CRM/McGill Applied Mathematics Seminar

On analysis and numerical treatment of Einstein's constraint equations

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March 13, 2009

Gravitational wave astronomy

Recently constructed gravitational wave detectors: LIGO, VIRGO, GEO600, TAMA300.

The two L-shaped LIGO observatories (in Washington and Louisiana), with legs at 4km, have phenomenal sensitivity, on the order of 10^{-15}m to 10^{-18}m. effective ranges (1.4Sol): 7-15MPc

Initial value formulation of the Einstein equations

The Lorentzian manifold $(M, \mathbf{g})$ satisfies

$$G(\mathbf{g}) := \mathrm{Ric}(\mathbf{g}) - \tfrac{1}{2}R(\mathbf{g})\mathbf{g} = 0.$$

Suppose $M = \mathbb{R} \times \Sigma$, each $\Sigma_t = \{t\} \times \Sigma$ is spacelike. On each Σ_t, one has

$$\begin{aligned} R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 &= 0, \\ \mathrm{div}_g K - d(\mathrm{tr}_g K) &= 0. \end{aligned} \tag{C}$$

Conversely, if (C) holds on some Riemannian manifold (Σ, g), then there are

- a Lorentzian manifold $(M, \mathbf{g})$
- and an embedding $\theta : \Sigma \to M$

such that $G(\mathbf{g}) = 0$ and that $\theta_* g$ and $\theta_* K$ are the first and second fundamental forms of $\theta\Sigma \subset M$ [Choquet-Bruhat '52].

The conformal method

Let $(\Sigma, \hat{g})$ be a Riemannian manifold, σ be a symmetric tensor with $\mathrm{div}_{\hat{g}}\,\sigma = 0$, $\mathrm{tr}_{\hat{g}}\,\sigma = 0$, and let $\tau \in C^\infty(\Sigma)$. With ϕ a positive scalar, and w a vector field, put

$$g = \phi^4 \hat{g}, \quad K = \phi^{-2}(\sigma + L_{\hat{g}} w) + \tfrac{1}{3}\tau\phi^4\hat{g},$$

where $L_{\hat{g}} w = \mathcal{L}_w \hat{g} - \frac{2}{3}\hat{g}\,\mathrm{div}_{\hat{g}} w$. Then (C) is equivalent to

$$\begin{aligned} -8\Delta_{\hat{g}}\phi + R(\hat{g})\phi + \tfrac{2}{3}\tau\phi^5 - \left|\sigma + L_{\hat{g}} w\right|_{\hat{g}}^2 \phi^{-7} &= 0, \\ -\mathrm{div}_{\hat{g}} L_{\hat{g}} w + \tfrac{3}{2}\phi^6 d\tau &= 0. \end{aligned}$$

Let us rewrite the above as

$$\begin{aligned} A\phi + R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} =: A\phi + f(w, \phi) &= 0, \\ Bw + \phi^6 d\tau &= 0. \end{aligned}$$

Note that $\mathrm{tr}_g K = \tau$ and that if $\tau = \mathrm{const}$ the system decouples.

Constant mean curvature solutions

[York, O'Murchadha, Isenberg, Marsden, Choquet-Bruhat, Moncrief, Maxwell, et al.]

$$A\phi + f(w, \phi) = 0, \qquad Bw = 0.$$

Sub- and super-solutions, or *barriers*:

$$A\phi_- + f(w, \phi_-) \leqslant 0, \qquad A\phi_+ + f(w, \phi_+) \geqslant 0.$$

For any $s > 0$, the constraint equation is equivalent to

$$A\phi + s\phi = s\phi - f(w, \phi) \quad \Leftrightarrow \quad \phi = (A + sI)^{-1}(s\phi - f(w, \phi)).$$

If $s > 0$ is sufficiently large, the map

$$T : [\phi_-, \phi_+] \to [\phi_-, \phi_+] : \phi \mapsto (A + sI)^{-1}(s\phi - f(w, \phi))$$

is monotone increasing. Also, $T(\phi_-) \geqslant \phi_-$ and $T(\phi_+) \leqslant \phi_+$. The iteration

$$\phi_{n+1} = T(\phi_n), \qquad \phi_0 = \phi_-,$$

converges to a fixed point of T.

Super-solution

We want to find $\phi > 0$ such that

$$A\phi + f(w,\phi) = A\phi + R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} \geqslant 0.$$

Recall $a(w) = \left|\sigma + L_{\hat{g}} w\right|^2_{\hat{g}}$, and assume that w is fixed ($w = 0$ in CMC case).
Assume that $\tau = \text{const} > 0$, $R = \text{const}$, and let $\phi = \text{const} > 0$.

$$\begin{aligned} R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} &\geqslant \tfrac{2}{3}\tau\phi^5 + R\phi - \phi^{-7}\sup a(w) \\ &= \phi^{-7}\left(\tfrac{2}{3}\tau\phi^{12} + R\phi^8 - \sup a(w)\right) \end{aligned}$$

Choosing $\phi > 0$ sufficiently large one can ensure that the above is nonnegative.

Near constant mean curvature solutions

[Isenberg, Moncrief, Choquet-Bruhat, York, Allen, Clausen, et al.]

$$A\phi + f(w, \phi) = 0, \qquad Bw + \phi^6 d\tau = 0.$$

With $S : \phi \mapsto -B^{-1}(\phi^6 d\tau)$ this can be written as

$$A\phi + f(S(\phi), \phi) = 0.$$

Sub- and super-solutions make sense, but in general

$$T : \phi \mapsto (A + sI)^{-1}(s\phi - f(S(\phi), \phi))$$

is not monotone. Nevertheless, when $d\tau$ is small T is almost monotone, and the iteration $\phi_{n+1} = T(\phi_n)$ converges.

Now one needs global sub- and super-solutions, e.g., $\phi_+ > 0$ such that

$$A\phi_+ + f(w, \phi_+) \geqslant 0,$$

for all $w \in S([0, \phi_+])$.

Global super-solution

We want to find $\phi > 0$ such that

$$A\phi + f(w, \phi) = A\phi + R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} \geqslant 0.$$

for all $w \in S([0, \phi])$. Recall that $a(w) = \left|\sigma + L_{\hat{g}} w\right|^2_{\hat{g}}$. Elliptic estimates give

$$a(w) \leqslant p + q\|\phi\|^{12}_{C^0}, \qquad \text{with } q \sim |d\tau|^2$$

Assume that $\tau = \text{const} > 0$, $R = \text{const}$, and let $\phi = \text{const} > 0$, so $\|\phi\|_{C^0} = \phi$.

$$\begin{aligned} R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} &\geqslant \tfrac{2}{3}\tau\phi^5 + R\phi - p\phi^{-7} - q\phi^{-7}\phi^{12} \\ &= (\tfrac{2}{3}\tau - q)\phi^5 + R\phi - p\phi^{-7}. \end{aligned}$$

If $q < \frac{2}{3}\tau$, choosing $\phi > 0$ sufficiently large one can ensure that the above is nonnegative.

Fixed point approach

[Holst, Nagy, GT '07, '08]

Let $0 < \phi_- \leqslant \phi_+ < \infty$ be global barriers, i.e.,

$$A\phi_- + f(w, \phi_-) \leqslant 0, \qquad A\phi_+ + f(w, \phi_+) \geqslant 0,$$

for all $w \in S([\phi_-, \phi_+])$. Then for $s > 0$ large, and any $w \in S([\phi_-, \phi_+])$

$$T_w : \phi \mapsto (A + sI)^{-1}(s\phi - f(w, \phi))$$

is monotone increasing on $U = [\phi_-, \phi_+]$, and for $\phi \in U$

$$T(\phi) \equiv T_{S(\phi)}(\phi) \leqslant T_{S(\phi)}(\phi_+) \leqslant \phi_+, \qquad T(\phi) \geqslant \phi_-,$$

so $T : U \to U$. Since T is compact, there is a fixed point in U.

Global super-solution

[Holst, Nagy, GT '07, '08]

We want to find $\phi > 0$ such that

$$A\phi + f(w, \phi) = A\phi + R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} \geqslant 0.$$

for all $w \in S([0, \phi])$. Recall that

$$a(w) \leqslant p + q\|\phi\|_{C^0}^{12}$$

Assume that $R = \text{const} > 0$, $\tau = \text{const}$, and let $\phi = \text{const} > 0$.

$$\begin{aligned} R\phi + \tfrac{2}{3}\tau\phi^5 - a(w)\phi^{-7} &\geqslant \tfrac{2}{3}\tau\phi^5 + R\phi - p\phi^{-7} - q\phi^{-7}\phi^{12} \\ &\geqslant \phi^{-7}\left(R\phi^8 - (q - \tfrac{2}{3}\tau)\phi^{12} - p\right) \end{aligned}$$

If p is small enough (depending on how large q is), choosing $\phi > 0$ sufficiently small one can ensure that the above is nonnegative.

Extensions

- The framework is extended to allow for rough data, e.g., metrics in H^s with $s > \frac{5}{2}$
- The global super-solution construction is extended to all metrics in the positive Yamabe class (closed manifolds)

Ongoing work / wish list

- Asymptotically flat manifolds
- Manifolds with boundary, black hole initial data
- Zero and negative Yamabe classes, large data
- Full parameterization of the solution space

Finite element methods

Model problem: $-\Delta u = f$, or

$$a(u, v) := (\nabla u, \nabla v) = (f, v) \qquad \text{for all } v \in H$$

Let $S \subset H$ be a linear subspace. Consider $\tilde{u} \in S$ such that

$$a(\tilde{u}, v) = (f, v) \qquad \text{for all } v \in S$$

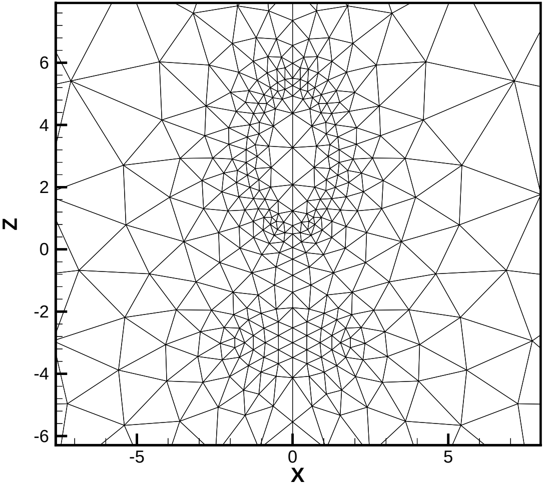
This gives the Galerkin orthogonality

$$a(u - \tilde{u}, v) = 0 \qquad \text{for all } v \in S$$

or $u - \tilde{u} \perp_a S$. $\tilde{u}$ is called the Galerkin approximation of u from S.

Typical finite element mesh

S is the space of continuous functions which are linear on each triangle.



Linear vs. nonlinear approximation

Let $S_0 \subset S_1 \subset \ldots \subset H$ with corresponding meshes $T_0, T_1, \ldots$, and Galerkin approximations $u_0, u_1, \ldots$.

$$\|u - u_i\|_a = \text{dist}(u, S_i) \leqslant C h_i^{s-1} \|u\|_{H^s}$$

where h_i is the maximum diameter of the triangles in T_i. If T_{j+1} is the uniform refinement of T_j, then $h_i \sim 2^{-i}$ and the number of vertices of T_i is $N_i \sim 2^{in}$ in n-dimension.

$$\|u - u_i\|_a = \text{dist}(u, S_i) \leqslant C 2^{-i(s-1)} \|u\|_{H^s} \leqslant C N_i^{-(s-1)/n} \|u\|_{H^s}$$

Is T_i optimal among meshes with N_i vertices?
Given a mesh, let $S(T)$ be the corresponding FE space. Let

$$\Sigma_N = \cup \{S(T) : T \text{ is a refinement of } T_0 \text{ and } \#T \leqslant N\}$$

Then with $\frac{1}{p} = \frac{1}{2} + \frac{s-1}{n}$

$$\text{dist}(u, \Sigma_N) \leqslant C N^{-(s-1)/n} \|u\|_{W^{s,p}}$$

Adaptive finite element methods

In a typical AFEM, the sequence u_i is generated as follows. Start with some initial mesh T_0. Set $i = 0$, and repeat

- Solve for u_i
- Estimate the distribution of $u_i - u$ over the triangles of T_i
- Refine the triangles of T_i with largest error, to get T_{i+1}
- $i{+}{+}$

We say the method is optimal if

$$\|u_i - u\|_a \leqslant C N^{-(s-1)/n} \|u\|_{W^{s,p}}$$

Linear convergence

From the Galerkin orthogonality

$$a(u - u_{i+1}, v) = 0 \qquad \text{for all } v \in S_{i+1},$$

taking $v = u_{i+1} - u_i$, we have

$$\|u - u_i\|_a^2 = \|u - u_{i+1}\|_a^2 + \|u_{i+1} - u_i\|_a^2.$$

So if

$$\|u_{i+1} - u_i\|_a \geqslant c\|u - u_i\|_a,$$

with constant $c \in (0, 1)$, we have

$$\|u - u_{i+1}\|_a^2 = \|u - u_i\|_a^2 - \|u_{i+1} - u_i\|_a^2 \leqslant (1 - c^2)\|u - u_i\|_a^2.$$

Quasi-orthogonality for semilinear problems

Let us consider

$$a(u,v)+(f(u),v)=0, \qquad \forall v\in H$$

We have

$$\|u-u_i\|_a^2=\|u-u_{i+1}\|_a^2+\|u_{i+1}-u_i\|_a^2+2a(u-u_{i+1},u_{i+1}-u_i)$$

$$\begin{aligned}
a(u-u_{i+1},u_{i+1}-u_i) &= (f(u)-f(u_{i+1}),u_{i+1}-u_i)\\
&\leqslant C\|f(u)-f(u_{i+1})\|\|u_{i+1}-u_i\|\\
&\leqslant C\|u-u_{i+1}\|\|u_{i+1}-u_i\|\\
&\leqslant Ch_{i+1}\|u-u_{i+1}\|_{H^1}\|u_{i+1}-u_i\|_{H^1}
\end{aligned}$$

Ongoing work / Open problems

- Geometry
- Boundary conditions
- Coupled system
- Fast solution of the discretized system
- Higher order elements, flexible mesh
- Problems with genuinely critical exponent

Manuscripts, Collaborators, Acknowledgments

HNT2 M. HOLST, G. NAGY, AND GT, Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics. *Phys. Rev. Let.*, 100:161101, 2008. Also available as arXiv:0802.1031 [gr-qc]

HNT1 M. HOLST, , G. NAGY, AND GT, Rough solutions of the Einstein constraint equations on closed manifolds without near-CMC conditions. arXiv:0712.0798 [gr-qc]. To appear in *Comm. Math. Phys.*.

HT1 M. HOLST, AND GT, Convergent adaptive finite element approximation of the Einstein constraints. In preparation.

Acknowledgments:

NSF: DMS 0411723, DMS 0715146 (Numerical geometric PDE)
DOE: DE-FG02-05ER25707 (Multiscale methods)