Absorbing Boundary Conditions for Nonlinear Wave Equations

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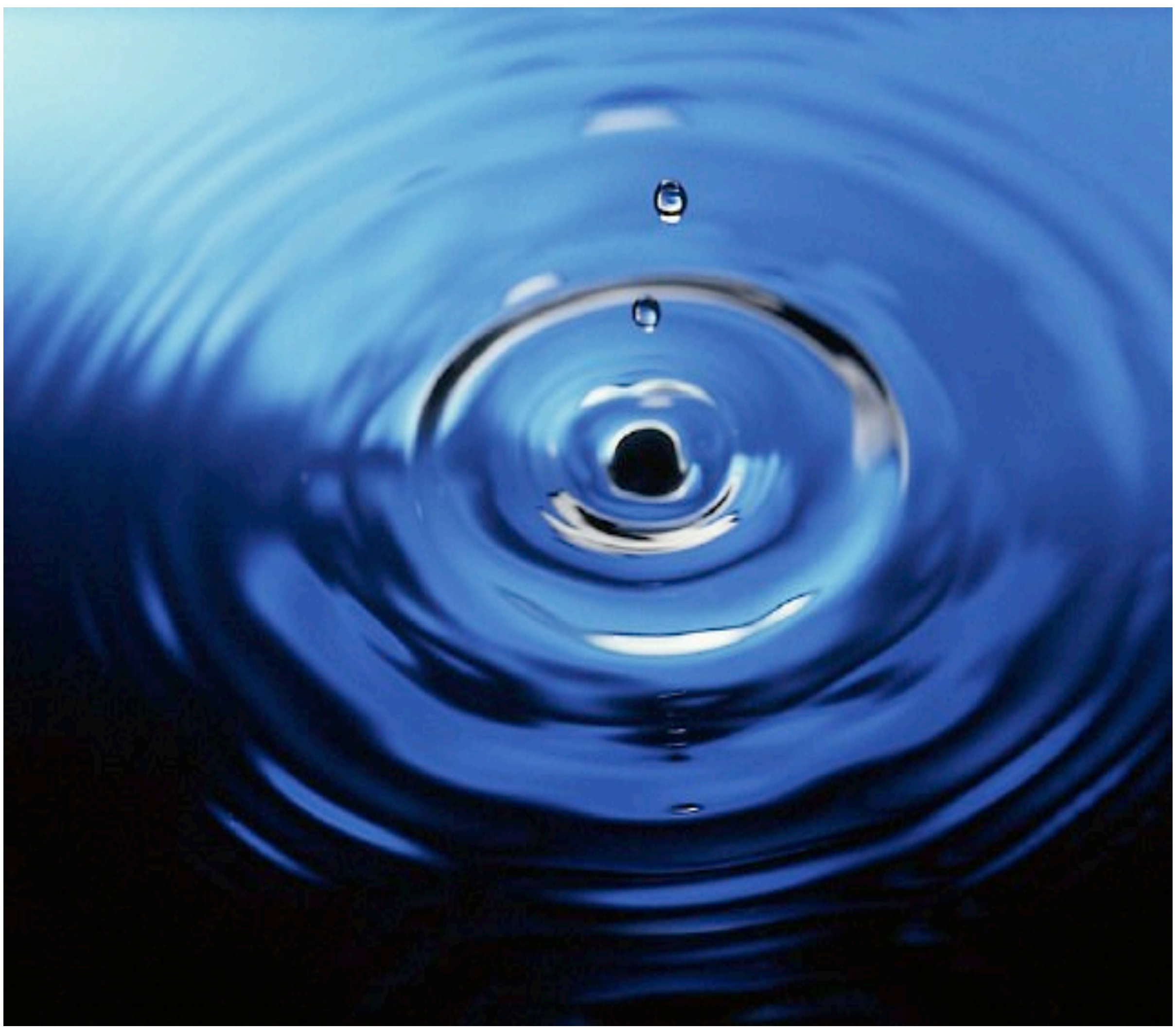

1 Linear Wave Equations

1.1 Boundary Condition 1

To begin, we wish to solve

$$u_{tt} = \nabla^2 u \qquad t, x \geq 0 \tag{1}$$

where

$$u = u(x, y, t)$$

with the boundary condition

$$\left[\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) u\right]_{x=0} = 0 \tag{2}$$

To solve this, we will use forward differences to approximate all derivatives. So, with U approximating the solution to **(1)**, it becomes

$$\frac{U_{j,k}^{n+2} - 2U_{j,k}^{n+1} + U_{j,k}^{n}}{\Delta t^2} = \frac{U_{j+2,k}^{n} - 2U_{j+1,k}^{n} + U_{j,k}^{n}}{\Delta x^2} + \frac{U_{j,k+2}^{n} - 2U_{j,k+1}^{n} + U_{j,k}^{n}}{\Delta y^2}$$

where $j = 0, 1, 2, \ldots$ and corresponds to x, $n = 0, 1, 2, \ldots$ and corresponds to t and where $k \in \mathbb{Z}$ and corresponds to y. Also, Δx is the mesh spacing for x and similarly for the rest. This simplifies to

$$U_{j,k}^{n+2} = 2U_{j,k}^{n+1} + (r_x^2 + r_y^2 - 1)U_{j,k}^{n} - r_x^2\left(2U_{j+1,k}^{n} - U_{j+2,k}^{n}\right) - r_y^2\left(2U_{j,k+1}^{n} - U_{j,k+2}^{n}\right) \tag{3}$$

where

$$r_x = \frac{\Delta t}{\Delta x} \quad \text{and} \quad r_y = \frac{\Delta t}{\Delta y}$$

Now, the boundary condition becomes

$$\frac{U_{1,k}^{n} - U_{0,k}^{n}}{\Delta x} - \frac{U_{0,k}^{n+1} - U_{0,k}^{n}}{\Delta t} = 0$$

which becomes

$$U_{0,k}^{n+1} = (1 - r_x)U_{0,k}^{n} + r_x U_{1,k}^{n} \tag{4}$$

Now we take a finite rectangle with $0 \leq x \leq L$ and $0 \leq y \leq W$. For simplicity, we'll use Dirichlet conditions on the remaining sides of this rectangle. We have

$$u(L, y, t) = f(y, t) \qquad u(x, 0, t) = g(x, t) \qquad u(x, W, t) = h(x, t) \tag{5}$$

which leads to

$$U_{J,k}^{n} = f(k\Delta y, n\Delta t) = f_k^n \tag{6}$$

$$U_{j,0}^{n} = g(j\Delta x, n\Delta t) = g_j^n \tag{7}$$

$$U^n_{j,K} = h(j\Delta x, n\Delta t) = h^n_j \tag{8}$$

where $j = 0, \ldots, J$ and $k = 0, \ldots, K$ and clearly we have that

$$J\Delta x = L$$

and

$$K\Delta y = W$$

Finally, we can assume that $u^0_{j,k}$ and $u^1_{j,k}$ are given as initial conditions. This allows us to to furnish the solution, so

$$U^0_{j,k} = p_{j,k} \qquad U^1_{j,k} = q_{j,k} \tag{9}$$

Now we let

$$\mathbf{U}^n = \left[U^n_{0,0}, \ldots, U^n_{J,0}, U^n_{0,1}, \ldots, U^n_{J,1}, \ldots, U^n_{0,K}, \ldots, U_{J,K}\right]^T$$

Given our initial and boundary conditions, we need a recurrence of the form

$$\mathbf{U}^{n+2} = A\mathbf{U}^{n+1} + B\mathbf{U}^n$$

where A and B are $(J+1)(K+1) \times (J+1)(K+1)$ matrices. From our recurrence, we can set

$$A = 2, \qquad B = \begin{bmatrix} \mathbf{C} & \mathbf{D} & \mathbf{E} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{C} & \mathbf{D} & \mathbf{E} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & & \mathbf{0} \\ \mathbf{0} & & & \ddots & \ddots & \ddots \\ \mathbf{0} & \cdots & \cdots & \mathbf{C} & \mathbf{D} & \mathbf{E} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \mathbf{C} \end{bmatrix} \tag{10}$$

where **C, D** and **E** are all $J+1 \times K+1$ matrices and $\mathbf{0}$ is the $J+1 \times K+1$ zero matrix. Also,

$$\mathbf{C} = \begin{bmatrix} a & b & c & \cdots & \cdots & 0 \\ 0 & a & b & c & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & 0 \\ 0 & & & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & a & b & c \\ 0 & \cdots & \cdots & \cdots & a & b \\ 0 & \cdots & \cdots & \cdots & \cdots & a \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} b & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} c & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c \end{bmatrix}$$

where

$$a = r_x^2 + r_y^2 - 1, \qquad b = -2r_x^2, \qquad c = r_y^2$$

1.2 Boundary Condition 2

Now we solve **(1)** with the following absorbing boundary condition

$$\left[\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)^2 u\right]_{x=0}=0 \tag{11}$$

this boundary condition expands and becomes

$$\left[\left(\frac{\partial^2}{\partial x^2}-2\frac{\partial^2}{\partial x\partial t}+\frac{\partial^2}{\partial t^2}\right)u\right]_{x=0}=0$$

Again, we use forward differences for all derivatives which gives on the boundary

$$\frac{U_{2,k}^n-2U_{1,k}^n+U_{0,k}^n}{\Delta x^2}-2\frac{U_{1,k}^{n+1}-U_{0,k}^{n+1}-U_{1,k}^n+U_{0,k}^n}{\Delta x\Delta t}+\frac{U_{0,k}^{n+2}-2U_{0,k}^{n+1}+U_{0,k}^n}{\Delta t^2}=0$$

Now, multiplying by $\Delta x^2\Delta t^2$, we get

$$U_{0,k}^{n+2}=2(1-r_x)U_{0,k}^{n+1}+2r_xU_{1,k}^{n+1}-r_x^2U_{2,k}^n+(2r_x-r_x^2-1)U_{0,k}^n+2r_x(r_x-1)-r_x^2U_{2,k}^n \tag{12}$$

Finally, with the same finite rectangle and some Dirichlet conditions to go with it as with the first boundary condition, we recover **(6)**, **(7)**, **(8)** and **(9)** and we can solve numerically. As before, we let

$$\mathbf{U}^n=\left[U_{0,0}^n,\ldots,U_{J,0}^n,U_{0,1}^n,\ldots,U_{J,1}^n,\ldots,U_{0,K}^n,\ldots,U_{J,K}\right]^T$$

We proceed here just as we did above, with the matrices A and B except for the difference on the $x=0$ side.

2 Linear Wave Equations In Polar Coordinates

2.1 One Spacial Dimension

We wish to solve

$$u_{tt}=u_{rr}$$

$$r\leq r_0,\quad t\geq 0$$

subject to

$$\left[\left(\frac{\partial}{\partial r}-\frac{\partial}{\partial t}\right)u\right]_{r=r_0}=0$$

We proceed with forward differences on temporal derivatives and centred differences for spacial derivatives. This yields

$$\frac{U_j^{n+2} - 2U_j^{n+1} + U_j^n}{\Delta t^2} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta r^2}$$

which leads to

$$U_j^{n+2} = 2U_j^{n+1} + r_r^2 U_{j-1}^n - (2r_r^2 + 1)U_j^n + r_r^2 U_{j+1}^n$$

where

$$r_r = \frac{\Delta t}{\Delta r}$$

We discretize the boundary condition with forward differences in time and backward differences in space. This yields

$$U_j^{n+1} = (1 + r_r)U_j^n - r_r U_{j-1}^n$$

and so when $r = r_0$ we have $j = J$ which gives

$$U_J^{n+1} = (1 + r_r)U_J^n - r_r U_{J-1}^n$$

We impose a boundary condition at $r = 0$ and an initial condition as follows

$$u(0,t) = f(t), \qquad u(r,0) = p(r)$$

and so

$$U_0^n = f(n\Delta t) = f^n$$

for $n \geq 0$ and

$$U_j^0 = p(j\Delta r) = p_j$$

for $j \leq J$ and

$$U_j^1 = q(j\Delta r) = q_j$$

for $j < J$. Now we define

$$\mathbf{U}^n = [U_1^n, \ldots, U_{J-1}^n]^T$$

and this gives

$$\mathbf{U}^{n+2} = 2\mathbf{U}^{n+1} + A\mathbf{U}^n$$

where A is a $J - 1 \times J - 1$ matrix and

$$A = \begin{bmatrix} a & b & c & \cdots & \cdots & 0 \\ 0 & a & b & c & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & 0 \\ 0 & & & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & a & b & c \\ 0 & \cdots & \cdots & \cdots & a & b \\ 0 & \cdots & \cdots & \cdots & \cdots & a \end{bmatrix}$$

where

$$a = r_r^2 \qquad b = -(2r_r^2 + 1) \qquad c = r_r^2$$

and from here, with initial conditions, we can solve numerically. Clearly, from the initial conditions, we know that

$$\mathbf{U}^0 = [p_1, \ldots, p_{J-1}]^T$$

and that

$$\mathbf{U}^1 = [q_1, \ldots, q_{J-1}]^T$$

so these two are given. We also know U_0^n for each time level n, so we can now just iterate for a solution with error $O(h^2)$.

3 (1+1)-Dimensional Nonlinear Wave Equations In Polar Coordinates

3.1 Theory

Here we'll solve the following problem

$$u_{tt} = u_{rr} + \frac{1}{r}u_r - \frac{\sin(2u)}{2r^2}$$

with

$$u(r_0, t) = 0$$

where r_0 is the radius, and

$$u(r, 0) = \varphi(r) \qquad u_t(r, 0) = \psi(r)$$

where φ and ψ are smooth. Also, we impose an absorbing boundary condition at $r = 0$ that simulates the wave propagating outward as opposed to being reflected. So

$$\left[\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial t}\right)^2 u\right]_{r=0} = 0$$

Here we'll handle these separately with high order discretizations. For the u_{rr} term, we'll use the Taylor expansions for $u(r-2h,t), u(r-h,t), u(r+h,t), u(r+2h)$ and solve the system of equations that arise from them. This yields

$$u_{rr}(r,t) = -\frac{u(r+2h,t) - 16u(r+h,t) + 30u(r,t) - 16u(r-h,t) + u(r-2h,t)}{12h^2} + O(h^4)$$

and using the same strategy for the u_r/r term, we get

$$\frac{1}{r}u_r(r,t) = -\frac{1}{r}\frac{u(r+2h,t) - 8u(r+h,t) + 8u(r-h,t) - u(r-2h,t)}{12h} + O(h^4)$$

Thus, we have

$$\begin{aligned} u_{tt} = & -\left(\frac{1}{12h^2} - \frac{1}{12hr}\right) u(r-2h,t) + \left(\frac{4}{3h^2} - \frac{2}{3hr}\right) u(r-h,t) - \frac{5}{2h^2} u(r,t) \\ & + \left(\frac{4}{3h^2} + \frac{2}{3hr}\right) u(r+h,t) - \left(\frac{1}{12h^2} + \frac{1}{12hr}\right) u(r+2h,t) \\ & - \frac{\sin(2u(r,t))}{2r^2} + O(h^4) \end{aligned} \tag{13}$$

And for simplicity, we'll say that

$$\begin{aligned} u_{tt} = & C_1 u(r-2h,t) + C_2 u(r-h,t) + C_3 u(r,t) + C_4 u(r+h,t) + C_5 u(r+2h,t) \\ & - \frac{\sin(2u(r,t))}{2r^2} + O(h^4) \end{aligned}$$

$$u_{tt}(jh, n\Delta t) = C_1 u^n_{j-2} + C_2 u^n_{j-1} + C_3 u^n_j + C_4 u^n_{j+1} + C_5 u^n_{j+2} - \frac{\sin(2u^n_j)}{2(jh)^2} + O(h^4)$$

where u^n_j is the approximation of $u(r,t)$ at $r = j\Delta r$ and $t = n\Delta t$. Also, $j = 0, \ldots, J$ and so $r_0 = Jh$.

We need a starter method that will allow us to have u^n_1 for each n. Thus, we will use Taylor's Theorem to derive a new 6 point approximation that only requires information from one step behind r (i.e. we need not go further back than $u(r-h,t)$ so that we can stay within the index range of j). This approximation uses the fact that

$$\begin{aligned} u_{rr}(r,t) = & \frac{1}{12} \frac{10u(r-h,t) - 15u(r,t) - 4u(r+h,t) + 14u(r+2h,t)}{h^2} \\ & + \frac{1}{12} \frac{-6u(r+3h,t) + u(r+4h,t)}{h^2} + O(h^4) \end{aligned}$$

and that

$$\frac{1}{r} u_r(r,t) = -\frac{1}{12} \frac{3u(r-h,t) + 10u(r,t) - 18u(r+h,t) + 6u(r+2h,t) - u(r+3h,t)}{hr} + O(h^4)$$

So, collecting like terms, we arrive at

$$\begin{aligned} u_{tt}(r,t) = & \left(\frac{5}{6h^2} - \frac{1}{4hr}\right) u(r-h,t) - \left(\frac{5}{4h^2} + \frac{5}{6hr}\right) u(r,t) - \left(\frac{1}{3h^2} - \frac{3}{2hr}\right) u(r+h,t) \\ & + \left(\frac{7}{6h^2} - \frac{1}{2hr}\right) u(r+2h,t) - \left(\frac{1}{2h^2} - \frac{1}{12hr}\right) u(r+3h,t) + \frac{1}{12h^2} u(r+4h,t) \\ & - \frac{\sin(2u(r,t))}{2r^2} + O(h^4) \end{aligned} \tag{14}$$

or simply

$$\begin{aligned} u_{tt}(r,t) = & D_1 u(r-h,t) + D_2 u(r,t) + D_3 u(r+h,t) + D_4 u(r+2h,t) + D_5 u(r+3h,t) \\ & + D_6 u(r+4h,t) - \frac{\sin(2u(r,t))}{2r^2} + O(h^4) \end{aligned}$$

And using the previous notation, we have

$$u_{tt}(jh, n\Delta t) = D_1 u_{j-1}^n + D_2 u_j^n + D_3 u_{j+1}^n + D_4 u_{j+2}^n + D_5 u_{j+3}^n + D_6 u_{j+4}^n - \frac{\sin(2u_j^n)}{2(jh)^2} + O(h^4)$$

Now we need a high order discretization near but not at the Dirichlet boundary. We need an approximation that gives u_{J-1}^n and needs only one spacial step forward. We do this in the usual fashion by solving some simultaneous equations that arise from Taylor expansions. So we have

$$\begin{aligned} u_{rr}(r,t) &= \frac{1}{12}\frac{u(r-4h,t) - 6u(r-3h,t) + 14u(r-2h,t) - 4u(r-h,t)}{h^2} \\ &\quad + \frac{1}{12}\frac{-15u(r,t) + 10u(r+h,t)}{h^2} + O(h^4) \\ \frac{1}{r}u_r(r,t) &= -\frac{1}{12}\frac{u(r-3h,t) - 6u(r-2h,t) + 18u(r-h,t)}{h^2} \\ &\quad - \frac{1}{12}\frac{-10u(r,t) - 3u(r+h,t)}{h} + O(h^4) \end{aligned}$$

and after simplifying and combining like terms, we arrive at

$$\begin{aligned} u_{tt}(r,t) = &\frac{1}{12h^2}u(r-4h,t) - \left(\frac{1}{2h^2} + \frac{1}{12hr}\right)u(r-3h,t) + \left(\frac{7}{6h^2} + \frac{1}{2hr}\right)u(r-2h,t) \\ &- \left(\frac{1}{3h^2} + \frac{3}{2hr}\right)u(r-h,t) - \left(\frac{5}{4h^2} - \frac{5}{6hr}\right)u(r,t) \\ &+ \left(\frac{5}{6h^2} + \frac{1}{4hr}\right)u(r+h,t) - \frac{\sin(2u(r,t))}{2r^2} + O(h^4) \end{aligned} \tag{15}$$

or for simplicity

$$\begin{aligned} u_{tt}(r,t) = E_1 u(r-4h,t) - E_2 u(r-3h,t) + E_3 u(r-2h,t) - E_4 u(r-h,t) \\ - E_5 u(r,t) + E_6 u(r+h,t) - \frac{\sin(2u(r,t))}{2r^2} + O(h^4) \end{aligned}$$

So now we wish to find a high order approximation for the absorbing boundary condition. Once again, we'll use Taylor's theorem to derive these approximations. First, we must expand and analyze this condition a the boundary $r = 0$. This expands to

$$u_{tt}(0,t) = 2u_{tr}(0,t) - u_{rr}(0,t)$$

However, if we let $u_t(r,t) = v(r,t)$ then we have the following system of differential equations

$$u_t(0,t) = v(0,t) \qquad u(0,0) = \varphi(0) = \varphi(0) = \varphi_0$$

$$v_t(0,t) = 2v_r(0,t) - u_{rr}(0,t) \qquad v(0,0) = \psi(0) = \psi(0) = \psi_0$$

Here, we'll approximate the spacial derivatives using the same process as above. We want an approximation that only requires information at $r = jh$ for $jh \geq r$. Thus, after solving the appropriate system of linear equations that arise from 5 taylor expansions, we have

$$\begin{aligned} v_r(0,t) = &-\frac{1}{12}\frac{25v(0,t) - 48v(h,t) + 36v(2h,t)}{h} \\ &-\frac{1}{12}\frac{-16v(3h,t) + 3v(4h,t)}{h} + O(h^4) \end{aligned} \tag{16}$$

and

$$\begin{aligned} u_{rr}(0,t) = &\frac{1}{12}\frac{45u(0,t) - 154u(h,t) + 214u(2h,t)}{h^2} \\ &\frac{1}{12}\frac{-156u(3h,t) + 61u(4h,t) - 10u(5h,t)}{h^2} + O(h^4) \end{aligned} \tag{17}$$

Now that we've found viable approximations, we'll deal with the time derivative. Because of the mesh spacing we've chosen, we can let $r = jh$ or $r = j\Delta r$ which gives

$$\begin{aligned} u_{tt}(jh,t) = &C_1u(jh-2h,t) + C_2u(jh-h,t) + C_3u(r,t) + C_4u(jh+h,t) + C_5u(jh+2h,t) \\ &- \frac{\sin(2u(jh,t))}{2(jh)^2} + O(h^4) \end{aligned}$$

and we also note that C_1, C_2, C_4 and C_5 will have $r = jh$ in the denominator of the second term of each. Now, for each j, we have an O.D.E. in time. So, letting $u_j(t) = u(jh,t)$, we have

$$u_j''(t) = C_1u_{j-2}(t) + C_2u_{j-1}(t) + C_3u_j(t) + C_4u_{j+1}(t) + C_5u_{j+2}(t) + f(u_j(t))$$

or

$$u_j''(t) = G_2(u_{j-2}(t), u_{j-1}(t), u_j(t), u_{j+1}(t), u_{j+2}(t))$$

for $j = 2, 3, \ldots, J-3, J-2$ from **(13)**. From **(14)**, we have

$$u_1''(t) = G_1(u_0(t), u_1(t), u_2(t), u_3(t), u_4(t), u_5(t))$$

for $j = 1$. Similarly, we have from **(15)** that

$$u_{J-1}''(t) = G_3(u_{J-5}(t), u_{J-4}(t), u_{J-3}(t), u_{J-2}(t), u_{J-1}(t), u_J(t))$$

when $j = J-1$, and for the boundary, we use **(16)** and **(17)** to obtain

$$u_J''(t) = G_4(u_0(t), \ldots, u_5(t), v_0(t), \ldots, v_4(t))$$

and so we let

$$\mathbf{U}(t) = \begin{bmatrix} u_0(t) \\ \vdots \\ u_J(t) \end{bmatrix} \qquad \mathbf{U}^n = \begin{bmatrix} u_0^n \\ \vdots \\ u_J^n \end{bmatrix}$$

where $u_j^n = u_j(n\Delta t)$ for each j and using similar notation, we have

$$\mathbf{V}(t) = \begin{bmatrix} v_0(t) \\ \vdots \\ v_J(t) \end{bmatrix} \qquad \mathbf{V}^n = \begin{bmatrix} v_0^n \\ \vdots \\ v_J^n \end{bmatrix}$$

and finally, for simplicity, we'll let

$$\mathbf{W}(t) = \begin{bmatrix} \mathbf{U}(t) \\ \mathbf{V}(t) \end{bmatrix} \qquad \mathbf{W}^n = \begin{bmatrix} \mathbf{U}^n \\ \mathbf{V}^n \end{bmatrix}$$

and for completeness, we have

$$\mathbf{W}(t) = \begin{bmatrix} u_0(t) \\ \vdots \\ u_J(t) \\ v_0(t) \\ \vdots \\ v_J(t) \end{bmatrix} \qquad \mathbf{W}^n = \begin{bmatrix} u_0^n \\ \vdots \\ u_J^n \\ v_0^n \\ \vdots \\ v_J^n \end{bmatrix}$$

and so we have one giant O.D.E. to handle now. We can write it as

$$\mathbf{W}'(t) = \mathbf{G}(\mathbf{W}(t))$$

with initial condition

$$\mathbf{W}(0) = \begin{bmatrix} \varphi_0 \\ \vdots \\ \varphi_J \\ \psi_0 \\ \vdots \\ \psi_J \end{bmatrix}$$

where

$$\mathbf{G}(\mathbf{W}(t)) = \begin{bmatrix} v_0(t) \\ \vdots \\ v_J(t) \\ G_4(u_0(t), \ldots, u_5(t), v_0(t), \ldots, v_4(t)) \\ G_1(u_0(t), u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)) \\ G_2(u_0(t), u_1(t), u_2(t), u_3(t), u_4(t)) \\ \vdots \\ G_2(u_{j-2}(t), u_{j-1}(t), u_j(t), u_{j+1}(t), u_{j+2}(t)) \\ \vdots \\ G_2(u_{J-4}(t), u_{J-3}(t), u_{J-2}(t), u_{J-1}(t), u_J(t)) \\ G_3(u_{J-5}(t), u_{J-4}(t), u_{J-3}(t), u_{J-2}(t), u_{J-1}(t), u_J(t)) \\ 0 \end{bmatrix}$$

And so now, $\mathbf{W}(t)$ and $\mathbf{G}(\mathbf{W}(t))$ are each collections of $2(J+1)$ elements. We can solve this matrix O.D.E. with a simple Runge-Kutta method. The R-K method that we'll use here is the fourth order four step method (RK4). So we have

$$\mathbf{W}^{n+1} = \mathbf{W}^n + \frac{\Delta t}{6}(\mathbf{M}_1 + 2\mathbf{M}_2 + 2\mathbf{M}_3 + \mathbf{M}_4)$$

where

$$\mathbf{M}_1 = \mathbf{G}(\mathbf{W}^n)$$
$$\mathbf{M}_2 = \mathbf{G}\left(\mathbf{U}^n + \frac{\Delta t}{2}\mathbf{M}_1\right)$$
$$\mathbf{M}_3 = \mathbf{G}\left(\mathbf{U}^n + \frac{\Delta t}{2}\mathbf{M}_2\right)$$
$$\mathbf{M}_4 = \mathbf{G}\left(\mathbf{U}^n + \Delta t\mathbf{M}_3\right)$$

and we solve for these at each time level.

3.2 Numerical Experiments

Here, we'll take

$$\varphi(r) = A(r_0 - r)^3 e^{-\frac{(r_0-r)^4}{\sigma}}$$

and

$$\psi(r) = 0$$

Where A is a damping constant for the amplitude of the wave. We wish to take A to be small to avoid blow-up. In the experiment here, we'll take $r_0 = 2$, $\sigma = 0.4$ and $A = -4$. Initially, we want the graph of $u(x, 0)$ to have a single peak so to monitor the evolution of the wave. The initial wave is as depicted in **Figure 1**.
We expect the left boundary (that is, the boundary at $r = 0$) to absorb the wave as though it was propagating out to infinity past the boundary.
In view of **Figure 2**, we observe first that the wave has collapsed in on itself forming a fold where the initial peak was while t ranged from 0 to 0.5. This is to be expected since there is no initial speed. From here, the two peaks are expected to propagate in the same direction as their position (that is, the left peak moves to the left and the right peak moves to the right) which can also be observed in **Figure 2** by looking at the difference in wave structure for $t \in [0.50, 0.70]$. Now, as t increases, **Figure 2** shows that the wave approaches the left boundary and is eventually absorbed at $t \approx 1.10$, while the right-moving peak continues toward the Dirichlet boundary at $r = r_0 = 2$.
From here, it is trivial to see that since the right-moving wave will be reflected by the Dirichlet boundary, it will then too be absorbed by the boundary at $r = 0$ so that as $t \to \infty$, we get that $u(r, t) = 0$ for all $r \in [0, 2]$.

It follows now that this method is a viable and accurate approximation to the solution to

$$u_{tt} = u_{rr} + \frac{1}{r}u_r - \frac{\sin(2u)}{2r^2}$$

up to small deviations in coarse oscillations.

4 Figures

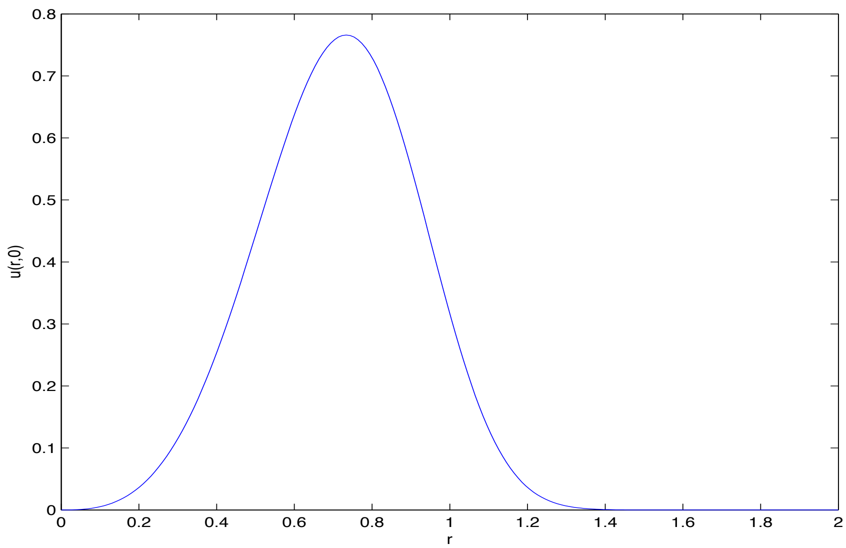

Figure 1: Initial Wave

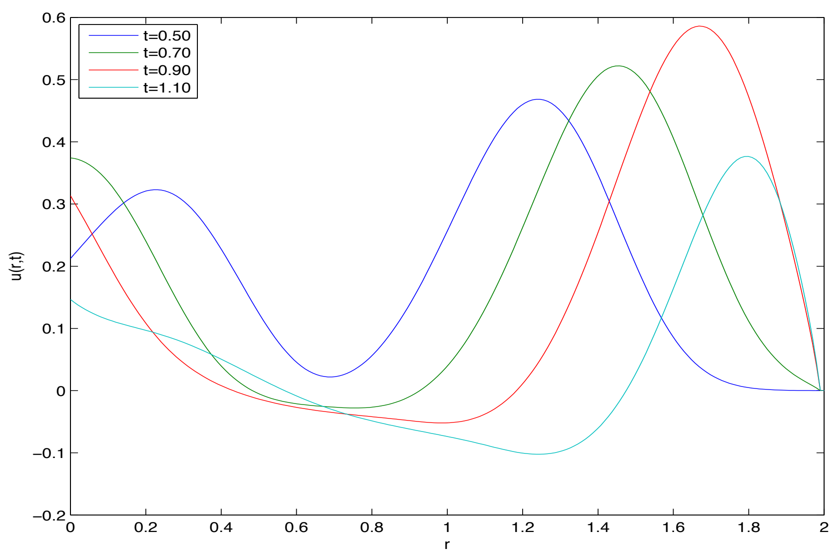

Figure 2: Wave At Time t

References

[1] Randall J. LeVeque: *Finite Difference Methods for Ordinary & Partial Differential Equations: Steady-State & Time-Dependent Problems*, First Edition; Siam (2007).

[2] Joseph E. Flaherty: *Numerical Partial Differential Equations (Course Notes)*; Rensselaer Polytechnic Institute (2005).

[3] Robert Plato: *Concise Numerical Mathematics*, First Edition; AMS (2000).

[4] Alfio Quarteroni, Riccardo Sacco & Fausto Saleri: *Numerical Mathematics*, Second Edition; Springer (2007).

[5] Piotr Bizoń, Tadeusz Chmaj & Zbislaw Tabor: *Dispersion & Collapse of Wave Maps*, Nonlinearity 13, pp. 1411-1423; AMS (2000).

[6] Piotr Bizoń, Tadeusz Chmaj & Zbislaw Tabor: *Formation og Singularities for Equivariant (2+1)-Dimensional Wave Maps Into the 2-Sphere*, Nonlinearity 14, pp. 1041-1053; AMS (2001).

[7] Bjorn Engquist & Andrew Majda: *Absorbing Boundary Conditions for the Numerical Simulation of Waves*, Mathematics of Computation 31, No. 139, pp. 629-651; AMS (1977).

[8] Adi Ditkowski & David Gottlieb: *On the Engquist-Majda Absorbing Boundary Conditions for Hyperbolic Systems*, Contemprary Mathematics, AMS (2000).

[9] Jérémie Szeftel: *Absorbing Boundary Conditions for Nonlinear Scalar Partial Differential Equations*, Computer Methods in Applied Mechanics & Engineering, No. 195, pp. 3760-3775; AMS (2006).