On the Navier-Stokes-$\alpha\beta$ equations with the wall-eddy boundary conditions

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The problem

The Navier-Stokes-$\alpha\beta$ equations:

$$\partial_t v - \Delta(1-\beta^2\Delta)u + (\operatorname{grad} v)u + (\operatorname{grad} u)^T v + \nabla p = 0,$$

$$v = (1-\alpha^2\Delta)u, \qquad \nabla\cdot u = 0,$$

with $\alpha > \beta > 0$. Wall-eddy boundary conditions:

$$\beta^2(1-n\otimes n)\left(\operatorname{grad}\omega + \gamma(\operatorname{grad}\omega)^T\right)n = \ell\omega, \qquad u = 0,$$

with $|\gamma| \leq 1$ and $\ell > 0$. [Fried&Gurtin'08]

Study the spatial principal part:

$$\Delta^2 u + \nabla p = f, \qquad \nabla\cdot u = 0, \qquad \&\ \text{b.c.}$$

Integration by parts

Let $G = \operatorname{grad}\omega + \gamma(\operatorname{grad}\omega)^T$, with $\omega = \operatorname{curl} u$. Then

$$\int_\Omega G : \operatorname{grad}\operatorname{curl}\phi = -\int_\Omega \operatorname{div} G \cdot \operatorname{curl}\phi + \int_{\partial\Omega} Gn \cdot \operatorname{curl}\phi$$
$$= -\int_\Omega \operatorname{curl}\operatorname{div} G \cdot \phi + \int_{\partial\Omega} Gn \cdot \operatorname{curl}\phi + (n \times \operatorname{div} G) \cdot \phi$$

Assume $\nabla u = 0$ and $\phi|_{\partial\Omega} = 0$. Then we have $\operatorname{curl}\operatorname{div} G = -\Delta^2 u$ and $g \cdot \operatorname{curl}\phi = -(n \times g) \cdot \partial_n \phi$, hence

$$\int_\Omega G : \operatorname{grad}\operatorname{curl}\phi = \int_\Omega \Delta^2 u \cdot \phi - \int_{\partial\Omega} (n \times Gn) \cdot \partial_n \phi.$$

The boundary condition is of the form $-n \times n \times Gn = k\omega$, which implies

$$kn \times \omega = n \times Gn \qquad (k = \ell / \beta^2).$$

If this is satisfied, and $\Delta^2 u = 0$, then

$$\int_\Omega G : \operatorname{grad}\operatorname{curl}\phi + k\int_{\partial\Omega} (n \times \omega) \cdot \partial_n \phi = 0, \qquad \forall \phi : \phi|_{\partial\Omega} = 0.$$

Variational formulation

Let $\mathcal{V} = \{u \in \mathcal{D}(\Omega) : \nabla \cdot u = 0\}$, $V = \operatorname{clos}_{H^1} \mathcal{V}$, and $V^s = V \cap H^s(\Omega)$. Define the continuous bilinear form $a : V^2 \times V^2 \to \mathbb{R}$ by

$$a(u,\phi) = \int_\Omega G : \operatorname{grad} \operatorname{curl} \phi + k \int_{\partial\Omega} (n \times \omega) \cdot \partial_n \phi,$$

where $k = \ell / \beta^2 > 0$. This bilinear form is symmetric, since

$$G : \operatorname{grad} \psi = \omega_{i,j} \psi_{i,j} + \gamma \omega_{j,i} \psi_{i,j} = \omega_{i,j} \psi_{i,j} + \gamma \omega_{i,j} \psi_{j,i},$$

and $(n \times \omega) \cdot \partial_n \phi = -\omega \cdot \operatorname{curl} \phi = -(n \times \partial_n u) \cdot (n \times \partial_n \phi)$, the latter inequality true provided $u|_{\partial\Omega} = 0$.

Let $u \in V^4$ satisfy $a(u,\phi) = (f,\phi)_{L^2}$ for all $\phi \in V^2$, where $f \in L^2$ is a given function. Then

$$\Delta^2 u + \nabla p = f \qquad \text{in} \quad \Omega,$$

$$u = n \times n \times Gn + k\omega = 0 \qquad \text{on} \quad \partial\Omega.$$

Coercivity: The volume term

We want to show that $a(u,u) \geq c\|u\|^2_{H^2} - C\|u\|^2_{L^2}$ for $u \in V^2$.
Case $\gamma = -1$:

$$\int_\Omega (\omega_{i,j} - \omega_{j,i})\omega_{i,j} = \frac{1}{2}\|\mathrm{curl}\,\mathrm{curl}\,u\|^2_{L^2} = \frac{1}{2}\|\Delta u\|^2_{L^2} \geq c\|u\|^2_{H^2}.$$

Case $\gamma = 1$: Korn's second inequality

$$\int_\Omega (\omega_{i,j} + \omega_{j,i})\omega_{i,j} \geq c\|\omega\|^2_{H^1}.$$

Case $|\gamma| < 1$:

$$\int_\Omega \omega_{i,j}\omega_{i,j} \leq \int_\Omega (\omega_{i,j} + \gamma\omega_{j,i})\omega_{i,j} + |\gamma| \int_\Omega \omega_{i,j}\omega_{i,j}$$

To conclude the latter two cases, note that

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2} = \|\mathrm{curl}\,\omega\|_{L^2} \leq \|\omega\|_{H^1}.$$

Coercivity: The boundary term

We have established

$$a(u,u) \geq c\|u\|_{H^2}^2 - k\int_{\partial\Omega} |n \times \partial_n u|^2 \geq c\|u\|_{H^2}^2 - kC\|u\|_{H^{3/2}}^2.$$

In order for this to be positive, we need

$$kC_P^2 C < c,$$

where C_P is the constant of the Friedrichs inequality

$$\|u\|_{H^{3/2}} \leq C_P \|u\|_{H^2},$$

that has the behaviour $C_P^2 \sim \mathrm{diam}(\Omega)$. To conclude, we have

$$a(u,u) \geq c\|u\|_{H^2}^2 - C\|u\|_{L^2}^2 \qquad \text{for} \quad u \in V^2,$$

and moreover there exists a constant $\delta > 0$ such that

$$\frac{\ell}{\beta} < \frac{\delta\beta}{\mathrm{diam}(\Omega)} \qquad \text{implies} \qquad C = 0.$$

Hilbert-Schmidt + elliptic regularity

Define the operator $A: V^2 \to (V^2)'$ by $(Au)(\phi) = a(u,\phi)$, and restrict its range to $H = \mathrm{close}_{L^2}\mathcal{V}$, i.e., consider A as an unbounded operator in H with the domain $\mathrm{dom}(A) = \{u \in V^2 : Au \in H\}$.

Then A is self-adjoint and has countably many eigenvalues $\lambda_1 \le \lambda_2 \le \ldots$, with $\lambda_n \to +\infty$ as $n \to \infty$. If $\ell > 0$ is sufficiently small, then $\lambda_1 > 0$.

Moreover, the corresponding eigenfunctions form both an orthonormal basis in H, and a basis in V^2, orthogonal with respect to $a(\cdot,\cdot) + \mu\langle\cdot,\cdot\rangle$ for some sufficiently large μ.

Regularity results on the solutions of $Au = f$ can be derived from the Agmon-Douglis-Nirenberg theory for elliptic systems.

One also has a functional calculus, e.g.,

$$g(A)\,u = \sum_n g(\lambda_n)\langle u, v_n\rangle v_n$$

Fixed-point formulation

In H, and with $f \in L^2H$, consider the initial value problem

$$\partial_t \Lambda u + \beta^2 A u = f, \qquad \text{where} \qquad \Lambda = 1 - \alpha^2 \Delta : V^2 \to H.$$

This is equivalent to

$$\partial_t v + \underbrace{\beta^2 \Lambda^{-\frac{1}{2}} A \Lambda^{-\frac{1}{2}}}_{D} v = \Lambda^{-\frac{1}{2}} f, \qquad \text{with} \qquad v = \Lambda^{\frac{1}{2}} u,$$

implying that

$$u(t) = \Lambda^{-\frac{1}{2}} e^{-tD} \Lambda^{\frac{1}{2}} u(0) + \int_0^t \Lambda^{-\frac{1}{2}} e^{(\tau - t)D} \Lambda^{-\frac{1}{2}} f(\tau) \mathrm{d}\tau.$$

Restricting attention to the time interval $[0, T]$, let us write it as

$$u = u_0 + \Phi f.$$

Let $B(v, u) = P[(\operatorname{grad} v) u + (\operatorname{grad} u)^T v]$, and let $P : L^2 \to H$ be the Leray projector. Then Navier-Stokes-$\alpha\beta$ equations are

$$\partial_t \Lambda u + \beta^2 A u - \Delta u + B(\Lambda u, u) = 0,$$

or equivalently

$$u = u_0 + \Phi \Delta u - \Phi B(\Lambda u, u).$$

Local existence and blow-up criterion

Recall the fixed-point formulation

$$u = u_0 + \Phi\Delta u - \Phi B(\Lambda u, u).$$

Noting that "$B(\Lambda u, u) = \partial(\Lambda u \cdot u)$", we can bound

$$\|B(\Lambda u, u)\|_{H^1} \lesssim \|u\|_{H^4}^2,$$

and show that $u \mapsto B(\Lambda u, u)$ is locally Lipschitz as a mapping $V^4 \to V^1$.

Hence we can design a Banach fixed point iteration in V^4, assuming that $T > 0$ is suitably small. This also gives the following blow-up criterion:

If there is a finite time $T^* < \infty$ beyond which the solution cannot be continued, then it is necessary that $\|u(t)\|_{H^4} \to \infty$ as $t \nearrow T^*$.

So global existence is proved if we show that $\|u(t)\|_{H^4}$ is bounded by a finite constant depending on the time of assumed existence.

A priori estimates and global well-posedness

Pairing

$$\partial_t \Lambda u + \beta^2 A u - \Delta u + B(\Lambda u, u) = 0, \qquad (*)$$

with u, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \Lambda u, u\rangle + \beta^2\langle Au, u\rangle + \langle \nabla u, \nabla u\rangle = 0,$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^1}^2 + c\|u\|_{H^2}^2 \le C\|u\|_{L^2}^2, \qquad \text{implying} \qquad u \in L^\infty V \cap L^2 V^2.$$

If we act on $(*)$ by A before pairing with u, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^3}^2 + c\|u\|_{H^4}^2 \le C\|u\|_{L^2}^2 + |\langle AB(\Lambda u, u), u\rangle|.$$

Taking into account the bound

$$|\langle B(\Lambda u, u), Au\rangle| \lesssim \|\Lambda u\|_{H^1}\|u\|_{H^2}\|Au\|_{L^2} \lesssim \varepsilon\|u\|_{H^4}^2 + C_\varepsilon\|u\|_{H^2}^2\|u\|_{H^3}^2,$$

we get $u \in L^\infty V^3$.

Similarly, if we act by A^2 before pairing with u, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^5}^2 + c\|u\|_{H^6}^2 \le C\|u\|_{L^2}^2 + |\langle A^2 B(\Lambda u, u), u\rangle|.$$

We have the bounds

$$|\langle A^{\frac{1}{2}} B(\Lambda u, u), A^{\frac{3}{2}} u\rangle| \lesssim \|B(\Lambda u, u)\|_{H^2}\|u\|_{H^6},$$

and

$$\|B(\Lambda u, u)\|_{H^2} \lesssim \|\Lambda u\|_{H^3}\|u\|_{H^3} \lesssim \|u\|_{H^5}\|u\|_{H^3},$$

giving rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^5}^2 + c\|u\|_{H^6}^2 \le C\|u\|_{L^2}^2 + \varepsilon\|u\|_{H^6}^2 + C_\varepsilon\|u\|_{H^3}^2\|u\|_{H^5}^2.$$

Thus $u \in L^\infty H^5$, and global existence follows.

Limit as $\alpha, \beta \to 0$

Let α_n and β_n be sequences satisfying $0 < \alpha_n \leq c\beta_n \to 0$, and consider

$$\partial_t \Lambda_n u_n + \beta_n^2 A u_n - \Delta u_n + B(\Lambda_n u_n, u_n) = 0,$$

where Λ_n has α_n in it, and $k = \ell_n / \beta_n^2$ is fixed, so that A does not change. Also, assume that the initial conditions are the same.

Then we can show that

$$u_n \in L^\infty H \cap L^2 V, \qquad \alpha u_n \in L^\infty V, \qquad \beta u_n \in L^2 V^2,$$

with uniformly bounded norms.

Hence there exists $u \in L^\infty H \cap L^2 V$ such that up to a subsequence

$$u_n \to u \quad \text{weak* in} \quad L^\infty L^2, \qquad \text{and} \qquad u_n \to u \quad \text{weakly in} \quad L^2 H^1.$$

Moreover, u is a weak solution of the Navier-Stokes equation. Note that the second order boundary condition will be lost under the limit.