$$\int 8|\bar{\nabla}(v-1)|^2=-\int 8(v-1)\bar{\Delta}v=-\int\alpha(v^5-1)(v-1)-\int\bar{\beta}(1-v^{-7})(v-1).$$

We conclude that $v=\text{const}$, and so $v\equiv 1$ unless $\alpha=\beta\equiv 0$. The latter condition would force $\bar{R}\equiv 0$ hence Yamabe null.

The scaling argument also implies nonexistence for Yamabe positive with $\beta\equiv 0$ and Yamabe negative with $\alpha\equiv 0$.

NonCMC case

Recall the constraint system

$$-8\Delta u + Ru + \alpha u^5 = |\sigma + Lw|^2 u^{-7}, \qquad \mathrm{div} Lw = \frac{2}{3} u^6 \mathrm{d}\tau,$$

with $\alpha = \frac{2}{3}\tau^2$. A picture to have in mind is

$$-8\Delta u + Ru + \alpha u^5 = c|\nabla(1-\Delta)^{-1}(u^6)|^2 u^{-7}.$$

- Isenberg-Moncrief '96: Near-CMC, Yamabe negative
- Allen-Clausen-Isenberg '07: Near-CMC, Yamabe nonnegative
- Holst-Nagy-Tsogtgerel '07: Small-TT, Yamabe positive, nonvacuum
- Maxwell '08: Small-TT, Yamabe positive, vacuum
- Maxwell '09: Model problem on $\mathbb{T}^n$
- Dahl-Gicquaud-Humbert '10: Near-CMC, compactness of the set of solutions, C^0-density of metrics that admit solution
- Tcheng '11: Model problem on $S^1 \times S^2$

Fixed point iterations

Let us write the constraint system

$$-8\Delta u + Ru + \alpha u^5 = |\sigma + Lw|^2 u^{-7}, \qquad \mathrm{div} Lw = \frac{2}{3} u^6 \mathrm{d}\tau,$$

with $\alpha = \frac{2}{3}\tau^2$, as

$$u = \mathscr{L}(|\sigma + Lw|^2), \qquad w = \mathscr{M}(u^6 \mathrm{d}\tau).$$

We assume g, α and σ are so that $\mathscr{L}(|\sigma|^2)$ is well-defined. For σ this means that $\sigma \not\equiv 0$ if g is Yamabe nonnegative. Since σ is L^2-orthogonal to Lw, $\sigma \not\equiv 0$ implies $\sigma + Lw \not\equiv 0$. The constraint system is equivalent to

$$u = \mathscr{N}(u) \qquad \text{with} \qquad \mathscr{N}(u) = \mathscr{L}(\mathscr{M}(u^6 \mathrm{d}\tau)).$$

This iteration was introduced by Isenberg and Moncrief in 96.

In [Holst-Nagy-Tsogtgerel '07] we inverted only a linear part of the Lichnerowicz equation.

Invariant set via supersolutions

To solve $u = \mathscr{N}(u)$, we need to establish an invariant set consisting of positive functions for the operator $\mathscr{N}$. For this purpose, global barriers have been used.

A positive function u_+ is called a global upper barrier if it is a supersolution of the Lichnerowicz equation with $\beta = |\sigma + L\mathscr{M}(u^6 \mathrm{d}\tau)|^2$ for all $0 < u \le u_+$. If u_+ is a global upper barrier, then $0 < \mathscr{N}(u) \le u_+$ pointwise for all $0 < u \le u_+$.

Example: Let $u_+ > 0$ be a constant. We want

$$Ru_+ + \alpha u_+^5 \ge |\sigma + L\mathscr{M}(u^6 \mathrm{d}\tau)|^2 u_+^{-7}, \qquad \text{for all} \qquad 0 < u \le u_+.$$

Since

$$|\sigma + L\mathscr{M}(u^6 \mathrm{d}\tau)| \lesssim |\sigma| + \|\mathrm{d}\tau\|_{L^p} \|u^6\|_{L^\infty} \lesssim |\sigma| + \|\mathrm{d}\tau\|_{L^p} \|u_+^6\|_{L^\infty},$$

provided that $\frac{\|\mathrm{d}\tau\|}{\min \tau}$ is smaller than some threshold value, any sufficiently large constant u_+ yields a global upper barrier.

The same constant u_+ also provides an *a priori* upper bound.

Maxwell's floor

Global lower barriers were used in [Holst-Nagy-Tsogtgerel '07] to bound the iteration from below. This restricted us to nonvacuum case for Yamabe positive metrics. Shortly after, Maxwell in '08 resolved the issue by the following elegant argument.

Let $V \geq 0$ and $V \not\equiv 0$. Then the Green function G of $-\Delta + V$ satisfies $G(x,y) \geq c$ for some constant $c > 0$.

For the solution u of $-\Delta u + Vu = f$ with $f \geq 0$, this implies

$$u(x) = \int G(x,y) f(y) \mathrm{d}y \geq c \int f = c\|f\|_{L^1}.$$

We saw that $\theta^{-1} cv$ is a subsolution (and so a lower bound), where

$$-8\bar{\Delta} v + (\bar{R} + \alpha) v = \bar{\beta} \equiv \theta^{-12} \beta,$$

and $c = \min\{1, \|v\|_{L^\infty}^{-1}\}$. Hence

$$\theta^{-1} cv \gtrsim \frac{\|\beta\|_{L^1}}{\|\beta\|_{L^p}} = \frac{\|\sigma\|_{L^2}^2 + \|Lw\|_{L^2}^2}{\|\beta\|_{L^p}} \geq \frac{\|\sigma\|_{L^2}^2}{\|\beta\|_{L^p}} \gtrsim \frac{\|\sigma\|_{L^2}^2}{1 + \|\mathrm{d}\tau\|_{2p}^2 \|u_+^{12}\|_{L^\infty}}$$

Compactness

One can apply the Schauder theorem with the invariant set $\{c \le u \le u_+\}$. Moreover, $\mathscr{N}$ is a contraction if $\frac{\|\mathrm{d}\tau\|}{\min \tau}$ is small enough.

However, we want the invariant set to be the L^r-ball $U = \{u : \|u\|_{L^r} \le M\}$.

The Lichnerowicz solution operator $\mathscr{L} : L^p \to W^{2,p}$ is C^1, [Maxwell '08].

For $u = \mathscr{L}(\beta)$, we have

$$\|\Delta u\|_{L^p} \lesssim \|u\|_{L^p} + \|\alpha\|_{L^\infty} \|u\|^5_{L^{5p}} + \|\beta\|_{L^p} \|u^{-7}\|_{L^\infty},$$

and u^{-1} is uniformly bounded, so $\mathscr{N} : U \to U$ is compact.

Invariant set via *a priori* estimates

From

$$-8\Delta u + Ru + \alpha u^5 = \beta u^{-7}, \qquad \text{i.e.,} \qquad -8u^7\Delta u + Ru^8 + \alpha u^{12} = \beta,$$

we get, with $\beta = |\sigma + L\mathscr{M}(v^6 \mathrm{d}\tau)|$

$$\int |\nabla u^4|^2 + Ru^8 + \alpha u^{12} \lesssim \|\beta\|_{L^1} \lesssim \|\sigma^2\| + \|\mathrm{d}\tau\|^2 \|v\|_{L^r}^{12}.$$

Supposing $R > 0$ and $\|v\|_{L^r} \leq M$, this yields

$$\|u\|_{L^8}^8 \lesssim \|\sigma\|^2 + \|\mathrm{d}\tau\|^2 M^{12}.$$

Similar argument gives

$$\|u\|_{L^{8+q}}^{8+q} \lesssim \left(\|\sigma\|^2 + \|\mathrm{d}\tau\|^2 M^{12}\right) \|u\|_{L^q}^q, \quad \text{hence} \quad \|u\|_{L^{8n}}^{8n} \lesssim \|\sigma\|^{2n} + \|\mathrm{d}\tau\|^{2n} M^{12n}.$$

Take $r = 8n$, and an invariant set is obtained if $\|\sigma\|$ is small enough (small-TT solutions).

Compact manifolds with boundary

What boundary conditions?

Outer boundary: Dirchlet $u = 1$ or Robin $\partial_r u + u = 1$.

Inner boundary: Trapped surface condition. We want

$$\theta_{\pm} \equiv \mp 2H + \tau - K(\nu, \nu) \leq 0.$$

In conformal quantities, this is

$$\theta_{\pm} \equiv \mp 2u^{-3}\left(2\partial_{\nu} u + Hu\right) + \frac{2}{3}\tau - u^{-6}\sigma(\nu, \nu),$$

and one ends up with a condition that looks like

$$2\partial_{\nu} u + Hu + \alpha u^{3} + \beta u^{-3} + \gamma u^{e} = 0.$$

The theory of Lichnerowicz equation can be extended to this case [Holst-Tsogtgerel].