An optimal adaptive wavelet method

without coarsening of the iterands

Tsogtgerel Gantumur
(joint work with H. Harbrecht and R.P. Stevenson)

"Recent Progress in Wavelet Analysis and Frame Theory"
23-26 January 2006 Bremen

Universiteit Utrecht

Overview

[Cohen, Dahmen, DeVore '02], [Stevenson '04], [Dahlke, Fornasier, Raasch '04] , [Werner]

- Wavelet frame Ψ: $Au = f \quad \dashrightarrow \quad \mathbf{Au} = \mathbf{f}$
- Richardson iteration: $\mathbf{u}^{(i+1)} := \mathbf{u}^{(i)} + \omega(\mathbf{f} - \mathbf{Au}^{(i)})$
- Coarsening after K iterations

[Cohen, Dahmen, DeVore '01], [Gantumur, Harbrecht, Stevenson '05]

- Galerkin approximation: $\mathbf{u}_\Lambda \in \ell_2(\Lambda)$ s.t.

$$\langle \mathbf{Au}_\Lambda, \mathbf{v}_\Lambda \rangle = \langle \mathbf{f}, \mathbf{v}_\Lambda \rangle \qquad \forall \mathbf{v}_\Lambda \in \ell_2(\Lambda)$$

- Expand Λ to $\tilde{\Lambda}$ s.t. $\|\mathbf{u} - \mathbf{u}_{\tilde{\Lambda}}\| \le \xi \|\mathbf{u} - \mathbf{u}_\Lambda\|$ with $\xi < 1$

+ Coarsening is not needed for the iterands $\mathbf{u}^{(i)}$ [GHS05]
- Using frames is problematic

Elliptic operator equation

- Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{H}'$ its dual
- $A : \mathcal{H} \to \mathcal{H}'$ linear, self-adjoint, $\mathcal{H}$-elliptic
  ($\langle Av, v\rangle \geq c\|v\|_{\mathcal{H}}^2 \qquad v \in \mathcal{H}$)

Find $u \in \mathcal{H}$ s.t. $\quad Au = f \qquad (f \in \mathcal{H}')$

- Example: Reaction-diffusion equation $\mathcal{H} = H_0^1(\Omega)$

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v + \kappa^2 uv$$

Equivalent discrete problem

[CDD01, CDD02]

- Wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ of $\mathcal{H}$
- Stiffness $\mathbf{A} = \langle A\psi_\lambda, \psi_\mu \rangle_{\lambda,\mu}$ and load $\mathbf{f} = \langle f, \psi_\lambda \rangle_\lambda$

Linear equation in $\ell_2(\nabla)$

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \qquad \mathbf{A} : \ell_2(\nabla) \to \ell_2(\nabla) \text{ SPD and } \mathbf{f} \in \ell_2(\nabla)$$

- $u = \sum_\lambda \mathbf{u}_\lambda \psi_\lambda$ is the solution of $Au = f$
- $\|\mathbf{u} - \mathbf{v}\|_{\ell_2} \eqsim \|u - v\|_{\mathcal{H}}$ with $v = \sum_\lambda \mathbf{v}_\lambda \psi_\lambda$

Galerkin solutions

- $\|\!|\cdot|\!\| := \langle \mathbf{A}\cdot, \cdot \rangle^{\frac{1}{2}}$ is a norm on ℓ_2
- $\Lambda \subset \nabla$
- $\mathbf{I}_\Lambda : \ell_2(\nabla) \to \ell_2(\Lambda)$ restr., $\qquad \mathbf{P}_\Lambda := \mathbf{I}_\Lambda^*$
- $\mathbf{A}_\Lambda := \mathbf{P}_\Lambda \mathbf{A} \mathbf{I}_\Lambda : \ell_2(\Lambda) \to \ell_2(\Lambda)$ SPD
- $\mathbf{f}_\Lambda := \mathbf{P}_\Lambda \mathbf{f} \in \ell_2(\Lambda)$

Lemma

A unique solution $\mathbf{u}_\Lambda \in \ell_2(\Lambda)$ to $\mathbf{A}_\Lambda \mathbf{u}_\Lambda = \mathbf{f}_\Lambda$ exists, and

$$\|\!|\mathbf{u} - \mathbf{u}_\Lambda|\!\| = \inf_{\mathbf{v} \in \ell_2(\Lambda)} \|\!|\mathbf{u} - \mathbf{v}|\!\|$$

Galerkin orthogonality

- $\operatorname{supp}\mathbf{w} \subset \Lambda, \qquad \mathbf{A}_\Lambda \mathbf{u}_\Lambda = \mathbf{f}_\Lambda$
- $\langle \mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda, \mathbf{v}_\Lambda \rangle = 0$ for $\mathbf{v}_\Lambda \in \ell_2(\Lambda)$

$$\|\!|\mathbf{u} - \mathbf{w}|\!\|^2 = \|\!|\mathbf{u} - \mathbf{u}_\Lambda|\!\|^2 + \|\!|\mathbf{u}_\Lambda - \mathbf{w}|\!\|^2$$

-

Error reduction

$$\|\mathbf{u} - \mathbf{u}_\Lambda\|^2 = \|\mathbf{u} - \mathbf{w}\|^2 - \|\mathbf{u}_\Lambda - \mathbf{w}\|^2$$

Lemma [CDD01]

Let $\mu \in (0, 1)$, and Λ be s.t.

$$\|\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{A}\mathbf{w})\| \geq \mu \|\mathbf{f} - \mathbf{A}\mathbf{w}\|$$

Then we have

$$\|\mathbf{u} - \mathbf{u}_\Lambda\| \leq \sqrt{1 - \kappa(\mathbf{A})^{-1}\mu^2}\, \|\mathbf{u} - \mathbf{w}\|$$

Ideal algorithm

SOLVE$[\varepsilon] \to \mathbf{u}_k$
$k := 0$; $\Lambda_0 := \emptyset$
`do`
`Solve` $\mathbf{A}_{\Lambda_k}\mathbf{u}_k = \mathbf{f}_{\Lambda_k}$
$\mathbf{r}_k := \mathbf{f} - \mathbf{A}\mathbf{u}_k$
`determine a set` $\Lambda_{k+1} \supset \Lambda_k$`, with minimal cardinality, such that` $\|\mathbf{P}_{\Lambda_{k+1}}\mathbf{r}_k\| \geq \mu\|\mathbf{r}_k\|$
$k := k + 1$
`while` $\|\mathbf{r}_k\| > \varepsilon$

Approximate Iterations

Approximate right-hand side

$$\mathbf{RHS}[\varepsilon] \to \mathbf{f}_\varepsilon \text{ with } \|\mathbf{f} - \mathbf{f}_\varepsilon\|_{\ell_2} \leq \varepsilon$$

Approximate application of the matrix

$$\mathbf{APPLY_A}[\mathbf{v}, \varepsilon] \to \mathbf{w}_\varepsilon \text{ with } \|\mathbf{Av} - \mathbf{w}_\varepsilon\|_{\ell_2} \leq \varepsilon$$

Approximate residual

$$\mathbf{RES}[\mathbf{v}, \varepsilon] := \mathbf{RHS}[\varepsilon/2] - \mathbf{APPLY_A}[\mathbf{v}, \varepsilon/2]$$

Best N-term approximation

Given $\mathbf{u} = (\mathbf{u}_\lambda)_\lambda \in \ell_2$, approximate $\mathbf{u}$ using N nonzero coeffs

$$\aleph_N := \bigcup_{\Lambda \subset \nabla : \#\Lambda = N} \ell_2(\Lambda)$$

- $\aleph_N$ is a nonlinear manifold
- Let $\mathbf{u}_N$ be a best approximation of $\mathbf{u}$ with $\#\mathrm{supp}\,\mathbf{u}_N \leq N$
- $\mathbf{u}_N$ can be constructed by picking N largest in modulus coeffs from $\mathbf{u}$

Nonlinear vs. linear approximation in $H^t(\Omega)$

Using wavelets of order d

Nonlinear approximation

If $u \in B_\tau^{t+ns}(L_\tau)$ with $\frac{1}{\tau} = \frac{1}{2} + s$ for some $s \in (0, \frac{d-t}{n})$

$$\varepsilon_N = \|\mathbf{u}_N - \mathbf{u}\| \leq \mathcal{O}(N^{-s})$$

Linear approximation

If $u \in H^{t+ns}$ for some $s \in (0, \frac{d-t}{n}]$, uniform refinement

$$\varepsilon_j = \|\mathbf{u}_j - \mathbf{u}\| \leq \mathcal{O}(N_j^{-s})$$

- [Dahlke, DeVore]: $u \in B_\tau^{t+ns}(L_\tau) \backslash H^{t+ns}$ "often"

Approximation spaces

- Approximation space $\mathcal{A}^s := \{\mathbf{v} \in \ell_2 : \|\mathbf{v} - \mathbf{v}_N\|_{\ell_2} \leq \mathcal{O}(N^{-s})\}$
- Quasi-norm $|\mathbf{v}|_{\mathcal{A}^s} := \sup_{N \in \mathbb{N}} N^s \|\mathbf{v} - \mathbf{v}_N\|_{\ell_2}$
- $u \in B_\tau^{t+ns}(L_\tau)$ with $\frac{1}{\tau} = \frac{1}{2} + s$ for some $s \in (0, \frac{d-t}{n}) \Rightarrow \mathbf{u} \in \mathcal{A}^s$

Assumption

$$\mathbf{u} \in \mathcal{A}^s \text{ for some } s > 0$$

Best approximation

$$\|\mathbf{u} - \mathbf{v}\| \leq \varepsilon \text{ satisfies } \#\mathrm{supp}\, \mathbf{v} \leq \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$$

Requirements on the subroutines

Complexity of $\mathbf{RHS}$

$\mathbf{RHS}[\varepsilon] \to \mathbf{f}_\varepsilon$ terminates with $\|\mathbf{f} - \mathbf{f}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\mathrm{supp}\,\mathbf{f}_\varepsilon \lesssim \varepsilon^{-1/s}|\mathbf{u}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim \varepsilon^{-1/s}|\mathbf{u}|_{\mathcal{A}^s}^{1/s} + 1$

Complexity of $\mathbf{APPLY_A}$

For $\#\mathrm{supp}\,\mathbf{v} < \infty$

$\mathbf{APPLY_A}[\mathbf{v}, \varepsilon] \to \mathbf{w}_\varepsilon$ terminates with $\|\mathbf{Av} - \mathbf{w}_\varepsilon\|_{\ell_2} \leq \varepsilon$

- $\#\mathrm{supp}\,\mathbf{w}_\varepsilon \lesssim \varepsilon^{-1/s}|\mathbf{v}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim \varepsilon^{-1/s}|\mathbf{v}|_{\mathcal{A}^s}^{1/s} + \#\mathrm{supp}\,\mathbf{v} + 1$

The subroutine $\mathbf{APPLY_A}$

- $\{\psi_\lambda\}$ are piecewise polynomial wavelets that are sufficiently smooth and have sufficiently many vanishing moments
- A is either differential or singular integral operator

Then we can construct $\mathbf{APPLY_A}$ satisfying the requirements.

Ref: [CDD01], [Stevenson '04], [Gantumur, Stevenson '05,'06], [Dahmen, Harbrecht, Schneider '05]

Optimal expansion

Lemma [GHS05]
Let $\mu \in (0, \kappa(\mathbf{A})^{-\frac{1}{2}})$. Then the smallest set $\Lambda \supset \operatorname{supp} \mathbf{w}$ with

$$\|\mathbf{P}_\Lambda(\mathbf{f} - \mathbf{A}\mathbf{w})\| \geq \mu \|\mathbf{f} - \mathbf{A}\mathbf{w}\|$$

satisfies

$$\#(\Lambda \setminus \operatorname{supp} \mathbf{w}) \lesssim \|\mathbf{u} - \mathbf{w}\|^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$$

Sketch of a proof

With $\nu > 0$, let N be the smallest integer s.t. a best N-term appr. $\mathbf{u}_N$ of $\mathbf{u}$ satisfies $\|\mathbf{u} - \mathbf{u}_N\| \leq \nu\|\mathbf{u} - \mathbf{w}\|$. Then we have

$$N \lesssim \|\mathbf{u} - \mathbf{w}\|^{-1/s}|\mathbf{u}|_{\mathcal{A}^s}^{1/s}$$

If ν s.t. $\nu^2 \leq \kappa(\mathbf{A})^{-1} - \mu^2$ then $\Sigma := \operatorname{supp}\mathbf{w} \cup \operatorname{supp}\mathbf{u}_N$ satisfies

$$\|\mathbf{P}_\Sigma(\mathbf{f} - \mathbf{A}\mathbf{w})\| \geq \mu\|\mathbf{f} - \mathbf{A}\mathbf{w}\|$$

By def. of Λ

$$\#(\Lambda \setminus \operatorname{supp}\mathbf{w}) \leq \#(\Sigma \setminus \operatorname{supp}\mathbf{w}) \leq N$$

Universiteit Utrecht

Adaptive Galerkin method

SOLVE$[\varepsilon] \to \mathbf{w}_k$
$k := 0$; $\Lambda_0 := \emptyset$
`do`
`Compute an appr.solution` $\mathbf{w}_k$ `of` $\mathbf{A}_{\Lambda_k}\mathbf{u}_k = \mathbf{f}_{\Lambda_k}$
`Compute an appr.residual` $\mathbf{r}_k$ `for` $\mathbf{w}_k$
`Determine a set` $\Lambda_{k+1} \supset \Lambda_k$`, with`
`modulo constant factor minimal cardinality,`
`such that` $\|\mathbf{P}_{\Lambda_{k+1}}\mathbf{r}_k\| \geq \mu\|\mathbf{r}_k\|$
$k := k+1$
`while` $\|\mathbf{r}_k\| > \varepsilon$

Optimal complexity

Theorem [GHS05]

SOLVE$[\varepsilon] \to \mathbf{w}$ terminates with $\|\mathbf{f} - \mathbf{A}\mathbf{w}\|_{\ell_2} \leq \varepsilon$.

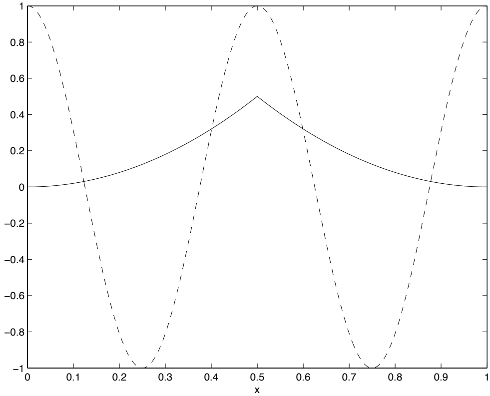
- $\#\mathrm{supp}\,\mathbf{w} \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\mathcal{A}^s}^{1/s}$
- flops, memory $\lesssim$ the same expression

Further result

- Can be extended to mildly nonsymmetric and indefinite problems [Gantumur '06]

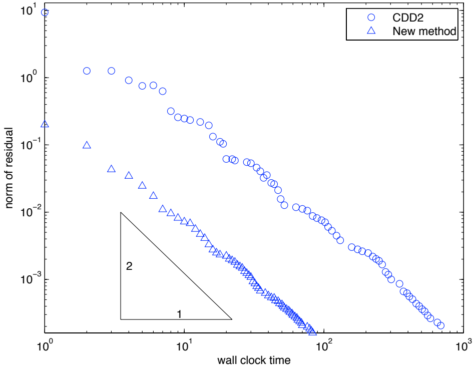
Numerical illustration

- The problem: $-\Delta u + u = f$ on $\mathbb{R}/\mathbb{Z}$ $\qquad$ ($t = 1$)
- $u \in H^{1+s}$ only for $s < \frac{1}{2}$; $\qquad$ $u \in B^{1+s}_{\tau,\tau}$ for any $s > 0$



Convergence histories

- B-spline wavelets of order d=3 with 3 vanishing moments from [Cohen, Daubechies, Feauveau '92] $\Rightarrow$ $\mathbf{u} \in \mathcal{A}^s$ for any $s < \frac{d-t}{n} = \frac{3-1}{1} = 2$



References

[CDD01] A. Cohen, W. Dahmen, R. DeVore. Adaptive wavelet methods for elliptic operator equations — Convergence rates. *Math. Comp.*, 70:27–75, 2001.

[GHS05] Ts. Gantumur, H. Harbrecht, R.P. Stevenson. An optimal adaptive wavelet method without coarsening of the iterands. Technical Report 1325, Utrecht University, March 2005. To appear in *Math. Comp.*.

[Gan05] Ts. Gantumur. An optimal adaptive wavelet method for nonsymmetric and indefinite elliptic problems. Technical Report 1343, Utrecht University, January 2006.

This work was supported by the Netherlands Organization for Scientific Research and by the EC-IHP project “Breaking Complexity”.

Universiteit Utrecht