On approximation classes of adaptive methods

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Plan of the talk

Background:

- approximation classes
- Besov spaces
- multilevel approximation

Ongoing work on

- approximation classes of adaptive finite element methods

Basic setup

Ω	polyhedral Lipschitz domain in $\mathbb{R}^n$
P_0	triangulation of Ω
$\mathscr{P}$	the family of all conforming triangulations obtained from P_0 by a sequence of newest vertex bisections
S_P	the Lagrange C^0 finite element space of piecewise polynomials of degree not exceeding m, subordinate to $P \in \mathscr{P}$
X_0	Examples: $X_0 = L^p(\Omega)$, $X_0 = H^1(\Omega)$

Let

$$E(u,P) = \min_{v \in S_P} \|u - v\|_{X_0}, \qquad E_j(u) = \inf_{\{P \in \mathscr{P} : \#P \leq 2^j\}} E(u,P),$$

and define the approximation class $\mathscr{A}^s_\infty(X_0)$ for $s > 0$ by

$$u \in \mathscr{A}^s_\infty(X_0) \qquad \Longleftrightarrow \qquad E_j(u) \lesssim 2^{-js} \qquad \Longleftrightarrow \qquad \left[2^{js} E_j(u)\right]_{j \in \mathbb{N}} \in \ell^\infty.$$

Adaptive approximation classes

Recall

$$E(u,P) = \min_{v\in S_P} \|u-v\|_{X_0}, \qquad E_j(u) = \inf_{\{P\in\mathscr{P}:\#P\le 2^j\}} E(u,P),$$

and

$$u \in \mathscr{A}^s_\infty(X_0) \qquad \Longleftrightarrow \qquad E_j(u) \lesssim 2^{-js} \qquad \Longleftrightarrow \qquad \left[2^{js}E_j(u)\right]_{j\in\mathbb{N}} \in \ell^\infty.$$

We extend this definition by introducing $\mathscr{A}^s_q(X_0)$ for $0<q\le\infty$ by

$$u \in \mathscr{A}^s_q(X_0) \qquad \Longleftrightarrow \qquad \left[2^{js}E_j(u)\right]_{j\in\mathbb{N}} \in \ell^q.$$

We have $\mathscr{A}^s_q(X_0) \subset \mathscr{A}^s_r(X_0)$ for $q \le r$, and $\mathscr{A}^s_q(X_0) \subset \mathscr{A}^\alpha_r(X_0)$ for $s>\alpha$ and for any $0<q,r\le\infty$. In a typical situation, it is a quasi-Banach space.

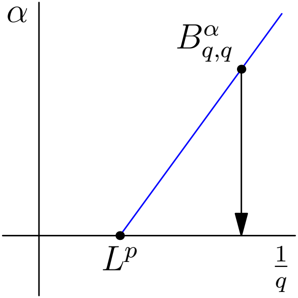
We would like to compare, say, $\mathscr{A}^s_q(L^p(\Omega))$ with known function spaces.

Besov spaces

For best N-term approximations in a wavelet basis, we have

$$\mathscr{A}_q^s(L^p(\Omega)) = B_{q,q}^{\alpha}(\Omega), \qquad \text{for} \qquad s = \frac{\alpha}{n} = \frac{1}{q} - \frac{1}{p} > 0,$$

where $B_{q,r}^{\alpha}(\Omega)$ is a Besov space $\left(B_{p,p}^{s} \approx W^{s,p}\right)$.



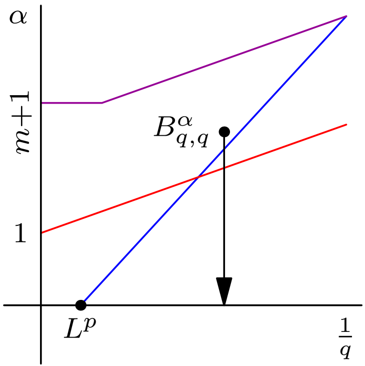
For $\frac{\alpha}{n} = \frac{1}{q} - \frac{1}{p}$ we have $B_{q,q}^{\alpha}(\Omega) \subset L^p(\Omega)$.

Less sharp characterizations are known for

- nonlinear spline approximations
- wavelet tree approximations
- adaptive finite element approximations

Direct and inverse embeddings

[Binev, Dahmen, DeVore, Petrushev '02], [Gaspoz, Morin '13]



$$B^{\alpha}_{q,q}(\Omega) \subset \mathscr{A}^{s}_{\infty}(L^p(\Omega))$$

with $s = \frac{\alpha}{n}$, if

$$\delta = \frac{\alpha}{n} + \frac{1}{p} - \frac{1}{q} > 0$$

and $0 < \alpha < m + \max\{1, \frac{1}{q}\}$.

On the other hand

$$\mathscr{A}^{s}_{q}(L^p(\Omega)) \subset B^{\alpha}_{q,q}(\Omega)$$

for

$$s = \frac{\alpha}{n} = \frac{1}{q} - \frac{1}{p} > 0,$$

and $\alpha < 1 + \frac{1}{q}$.

Main ingredients of the direct theorem

Direct estimate [BDDP02,GM13]

Let $\delta = \frac{\alpha}{n} + \frac{1}{p} - \frac{1}{q} > 0$ and $0 < \alpha < m + \max\{1, \frac{1}{q}\}$. Then for $u \in B^{\alpha}_{q,q}(\Omega)$ and $P \in \mathscr{P}$, there exists $v \in S_P$ such that

$$\|u - v\|^p_{L^p(\Omega)} \lesssim \sum_{\tau \in P} |\tau|^{p\delta} |u|^p_{B^{\alpha}_{q,q}(\hat{\tau})},$$

where $\hat{\tau}$ is the patch of triangles that touch τ.

Proof: Quasi-interpolator, Whitney estimates, Besov-Sobolev embedding.

Mesh construction [BDDP02]

For any $u \in B^{\alpha}_{q,q}(\Omega)$ and N, there exists $P \in \mathscr{P}$ with $\#P \leq N$ such that

$$\sum_{\tau \in P} |\tau|^{p\delta} |u|^p_{B^{\alpha}_{q,q}(\hat{\tau})} \lesssim N^{-sp} \|u\|^p_{B^{\alpha}_{q,q}(\Omega)},$$

where $s = \frac{\alpha}{n}$.

Proof: Greedy algorithm to reduce $e(\tau, P) = |\tau|^{\delta} |u|_{B^{\alpha}_{q,q}(\hat{\tau})}$.

Main ingredients of the inverse theorem

Inverse estimate [BDDP02]

Let $s = \frac{\alpha}{n} = \frac{1}{q} - \frac{1}{p} > 0$ and $\alpha < 1 + \frac{1}{q}$. Then we have

$$\|v\|_{B^{\alpha}_{q,q}(\Omega)} \lesssim (\#P)^{s} \|v\|_{L^{p}(\Omega)}, \qquad P \in \mathscr{P}, \qquad v \in S_P.$$

Proof: Multiscale decomposition of v.

Corollary [BDDP02]

For $s = \frac{\alpha}{n} = \frac{1}{q} - \frac{1}{p} > 0$ and $\alpha < 1 + \frac{1}{q}$ we have $\mathscr{A}^{s}_{q}(L^{p}(\Omega)) \subset B^{\alpha}_{q,q}(\Omega)$.

Proof: Real interpolation.

The embedding $\mathscr{A}^{s}_{q}(L^{p}(\Omega)) \subset B^{\alpha}_{q,q}(\Omega)$ cannot hold for $\alpha \geq 1 + \frac{1}{q}$ because in this range we have $S_P \subsetneq B^{\alpha}_{q,q}(\Omega)$.

This problem was dealt with in [GM13] by introducing generalized Besov spaces $A^{\alpha}_{q,q}(\Omega)$, and showing that $\mathscr{A}^{s}_{q}(L^{p}(\Omega)) \subset A^{\alpha}_{q,q}(\Omega)$ for all $\alpha > 0$. We call $A^{\alpha}_{q,q}(\Omega)$ multilevel approximation spaces.

Multilevel approximation spaces

- For $j = 1, 2, \ldots$, let P_j be the uniform refinement of P_{j-1}.
- Let $G \subset \Omega$ be a domain consisting of elements from some P_j.
- With $S_j = S_{P_j}$, and $0 < p < \infty$, we let

$$E(u, S_j, G)_p = \inf_{v \in S_j} \|u - v\|_{L^p(G)}, \qquad u \in L^p(G).$$

- Define the multilevel approximation spaces $A^\alpha_{p,q}(G) = A^\alpha_{p,q}(\{S_j\}, G)$ by

$$u \in A^\alpha_{p,q}(\{S_j\}, G) \qquad \Longleftrightarrow \qquad \left(\lambda^{j\alpha} E(u, S_j, G)_p\right)_{j \geq 0} \in \ell^q,$$

where $\lambda = \sqrt[n]{2}$.
- Note that $u \in A^\alpha_{p,q}(G)$ implies $E(u, S_j, G)_p \lesssim 2^{-\alpha j/n} \sim h_j^\alpha$, with h_j the typical meshwidth of P_j.

Multilevel approximation spaces II

- We have $B^{\alpha}_{q,q}(\Omega) \subset A^{\alpha}_{q,q}(\Omega)$ for $0 < q < \infty$ and $0 < \alpha < m + \max\{1, \frac{1}{q}\}$.
- In the other direction, we have $A^{\alpha}_{q,q}(\Omega) \subset B^{\alpha}_{q,q}(\Omega)$ for $0 < q < \infty$ and $0 < \alpha < 1 + \frac{1}{q}$.
- So in most interesting situations, we have $B^{\alpha}_{q,q}(\Omega) \subsetneq A^{\alpha}_{q,q}(\Omega)$.
- Gaspoz-Morin's inverse theorem says that $\mathscr{A}^{s}_{q}(L^p(\Omega)) \subset A^{\alpha}_{q,q}(\Omega)$ for $s = \frac{\alpha}{n} = \frac{1}{q} - \frac{1}{p} > 0$. Recall the inclusion $\mathscr{A}^{s}_{q}(L^p(\Omega)) \subset B^{\alpha}_{q,q}(\Omega)$ cannot hold above the red line.
- Their direct theorem says that $B^{\alpha}_{q,q}(\Omega) \subset \mathscr{A}^{s}_{\infty}(L^p(\Omega))$ for $\frac{\alpha}{n} > \frac{1}{q} - \frac{1}{p}$ and $0 < \alpha < m + \max\{1, \frac{1}{q}\}$.
- Question I: What is the difference between $A^{\alpha}_{q,q}$ and $B^{\alpha}_{q,q}$?
- Question II: Do we have $A^{\alpha}_{q,q}(\Omega) \subset \mathscr{A}^{s}_{\infty}(L^p(\Omega))$?

Multilevel approximation spaces III

Conjecture: If $u \in A^{\alpha}_{p,q}(\{S_j\},\Omega)$ for all possible initial triangulations P_0 of Ω, then $u \in B^{\alpha}_{p,q}(\Omega)$.

Lemma

Let $\phi \in S_k$ be such that $\phi \notin C^1(\Omega)$ for some k. Then there exists an initial triangulation $\overline{P}_0$ of Ω, such that $E(\phi, \overline{S}_j)_p \gtrsim \lambda^{-(1+\frac{1}{p})j}$ for $0 < p < \infty$, where $\{\overline{S}_j\}$ is the sequence analogous to $\{S_j\}$ with P_0 replaced by $\overline{P}_0$.

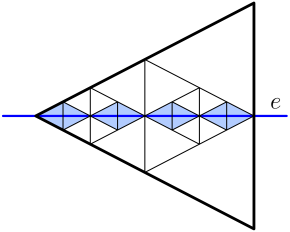
Proof $(n=2)$:

- There is an edge e of P_k, such that $|\phi(x,y)| \sim \max\{0,y\}$ under suitable transformation, where y is the coordinate normal to e.
- We choose $\overline{P}_0$ so that e cuts through the "middle" of each triangle in any refinement of $\overline{P}_0$.

Multilevel approximation spaces IV

Proof ($n = 2$):

- There is an edge e of P_k, such that $|\phi(x,y)| \sim \max\{0,y\}$ under a suitable transformation, where y is the coordinate normal to e.
- We choose $\overline{P}_0$ so that e cuts through the "middle" of each triangle in any refinement of $\overline{P}_0$.



We have

$$\|\phi\|^p_{L^p(V_j)} \sim \int_0^{h_j} y^p \mathrm{d}y \sim h_j^{p+1} \sim \lambda^{-j(p+1)},$$

where V_j is the shaded area, and

$$E(\phi, \overline{S}_j)_p \gtrsim \|\phi\|_{L^p(V_j)} \sim \lambda^{-j(1+\frac{1}{p})}.$$

Direct embeddings II

Theorem: We have $A^{\alpha}_{q,q}(\Omega) \subset \mathscr{A}^{s}_{\infty}(L^{p}(\Omega))$ for $s = \frac{\alpha}{n} > \frac{1}{q} - \frac{1}{p} \geq 0$.

Proof: The two ingredients are the same as before.

Mesh construction

For any $u \in A^{\alpha}_{q,q}(\Omega)$ and N, there exists $P \in \mathscr{P}$ with $\#P \leq N$ such that

$$\sum_{\tau \in P} |\tau|^{p\delta} |u|^{p}_{A^{\alpha}_{q,q}(\hat{\tau})} \lesssim N^{-sp} \|u\|^{p}_{A^{\alpha}_{q,q}(\Omega)},$$

where $s = \frac{\alpha}{n}$.

Proof: The same argument works basically because the spaces $A^{\alpha}_{q,q}(G)$ enjoy the locality property

$$\sum_{\tau \in P} |u|^{q}_{A^{\alpha}_{q,q}(\hat{\tau})} \lesssim \|u\|^{q}_{A^{\alpha}_{q,q}(\Omega)}.$$

Direct estimate

Lemma: Let $\delta = \frac{\alpha}{n} + \frac{1}{p} - \frac{1}{q} > 0$. Then for $u \in A^{\alpha}_{q,q}(\Omega)$ and $P \in \mathscr{P}$ we have

$$\|u - Q_P u\|^p_{L^p(\Omega)} \lesssim \sum_{\tau \in P} |\tau|^{p\delta} |u|^p_{A^{\alpha}_{q,q}(\hat{\tau})},$$

where Q_P is the quasi-interpolation operator from [GM13].

Proof $(q \leq 1)$: We have

$$\|u - Q_P u\|^p_{L^p(\Omega)} = \sum_{\tau \in P} \|u - Q_P u\|^p_{L^p(\tau)} \lesssim \sum_{\tau \in P} \inf_{v \in S_P} \|u - v\|^p_{L^p(\hat{\tau})}.$$

Every triangle $\sigma \in P$ belongs to a unique P_j. Given $\tau \in P$ denote by $j(\tau)$ the highest index j that occurs in the local patch surrounding τ. We have

$$\inf_{v \in S_P} \|u - v\|_{L^p(\hat{\tau})} \leq \inf_{v \in S_{j(\tau)}} \|u - v\|_{L^p(\hat{\tau})},$$

because in $\hat{\tau}$, $P_{j(\tau)}$ is more refined that P.

Proof of direct estimate continued

So far, we have

$$\|u - Q_P u\|_{L^p(\Omega)}^p \lesssim \sum_{\tau \in P} \inf_{v \in S_{j(\tau)}} \|u - v\|_{L^p(\hat{\tau})}^p.$$

For each j, let $u_j \in S_j$ be such that $\|u - u_j\|_{L^p(\hat{\tau})} = \inf_{v \in S_j} \|u - v\|_{L^p(\hat{\tau})}$. We have

$$\|u - u_{j(\tau)}\|_{L^p(\hat{\tau})}^{p^*} \le \sum_{j=j(\tau)}^{\infty} \|u_{j+1} - u_j\|_{L^p(\hat{\tau})}^{p^*} \lesssim \sum_{j=j(\tau)}^{\infty} \lambda^{(\frac{1}{q} - \frac{1}{p}) jnp^*} \|u_{j+1} - u_j\|_{L^q(\hat{\tau})}^{p^*},$$

with $p^* = \min\{1, p\}$. Putting $\frac{1}{q} - \frac{1}{p} = \frac{\alpha}{n} - \delta$, we get

$$\begin{aligned} \|u - u_{j(\tau)}\|_{L^p(\hat{\tau})}^{p^*} &\lesssim \sum_{j=j(\tau)}^{\infty} \lambda^{-jn\delta p^*} \lambda^{j\alpha p^*} \|u - u_j\|_{L^q(\hat{\tau})}^{p^*} \\ &\le \lambda^{-j(\tau) n\delta p^*} \sum_{j=j(\tau)}^{\infty} \lambda^{j\alpha p^*} \|u - u_j\|_{L^q(\hat{\tau})}^{p^*} \lesssim |\tau|^{\delta p^*} |u|_{A^\alpha_{p,p^*}}^{p^*}. \end{aligned}$$

Adaptive finite element methods

Consider the boundary value problem

$$\Delta u = f \qquad \text{in} \quad \Omega, \qquad u = 0 \qquad \text{on} \quad \partial\Omega.$$

A typical *a posteriori* error estimate satisfies

$$\left[\eta(u,P)\right]^2 \sim \|u - u_P\|^2_{H^1(\Omega)} + \sum_{\tau\in P} h_\tau^2 \|f - \Pi_\tau f\|^2_{L^2(\tau)},$$

where $u_P \in S_P$ is the Galerkin solution on P, and $\Pi_\tau : L^2(\tau) \to \mathbb{P}_d$ is the $L^2(\tau)$-orthogonal projection onto $\mathbb{P}_d$, $d \geq m-2$.

It is known that certain practical adaptive FEM converges optimally w.r.t. approximation classes associated to

$$E(u,P) = \left(\min_{v\in S_P} \|u - v\|^2_{H^1(\Omega)} + \sum_{\tau\in P} h_\tau^2 \|f - \Pi_\tau f\|^2_{L^2(\tau)}\right)^{\frac{1}{2}}.$$

Generalized approximation classes

Let
$$\rho(u,v,P) = \left(\|u-v\|^2_{H^1(\Omega)} + \sum_{\tau\in P} h_\tau^2 \|f-\Pi_\tau f\|^2_{L^2(\tau)} \right)^{\frac{1}{2}},$$
and define
$$E(u,P) = \min_{v\in S_P} \rho(u,v,P), \qquad E_j(u) = \inf_{\{P\in\mathscr{P}:\#P\le 2^j\}} E(u,P).$$

We introduce the approximation class $\mathscr{A}^s_q(\rho)$ given by
$$u \in \mathscr{A}^s_q(\rho) \qquad \Longleftrightarrow \qquad \left[2^{js}E_j(u)\right]_{j\in\mathbb{N}} \in \ell^q.$$

Also, define the oscillation class $\mathscr{O}^s$ by
$$f\in\mathscr{O}^s_q \qquad \Longleftrightarrow \qquad \inf_{\{P\in\mathscr{P}:\#P\le 2^j\}} \sum_{\tau\in P} h_\tau^2 \|f-\Pi_\tau f\|^2_{L^2(\tau)} \lesssim 2^{-2js}.$$

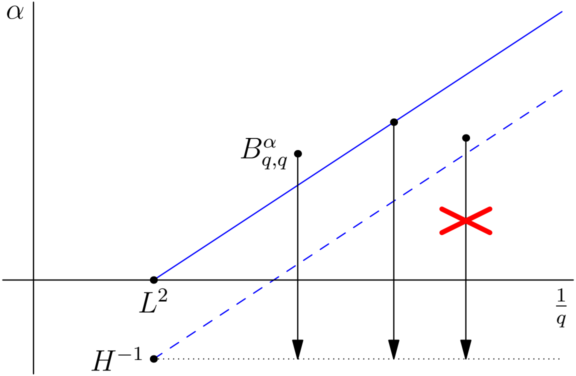
Lemma: If $u\in\mathscr{A}^s_\infty(H^1_0(\Omega))$ and $f\in\mathscr{O}^s$ then $u\in\mathscr{A}^s_\infty(\rho)$.

Proof: Overlay of meshes.
Example:
$H^\alpha(\Omega)\subset\mathscr{O}^{1+\alpha}$ for $\alpha\ge 0$, so $\mathscr{A}^s_\infty(H^1_0(\Omega))\cap\Delta^{-1}(H^{s-1}(\Omega))\subset\mathscr{A}^s_\infty(\rho)$ for $s\ge 1$.

Direct embeddings III

Morally, $\mathscr{O}^s \approx \mathscr{A}^s_\infty(H^{-1}(\Omega))$, so we expect $B^\alpha_{q,q}(\Omega) \subset \mathscr{O}^{1+\alpha}$.

Theorem: We have $B^\alpha_{q,q}(\Omega) \subset \mathscr{O}^{1+\alpha}$ for $\frac{\alpha}{n} \geq \frac{1}{q} - \frac{1}{2}$,
hence $\mathscr{A}^s_\infty(H^1_0(\Omega)) \cap \Delta^{-1}(B^{s-1}_{q,q}(\Omega)) \subset \mathscr{A}^s_\infty(\rho)$ for $\frac{s-1}{n} \geq \frac{1}{q} - \frac{1}{2}$.



Direct embeddings III

Theorem: We have $B^{\alpha}_{q,q}(\Omega) \subset \mathscr{O}^{1+\alpha}$ for $\frac{\alpha}{n} \geq \frac{1}{q} - \frac{1}{2}$,
hence $\mathscr{A}^s_\infty(H^1_0(\Omega)) \cap \Delta^{-1}(B^{s-1}_{q,q}(\Omega)) \subset \mathscr{A}^s_\infty(\rho)$ for $\frac{s-1}{n} \geq \frac{1}{q} - \frac{1}{2}$.

Proof: The mesh construction part works the same as before. For the direct estimate, with $\delta = \frac{\alpha}{n} - \frac{1}{q} + \frac{1}{2} \geq 0$, we have

$$\|f - \Pi_\tau f\|_{L^2(\tau)} \leq \|f - p\|_{L^2(\tau)} \lesssim |\tau|^\delta \|f - p\|_{L^q(\tau)} + |\tau|^\delta |f|_{B^{\alpha}_{q,q}(\tau)},$$

for any $p \in \mathbb{P}_d$, and

$$\min_{p \in \mathbb{P}_d} \|f - p\|_{L^q(\tau)} \lesssim \omega_{d+1}(f, \tau)_q \lesssim |f|_{B^{\alpha}_{q,q}(\tau)},$$

which gives

$$\sum_{\tau \in P} h_\tau^2 \|f - \Pi_\tau f\|^2_{L^2(\tau)} \lesssim \sum_{\tau \in P} |\tau|^{2\delta + 2/n} |f|^2_{B^{\alpha}_{q,q}(\tau)}.$$

Concluding remarks

The arguments can be adapted to

- red refinements,
- splines,
- higher order problems,
- Stokes equations, etc.
- For variable coefficient equations, one loses “an epsilon” in the convergence rate.

Plans:

- inverse theorems for adaptive FEM
- boundary elements
- finite element exterior calculus