# Computation of operators in wavelet coordinates

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# Linear operator equations

- Let $\Omega$ be an $n$-dimensional domain or smooth manifold
- $H^t \subset H^t(\Omega)$ be a subspace, and $H^{-t}$ be its dual space
- Consider the problem of finding $u$ from

$$Lu = g$$

- where $L : H^t \to H^{-t}$ is a self-adjoint elliptic operator of order $2t$
- and $g \in H^{-t}$ is a linear functional

# Differential operators

- Partial differential operators of order $2t$

$$\langle v, Lu\rangle = \sum_{|\alpha|,|\beta|\leq t} \langle \partial^\alpha v, a_{\alpha\beta}\partial^\beta u\rangle,$$

- Example: The reaction-diffusion equation ($t = 1$)

$$\langle v, Lu\rangle = \int_\Omega \nabla v \cdot \nabla u + \kappa^2 vu,$$

# Singular integral operators

- Boundary integral operators

$$[Lu](x) = \int_\Omega K(x,y)u(y)d\Omega_y$$

  with the kernel $K(x,y)$ singular at $x = y$

- Example: The single layer operator for the Laplace BVP in $3$-d domain ($t = -\frac{1}{2}$)

$$K(x,y) = \frac{1}{4\pi|x-y|}$$

# Wavelet bases

- Multiresolution: $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset H^t$
- $\dim \mathcal{S}_j = \mathcal{O}(2^{jn})$ (dyadic refinements)
- $\mathcal{S}_j$ contains all piecewise pols of degree $\boxed{d-1}$
- $\mathcal{S}_j$ is globally $C^r$-smooth $\boxed{\gamma := r + \frac{3}{2}}$
- $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ is a Riesz basis for $H^t$
- $\operatorname{span}\{\psi_\lambda : |\lambda| \leq j\} = \mathcal{S}_j$
- $\operatorname{diam}(\operatorname{supp} \psi_\lambda) \eqsim 2^{-|\lambda|}$
- $\Psi$ has $\boxed{\tilde{d}}$ vanishing moments

# Nonlinear approximation

- $\mathbf{u} = (\mathbf{u}_\lambda)_\lambda \in \ell_2$ s.t. $u = \sum_\lambda \mathbf{u}_\lambda \psi_\lambda$
- $\mathbf{u}_N$ best approximation of $\mathbf{u}$ with $\#\mathrm{supp}\,\mathbf{u}_N \leq N$
- If $u \in B^{t+ns}_{\tau,\tau}$ with $\frac{1}{\tau} = \frac{1}{2} + s$ for some $s < \frac{d-t}{n}$

$$\varepsilon_N = \| \textstyle\sum_\lambda [\mathbf{u}_N]_\lambda \psi_\lambda - u \|_{H^t} \eqsim \|\mathbf{u}_N - \mathbf{u}\| \lesssim N^{-s}$$

- $u$ is given as the solution to $Lu = g$.
- [Dahlke, DeVore]: $u \in B^{t+ns}_{\tau,\tau}$ under mild requirements

# Equivalent problem in $\ell_2$

- Wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$
- Stiffness $\mathbf{L} = \langle L\psi_\lambda, \psi_\mu \rangle_{\lambda,\mu}$ and load $\mathbf{g} = \langle g, \psi_\lambda \rangle_\lambda$
- Linear equation in $\ell_2(\Lambda)$

$$\mathbf{L}\mathbf{u} = \mathbf{g}$$

- $\mathbf{L} : \ell_2(\Lambda) \to \ell_2(\Lambda)$ SPD and $\mathbf{g} \in \ell_2(\Lambda)$
- $u = \sum_\lambda \mathbf{u}_\lambda \psi_\lambda$ is the solution of $Lu = g$

# Richardson iterations in $\ell_2$

[Cohen, Dahmen, DeVore '02]

- $\mathbf{u}^{(0)} = \mathbf{0}$
- $\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \alpha[\mathbf{g} - \mathbf{L}\mathbf{u}^{(i)}] \qquad i = 0, 1, \ldots$
- $\mathbf{g}$ and $\mathbf{L}\mathbf{u}^{(i)}$ are infinitely supported
- Approximate them by finitely supported sequences
- Algorithm $\mathbf{SOLVE}[N, \mathbf{L}, \mathbf{g}] \to \mathbf{u}_{[N]}$ $\quad$ ($N$ operations)
- $\#\mathrm{supp}\,\mathbf{u}_{[N]} \lesssim N$ and

$$\varepsilon_{[N]} = \|\mathbf{u}_{[N]} - \mathbf{u}\| \to 0 \quad \text{as } N \to \infty$$

- $\varepsilon_{[N]}$ speed of convergence?

# Complexity of SOLVE

- Matrix $\mathbf{L}$ is called $q^*$-computable, when for each $N$ one can construct an infinite matrix $\mathbf{L}_N^*$ s.t.
  - for any $q < q^*$, $\|\mathbf{L}_N^* - \mathbf{L}\| \lesssim N^{-q}$
  - having in each column $\mathcal{O}(N)$ non-zero entries
  - whose computation takes $\mathcal{O}(N)$ operations
- [CDD'02]: Suppose that
  - $\|\mathbf{u}_N - \mathbf{u}\| \lesssim N^{-s}$ $\qquad [s < \frac{d-t}{n}]$
  - $\mathbf{L}$ is $q^*$-computable with $q^* > s$ $\quad [q^* \geq \frac{d-t}{n}$ is suff. $]$
  - Some condition on $\mathbf{g}$

  then $\mathbf{u}_{[N]} = \mathbf{SOLVE}[N, \mathbf{L}, \mathbf{g}]$ satisfies

$$\|\mathbf{u}_{[N]} - \mathbf{u}\| \lesssim N^{-s}$$

# Compressibility of diff. ops.

[Stevenson '04]: Suppose $L$ is a diff. op.

- $L, L' : H^{t+\sigma} \to H^{-t+\sigma}$ are bounded with $\sigma \geq d - t$
- $\Psi$ are piecewise polynomial wavelets that
  - are smooth: $\gamma \geq d - \frac{d-t}{n}$
  - have vanishing moments: $\tilde{d} \geq d - 2t$

then we can construct $\mathbf{L}_N$ by dropping entries from $\mathbf{L}$, s.t. with some $q^* \geq \frac{d-t}{n} \qquad (> s)$

- for any $q < q^*$, $\|\mathbf{L}_N - \mathbf{L}\| \lesssim N^{-q}$
- $\mathbf{L}_N$ has $\lesssim N$ non-zeros in each column

Need to spend $\mathcal{O}(N)$ ops. for each column of $\mathbf{L}_N$

# Quadrature for diff. ops.

- $\mathbf{L}_{\lambda,\mu} = \sum_{\alpha\beta} \int_{\Theta} a_{\alpha\beta} \partial^{\alpha}\psi_{\lambda} \partial^{\beta}\psi_{\mu} \qquad \Theta = \operatorname{supp}\psi_{\lambda} \cap \operatorname{supp}\psi_{\mu}$
- $a_{\alpha\beta}$ are piecewise smooth
- $\Psi$ are piecewise pols of order $e$ (degree $e-1$)
- Internal, positive, interpolatory quadratures
- Composite quadrature of rank $W$:
  - Subdivide $\Theta$ into $W$ subdomains
  - Apply quadrature of order $p$ to each subdomain
- Fixed order $p$, variable rank $W \quad \rightsquigarrow \quad$ work $= \mathcal{O}(W)$

# Computability of diff. ops.

[T.G., Stevenson '04]

- Fix quadrature order $p > q^* n + e - 1 - t$
- Fix $\theta$ and $\varrho$ satisfying $\frac{q^* n}{p} \leq \theta \leq \varrho < 1 - \frac{e-1-t}{p}$
- $\mathbf{L}_N^*$ — computed approximation of $\mathbf{L}_N$
- Work $W_{N,\lambda,\mu} \eqsim \max\{1, N^\theta 2^{-||\lambda|-|\mu||n\varrho}\}$ for $[\mathbf{L}_N^*]_{\lambda,\mu}$
- Then
  - $\|\mathbf{L}_N - \mathbf{L}_N^*\| \lesssim N^{-q^*}$ $(\Rightarrow \forall q < q^* : \|\mathbf{L} - \mathbf{L}_N^*\| \lesssim N^{-q})$
  - Work for each column of $\mathbf{L}_N^*$ is $\mathcal{O}(N)$
- Therefore $\mathbf{L}$ is $q^*$-computable with $q^* \geq \frac{d-t}{n}$

# Compressibility of b.i.o.

[Stevenson '04]: Suppose $L$ is a b.i.o.

- $\Omega$ is sufficiently smooth
- $K(x, y)$ satisfies the Calderon-Zygmund estimate
- $\Psi$ are sufficiently smooth and have sufficiently many vanishing moments

Then we can construct $\mathbf{L}_N$ by dropping entries from $\mathbf{L}$, s.t. with some $q^* \geq \frac{d-t}{n} \quad (> s)$

- for any $q < q^*$, $\|\mathbf{L}_N - \mathbf{L}\| \lesssim N^{-q}$
- $\mathbf{L}_N$ has $\lesssim N(\log_2 N)^{-2n-1}$ non-zeros in each column

Need to spend $\mathcal{O}(N)$ ops. for each column of $\mathbf{L}_N$

# Quadrature for b.i.o.

- $\mathbf{L}_{\lambda,\mu} = \int_{\Theta} \int_{\Theta'} K(x,y)\psi_\lambda\psi_\mu \qquad \Theta = \operatorname{supp}\psi_\lambda,\ \Theta' = \operatorname{supp}\psi_\mu$
- Far field: $\operatorname{dist}(\Theta,\Theta') \gtrsim 2^{-\min\{|\lambda|,|\mu|\}}$
  Uniform mesh, variable order $p \quad \rightsquigarrow \quad$ work $= \mathcal{O}(p^{2n})$
- Near field: $\operatorname{dist}(\Theta,\Theta') \lesssim 2^{-\min\{|\lambda|,|\mu|\}}$
  Adaptive (non-uniform) mesh, variable order $p \quad \rightsquigarrow$

  $$\text{work} = \mathcal{O}(p^{2n}(1 + ||\lambda| - |\mu||))$$

  [Schneider '95], [von Petersdorff, Schwab '97], [Lage, Schwab '99], [Harbrecht '01]
- Singular integrals: $\operatorname{dist}(\Theta,\Theta') = 0$
  Duffy's transformation [Duffy '82], [Sauter '96], [vPS'97]

# Computability of b.i.o.

[T.G. '04]

- $\mathbf{L}_N^*$ — computed approximation of $\mathbf{L}_N$
- With sufficiently large, fixed $\theta$ and $\varrho$
- Choose the quadrature order

$$p = \theta \log_2 N + \varrho ||\lambda| - |\mu|| + const. \quad \text{for } [\mathbf{L}_N^*]_{\lambda,\mu}$$

- Then
  - $\|\mathbf{L}_N - \mathbf{L}_N^*\| \lesssim N^{-q^*} \qquad (\Rightarrow \forall q < q^* : \|\mathbf{L} - \mathbf{L}_N^*\| \lesssim N^{-q})$
  - Work for each column of $\mathbf{L}_N^*$ is $\mathcal{O}(N)$
- Therefore $\mathbf{L}$ is $q^*$-computable with $q^* \geq \frac{d-t}{n}$

# Conclusion

Given $d$ and $t$, for wavelets that

- are sufficiently smooth
- have sufficiently many vanishing moments
- on sufficiently smooth manifolds

any internal, positive, interpolatory quadrature formula yields computational schemes for the infinite stiffness matrix $\mathbf{L}$, such that the CDD2 algorithm converges at the same rate as that of best $N$-term approximation.

# References


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# (extra slide)

Suppose $\forall q < q^* : \|\mathbf{L}_N - \mathbf{L}\| \lesssim N^{-q}$
and $\#$ of non-zeros in each column of $\mathbf{L}_N$ is $\lesssim N$.

With $\alpha_N = (\log_2 N)^{-c}$, define $\tilde{\mathbf{L}}_N := \mathbf{L}_{[N\alpha_N]}$
Non-zeros per column for $\tilde{\mathbf{L}}_N$:

$$N\alpha_N = N(\log_2 N)^{-c}$$

For arbitrary $q < q^*$, choose $q' \in (q, q^*)$

$$\begin{aligned} \|\tilde{\mathbf{L}}_N - \mathbf{L}\| &= \|\mathbf{L}_{[N\alpha_N]} - \mathbf{L}\| \lesssim (N\alpha_N)^{-q'} \\ &= N^{-q'}(\log_2 N)^{cq'} \lesssim N^{-q} \end{aligned}$$