CRM/McGill Applied Mathematics Seminar

Local convergence of adaptive finite element methods for nonlinear problems

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Model problem

$$-u''(x) = f(x) \quad \text{in} \quad I = (0,1), \qquad \text{and} \qquad u(0) = u(1) = 0.$$

Define $A : C^2(I) \to C^0(I)$ by

$$Au = -u'' \qquad \text{for} \quad u \in C^2(I).$$

Let $D = C_0^\infty(I)^*$. Then $C^0(I) \subset L^1(I) \subset D$ by

$$\langle w, v \rangle = \int_0^1 wv \leqslant \|w\|_{L^1} \|v\|_{C^0} \qquad \text{for} \quad w \in L^1(I),\, v \in C_0^\infty(I).$$

We can extend A to $A : D \to D$ by

$$\langle Au, v \rangle = -\langle u, v'' \rangle \qquad \text{for any} \quad v \in C_0^\infty(I).$$

Note that if $u \in C^2(I)$ we have

$$\int_0^1 -u''v = -u''v\Big|_0^1 + \int_0^1 u'v' = u'v'\Big|_0^1 - \int_0^1 uv''.$$

Weak formulation

$$\langle Au, v\rangle = -\int_0^1 u''v = uv\Big|_0^1 + \int_0^1 u'v' \leqslant \|u'\|_{L^2}\|v'\|_{L^2}$$

is an inner product on $C_0^\infty(I)$. The induced norm is

$$\|u\|_{H^1}^2 = \langle Au, u\rangle = \int_0^1 (u')^2$$

and the completion of $C_0^\infty(I)$ wrt to this norm is denoted by $H_0^1(I)$.
Thus $A : H_0^1(I) \to H^{-1}(I) \equiv H_0^1(I)^*$ is bounded, linear, and invertible.
In particular, for any $f \in H^{-1}(I)$ the following equation has a unique solution

$$Au = f.$$

Galerkin method

Let X be a Hilbert space, and $A : X \to X^*$ be a bounded linear operator satisfying

$$\langle Av, v\rangle \geqslant \alpha^2 \|v\|_X^2 \qquad \text{for any} \quad v \in X \quad (\alpha > 0).$$

Then $\langle A\cdot, \cdot\rangle$ is an inner product on X, inducing a norm $\|\cdot\|_A$ equivalent to $\|\cdot\|_X$. So A is invertible.

$$Au = f \qquad \Leftrightarrow \qquad \langle Au, v\rangle = \langle f, v\rangle \quad \text{for all} \quad v \in X.$$

Let $X_h \subset X$ be a linear subspace. Consider $u_h \in X_h$ such that

$$\langle Au_h, v\rangle = \langle f, v\rangle \qquad \text{for all} \quad v \in X_h.$$

This gives the Galerkin orthogonality

$$\langle A(u - u_h), v\rangle = 0 \qquad \text{for all} \quad v \in X_h$$

or $u - u_h \perp_A X_h$.

$$\|u - u_h\|_A = \inf_{v \in X_h} \|u - v\|_A.$$

Finite element method

$X = H_0^1(I)$. Let $T_h = \{[0,h],[h,2h],\ldots,[1-h,1]\}$, and

$$X_h = \{v \in C_0^0(I) : v \text{ is linear on each of } e \in T_h\}.$$

Let $\{\phi_i\}$ be a basis of X_h, and put $u_h = \sum_i U_i \phi_i$

$$\sum_i U_i \langle A\phi_i, \phi_k\rangle = \langle f, \phi_k\rangle \qquad k = 1,\ldots,m.$$

$$\|u-u_h\|_A = \inf_{v\in X_h} \|u-v\|_A \lesssim h^{s-1}\|u\|_{H^s} \qquad \text{for any} \quad s \leqslant 2.$$

In general, and piecewise polynomial elements of order d, we have

$$\|u-u_h\|_{H^1} \lesssim h^{s-1}\|u\|_{H^s} \qquad \text{for any} \quad s \leqslant d.$$

In n-dimension, the number of degrees of freedom $N \sim h^{-n}$, so

$$\|u-u_h\|_{H^1} \lesssim N^{-\frac{s-1}{n}}\|u\|_{H^s} \qquad \text{for any} \quad s \leqslant d.$$

Saddle point problems

$$\left.\begin{aligned} -\Delta \mathbf{u} + \operatorname{grad} p &= \mathbf{f} \\ \operatorname{div} \mathbf{u} &= 0 \end{aligned}\right\} \quad \text{or} \quad Au = f \quad \text{with} \quad A = \begin{pmatrix} -\Delta & \operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}.$$

Set $X = (H_0^1)^n \times L^2$. Let $X_h \subset X$, and consider $u_h \in X_h$ such that

$$\langle Au_h, v \rangle = \langle f, v \rangle \qquad \text{for all} \quad v \in X_h.$$

The following theorem is due to Jinchao Xu and Ludmil Zikatonov.

Theorem (Babuška-Brezzi-Ladyzhenskaya condition)

The above problem is uniquely solvable if and only if

$$\inf_{w \in X_h} \sup_{v \in X_h} \frac{\langle Aw, v \rangle}{\|w\|_X \|v\|_X} = \alpha_h > 0.$$

Moreover, the latter implies

$$\|u - u_h\| \leqslant \frac{\|A\|}{\alpha_h} \inf_{v \in X_h} \|u - v\|_X.$$

Semilinear problems

$$-\Delta u + u^q = f \quad \text{in} \quad \Omega \subset \mathbb{R}^3, \qquad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

$$u \in H^1 \Rightarrow u \in L^6 \Rightarrow u^q \in L^{6/q} \Rightarrow u^q \in H^{-1+s} \quad \text{with} \quad s = \frac{5-q}{2}.$$

If $f \in L^2$, then $(-\Delta)^{-1}(f - u^q) \in H^{1+s}$. Hence

$$\phi : H_0^1 \to H_0^1 : u \mapsto (-\Delta)^{-1}(f - u^q)$$

is compact if $q < 5$. By Schauder, there exists $u \in H_0^1$ such that $\phi(u) = u$.

Galerkin approximation of a locally unique solution is locally quasi-optimal.

A posteriori error estimates

$X = H_0^1(I)$. Let $T_h = \{[0, x_1], [x_1, x_2], \ldots, [x_m, 1]\}$, and

$$X_h = \{v \in C_0^0(I) : v \text{ is linear on each of } e \in T_h\}.$$

For any $v \in X$ and its linear interpolant $v_h \in X_h$, we have

$$\begin{aligned}
\langle f - Au_h, v\rangle = \langle f - Au_h, v - v_h\rangle &= \int_0^1 f(v - v_h) - u_h'(v - v_h)' \\
&= \sum_i \int_{x_i}^{x_{i+1}} f(v - v_h) - u_h'(v - v_h)\Big|_{x_i}^{x_{i+1}} \\
&\leqslant C \sum_i \|f\|_{L^2(x_i, x_{i+1})} \|v - v_h\|_{L^2(x_i, x_{i+1})} \\
&\leqslant C \sum_i \|f\|_{L^2(x_i, x_{i+1})} h_i \|v\|_{H^1(x_i, x_{i+1})}
\end{aligned}$$

implying that

$$\|f - Au_h\|_{H^{-1}}^2 \leqslant C \sum_i h_i^2 \|f\|_{L^2(x_i, x_{i+1})}^2 = \sum_i \eta_i^2.$$

Lower bound

Let f be piecewise constant wrt T_h, and φ be a "bubble" function supported in (x_i, x_{i+1}).

$$\begin{aligned}
\|f\|^2_{L^2(x_i,x_{i+1})} &\lesssim \int_{x_i}^{x_{i+1}} f \cdot f\varphi = \int_{x_i}^{x_{i+1}} (u - u_h)'(f\varphi)' \\
&\leqslant \|u - u_h\|_{H^1(x_i,x_{i+1})} \|f\varphi\|_{H^1} \\
&\lesssim \|u - u_h\|_{H^1(x_i,x_{i+1})} \|f\|_{L^2(x_i,x_{i+1})} h_i^{-1}
\end{aligned}$$

implying that

$$\eta_i = h_i \|f\|_{L^2(x_i,x_{i+1})} \lesssim \|u - u_h\|_{H^1(x_i,x_{i+1})}.$$

Interior node property

Let f be piecewise constant wrt T_h, and $\varphi \in X_\ell$ ($\ell < h$) be a “bubble” function supported in (x_i, x_{i+1}).

$$\begin{aligned}\|f\|^2_{L^2(x_i,x_{i+1})} &\lesssim \int_{x_i}^{x_{i+1}} f \cdot f\varphi = \int_{x_i}^{x_{i+1}} (u_\ell - u_h)'(f\varphi)' \\ &\leqslant \|u_\ell - u_h\|_{H^1(x_i,x_{i+1})} \|f\varphi\|_{H^1} \\ &\lesssim \|u_\ell - u_h\|_{H^1(x_i,x_{i+1})} \|f\|_{L^2(x_i,x_{i+1})} h_i^{-1}\end{aligned}$$

implying that

$$\eta_i = h_i \|f\|_{L^2(x_i,x_{i+1})} \lesssim \|u_\ell - u_h\|_{H^1(x_i,x_{i+1})}.$$

Adaptive finite element method

Start with some initial mesh T_0. Set $k = 0$, and repeat

- Solve for the Galerkin solution u_k on the mesh T_k
- Estimate the error indicators $\{\eta_i\}$ over the elements of T_k
- Refine the elements of T_k with largest error, to get T_{k+1}
- $k++$

Questions:

- $u_k \to u$?
- $\|u_k - u\|_{H^1} \lesssim \rho^k$ with $\rho < 1$?
- $\dim X_k \sim$?

Linear convergence

From the Galerkin orthogonality

$$\langle A(u-u_{i+1}), v\rangle = 0 \qquad \text{for all } v \in X_{i+1},$$

taking $v = u_{i+1} - u_i$, we have

$$\|u-u_i\|_A^2 = \|u-u_{i+1}\|_A^2 + \|u_{i+1}-u_i\|_A^2.$$

So if

$$\|u_{i+1}-u_i\|_A \geqslant c\|u-u_i\|_A,$$

with constant $c \in (0, 1)$, we have

$$\|u-u_{i+1}\|_A^2 = \|u-u_i\|_A^2 - \|u_{i+1}-u_i\|_A^2 \leqslant (1-c^2)\|u-u_i\|_A^2.$$

Quasi-orthogonality for semilinear problems

Let us consider

$$a(u, v) + (f(u), v) = 0, \qquad \forall v \in H$$

We have

$$\|u - u_i\|_a^2 = \|u - u_{i+1}\|_a^2 + \|u_{i+1} - u_i\|_a^2 + 2a(u - u_{i+1}, u_{i+1} - u_i)$$

$$\begin{aligned} a(u - u_{i+1}, u_{i+1} - u_i) &= (f(u) - f(u_{i+1}), u_{i+1} - u_i) \\ &\leqslant C\|f(u) - f(u_{i+1})\|_{L^2}\|u_{i+1} - u_i\|_{L^2} \\ &\leqslant C\|u - u_{i+1}\|_{L^2}\|u_{i+1} - u_i\|_{L^2} \\ &\leqslant Ch_{i+1}\|u - u_{i+1}\|_{H^1}\|u_{i+1} - u_i\|_{H^1} \end{aligned}$$

Abstract argument

Let $F : X \to X^*$ be continuous, and consider

$$\langle F(u), v \rangle = 0 \qquad \text{for all} \quad v \in X.$$

Suppose that an AFEM generated the following

- subspaces $X_0, X_1, \ldots \subset X$
- approximations $u_0, u_1, \ldots$, solutions to $\langle F(u_k), v_k \rangle = 0 \; \forall v_k \in X_k$

Let $X_\infty = \cup_i X_i$, and let $u_\infty \in X_\infty$ be such that $\langle F(u_\infty), v \rangle = 0 \; \forall v \in X_\infty$.
If u_k are locally quasi-optimal, then

$$\|u_\infty - u_k\|_X \lesssim \inf_{v \in X_k} \|u_\infty - v\|_X \to 0 \qquad \text{as} \quad k \to \infty.$$

For any $v \in X$, we have

$$\begin{aligned}\langle F(u_\infty), v \rangle &= \langle F(u_\infty) - F(u_k), v \rangle + \langle F(u_k), v \rangle \\ &\leqslant \|F(u_\infty) - F(u_k)\|_{X^*} \|v\|_X + |\langle F(u_k), v \rangle| \to 0,\end{aligned}$$

provided $\lim_{k \to \infty} \langle F(u_k), v \rangle = 0$.

Error estimate and marking

$X = H_0^1(\Omega)$. $\eta_k : T_k \to \mathbb{R}$ error estimator corresponding to T_k and u_k.
If $D \subset \Omega$ is a union of some elements in T_k, then

$$\langle F(u_k), v\rangle \lesssim \eta_k(D)\|v\|_{H^1(D)} + \eta_k(\Omega \setminus D)\|v\|_{H^1(\Omega\setminus D)},$$

and for some $f \in L^2(\Omega)$

$$\eta_k(D) \lesssim \|u_k\|_{H^1(D)} + \|f\|_{L^2(D)}.$$

Let $M_k \subset T_k$ be the marked elements to refine. Then

$$\eta_k(\tau) \leqslant C \max_{\sigma\in M_k} \eta_k(\sigma) \qquad \tau \in T_k \setminus M_k$$

Convergence

The elements in T_i that are not refined anymore, and the others

$$T_i^+ = \{\tau \in T_i : \tau \in T_j, \quad \forall j \geqslant i\}, \qquad T_i^0 = T_i \setminus T_i^+,$$

$$\Omega_i^+ = \bigcup_{\tau \in T_i^+} \tau, \qquad \Omega_i^0 = \bigcup_{\tau \in T_i^0} \tau.$$

For $j < i$ we have

$$T_j^+ \subset T_i^+ \subset T_i \quad \text{and} \quad \Omega_j^0 = \bigcup_{\tau \in T_i \setminus T_j^+} \tau.$$

For any $v \in X$ and $\bar{v} \in X_j$ such that $\|v - \bar{v}\|_{H^1(\Omega^0)} \leqslant \varepsilon$ with $\Omega^0 = \cap_j \Omega_j^0$

$$\langle F(u_i), v \rangle = \langle F(u_i), v - \bar{v} \rangle \lesssim \eta_i(\Omega_j^0) \|v - \bar{v}\|_{H^1(\Omega_j^0)} + \eta_i(\Omega_j^+) \|v - \bar{v}\|_{H^1(\Omega_j^+)}.$$

$$\|v - \bar{v}\|_{H^1(\Omega_j^0)} \lesssim \|v - \bar{v}\|_{H^1(\Omega^0)} + \|v - \bar{v}\|_{H^1(\Omega_j^0 \setminus \Omega^0)} \leqslant 2\varepsilon.$$

Convergence

$$\begin{aligned}\eta_i(\Omega_j^0) &\lesssim \|u_i\|_{H^1(\Omega_j^0)} + \|f\|_{L^2(\Omega_j^0)} \leqslant \|u_i\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} \\ &\leqslant \|u_i - u_\infty\|_{H^1(\Omega)} + \|u_\infty\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} \\ &\leqslant 2\|u_\infty\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}.\end{aligned}$$

We have for $\tau \in T_j^+$

$$\eta_i(\tau) \leqslant C \max_{\sigma \in T_i^0} \eta_i(\sigma).$$

and for $\sigma \in T_i^0$

$$\begin{aligned}\eta_i(\sigma) &\lesssim \|u_i\|_{H^1(\sigma)} + \|f\|_{L^2(\sigma)} \\ &\leqslant \|u_i - u_\infty\|_{H^1(\sigma)} + \|u_\infty\|_{H^1(\sigma)} + \|f\|_{L^2(\sigma)}.\end{aligned}$$

Manuscripts, Collaborators, Acknowledgments

HTZ MICHAEL HOLST, GT, AND YUNRONG ZHU, Local convergence of adaptive methods for nonlinear PDE's. In preparation.

HT MICHAEL HOLST, AND GT, Convergent adaptive finite element approximation of the Einstein constraints. In preparation.

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