MATH 20F WINTER 2007 MIDTERM EXAM II

FEBRUARY 28

PROBLEM 1: Let B be a 3×3 matrix such that $\det B = 2$, and let A be given by

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 8 & 64 \\ 1 & 13 & 169 \end{bmatrix}.$$

a). Calculate $\det(BAB)$.

b). Write A in LU form, that is, find an upper triangular matrix U and a unit lower triangular matrix L such that $A = LU$.

SOLUTION:

1a). A property of determinant implies

$$\det(BAB) = (\det B)(\det A)(\det B) = (\det B)^2 \det A = 4 \cdot \det A,$$

so if we know the determinant of A we can calculate the determinant of BAB. Let us use row reduction to find $\det A$.

$$\begin{bmatrix} \mathbf{1} & 3 & 9 \\ 1 & 8 & 64 \\ 1 & 13 & 169 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 \\ 0 & \mathbf{5} & 55 \\ 0 & 10 & 160 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 3 & 9 \\ 0 & \mathbf{5} & 55 \\ 0 & 0 & \mathbf{50} \end{bmatrix}.$$

Taking the product of the diagonal entries of the echelon form, $\det A = 1 \cdot 5 \cdot 50 = 250$, and so $\det(BAB) = 4 \cdot 250 = 1000$.

1b). In the above row reduction, we used only row replacements, and the multiplying numbers we used were $-\mathbf{1}$, $-\mathbf{1}$, and $-\mathbf{2}$. We can take U equal to the above echelon form and construct L from the (negatives of the) multiplying numbers as follows.

$$U = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 5 & 55 \\ 0 & 0 & 50 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{1} & 1 & 0 \\ \mathbf{1} & \mathbf{2} & 1 \end{bmatrix}.$$

PROBLEM 2: Let the matrix A and the vectors $\mathbf{u} \in \mathbb{R}^4$ and $\mathbf{w} \in \mathbb{R}^3$ be given by

$$A = \begin{bmatrix} 3 & -2 & 4 & 4 \\ -2 & 6 & 2 & 0 \\ 4 & 2 & 3 & 9 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 14 \\ 10 \\ -1 \\ -14 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 23 \\ 43 \\ 0 \end{bmatrix}.$$

a). Find a *basis* for and the *dimension* of $\operatorname{Nul} A$. Is $\mathbf{u}$ in $\operatorname{Nul} A$? (*Hint on row reducing A*: Scale the second row by factor $\frac{1}{2}$ and interchange the first two rows. Then do not use scaling until the last moment. This will save some arithmetics on fractions.)

b). Find a *basis* for and the *dimension* of $\operatorname{Col} A$. Is $\mathbf{w}$ in $\operatorname{Col} A$?

SOLUTION:

2a). A basis for $\operatorname{Nul} A$ can be extracted from writing the general solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form. We can solve this system by row reducing A into its reduced echelon form. The right hand side is zero, there is no need to carry those zeroes around when you do the row reduction. Following *Hint*, we find

$$\begin{bmatrix} 3 & -2 & 4 & 4 \\ -2 & 6 & 2 & 0 \\ 4 & 2 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 1 & 0 \\ 0 & 7 & 7 & 4 \\ 0 & 14 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 1 & 0 \\ 0 & 7 & 7 & 4 \\ 0 & 0 & -7 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 3 & 0 & \frac{1}{7} \\ 0 & 7 & 0 & 5 \\ 0 & 0 & -7 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & 7 & 0 & 5 \\ 0 & 0 & -7 & 1 \end{bmatrix}.$$

Before the second from last step we could multiply the first row by 7 to avoid some fractions. Now the general solution to $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 = -2x_4 \\ x_2 = -\frac{5}{7}x_3 \\ x_3 = \frac{1}{7}x_4 \\ x_4 \text{ is free} \end{cases} \qquad \text{or} \qquad \mathbf{x} = x_4 \cdot \begin{bmatrix} -2 \\ -\frac{5}{7} \\ \frac{1}{7} \\ 1 \end{bmatrix}.$$

The vector that multiplies x_4 in the last equation forms a basis for $\operatorname{Nul} A$. The dimension of $\operatorname{Nul} A$ equals 1. Since multiplying it by 7 does not change the span,

$$\mathbf{b} = \begin{bmatrix} -14 \\ -5 \\ 1 \\ 7 \end{bmatrix}$$

also forms a basis for $\operatorname{Nul} A$. The vector $\mathbf{u}$ is not a multiple of $\mathbf{b}$, so $\mathbf{u}$ is *not* in $\operatorname{Nul} A$. Note that $\mathbf{b}$ is introduced here merely for the purpose of having integer entries in the vector. Note also that we could check if $\mathbf{u}$ is in $\operatorname{Nul} A$ by taking the product $A\mathbf{u}$ and comparing the result to $\mathbf{0}$.

2b). The pivot columns of A form a basis for $\operatorname{Col} A$. From a) we already know that the first three columns of A are pivot columns. Therefore the set

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Col} A$. The dimension of $\operatorname{Col} A$ (i.e., rank of A) is 3. The only 3 dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself, meaning that $\operatorname{Col} A = \mathbb{R}^3$. The vector $\mathbf{w}$ is in $\mathbb{R}^3$, so $\mathbf{w}$ *is* in $\operatorname{Col} A$. Note that if U is an echelon form of A, the pivot columns of U do *not* necessarily form a basis for $\operatorname{Nul} A$. In this particular case they do, because both $\operatorname{Col} A$ and $\operatorname{Col} U$ are equal to $\mathbb{R}^3$.

PROBLEM 3: Let H be a vector space and let $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for H. Let the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ be given by

$$\mathbf{u}_1 = \mathbf{v}_1 - 2\mathbf{v}_3, \qquad \mathbf{u}_2 = -3\mathbf{v}_1 + \mathbf{v}_2 + 4\mathbf{v}_3, \qquad \mathbf{u}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2 + 4\mathbf{v}_3.$$

a). Is $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ a basis for H? If so, find the *change-of-coordinate matrix* from $\mathcal{V}$ to $\mathcal{U}$, that is, find the matrix Q such that $[\mathbf{x}]_{\mathcal{U}} = Q[\mathbf{x}]_{\mathcal{V}}$ for all $\mathbf{x} \in H$. Here $[\mathbf{x}]_{\mathcal{V}}$ and $[\mathbf{x}]_{\mathcal{U}}$ denote the *coordinate vectors* of $\mathbf{x}$ relative to the bases $\mathcal{V}$ and $\mathcal{U}$, respectively.

b). If $\mathbf{x} = 4\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3$, find $[\mathbf{x}]_{\mathcal{U}}$.

SOLUTION:

3a). Writing the $\mathcal{V}$-coordinates of the vectors in $\mathcal{U}$ as columns, define the following matrix

$$P = \begin{bmatrix} [\mathbf{u}_1]_{\mathcal{V}} & [\mathbf{u}_2]_{\mathcal{V}} & [\mathbf{u}_3]_{\mathcal{V}} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -3 \\ -2 & 4 & 4 \end{bmatrix}.$$

Since $\mathcal{V}$ is a basis, the linear independence of $\mathcal{U}$ is equivalent to the linear independence of the columns of P (i.e., the $\mathcal{V}$-coordinate vectors of the vectors in $\mathcal{U}$). So if P is invertible, $\mathcal{U}$ is a basis for H.

If $\mathcal{U}$ is a basis for H, the change-of-coordinate matrix from $\underline{\mathcal{U} \text{ to } \mathcal{V}}$ would be P. The change-of-coordinate matrix from $\underline{\mathcal{V} \text{ to } \mathcal{U}}$ would be the inverse of P, i.e., $Q = P^{-1}$.

The problem is reduced to determining whether P is invertible, and if it is invertible, finding the inverse. We will use the row reduction algorithm:

$$\begin{bmatrix} \mathbf{1} & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ -2 & 4 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & \mathbf{1} & -3 & 0 & 1 & 0 \\ 0 & -2 & 8 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{2} & 2 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & -1 & -2 & -1 \\ 0 & \mathbf{1} & 0 & 3 & 4 & \frac{3}{2} \\ 0 & 0 & 2 & 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 10 & \frac{7}{2} \\ 0 & 1 & 0 & 3 & 4 & \frac{3}{2} \\ 0 & 0 & 2 & 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 10 & \frac{7}{2} \\ 0 & 1 & 0 & 3 & 4 & \frac{3}{2} \\ 0 & 0 & 1 & 1 & 1 & \frac{1}{2} \end{bmatrix}.$$

Already the result of the second step reveals that P is invertible since is has 3 pivots. So $\mathcal{U}$ is a basis for H. Moreover, we conclude that

$$Q = P^{-1} = \begin{bmatrix} 8 & 10 & 7/2 \\ 3 & 4 & 3/2 \\ 1 & 1 & 1/2 \end{bmatrix}.$$

3b). Using the definition of Q, we find

$$[\mathbf{x}]_{\mathcal{U}} = Q[\mathbf{x}]_{\mathcal{V}} = \begin{bmatrix} 8 & 10 & 3.5 \\ 3 & 4 & 1.5 \\ 1 & 1 & 0.5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 31.5 \\ 11.5 \\ 3.5 \end{bmatrix}.$$

PROBLEM 4: Mark each statement TRUE or FALSE. Briefly justify each answer.

a). If A and B are invertible $n \times n$ matrices, then $A^{-1}B^{-1}$ is the inverse of AB.
b). If A is invertible, then elementary row operations that reduce A to I also reduce I to A^{-1}.
c). The determinant of a matrix in echelon form is the product of its pivot entries.
d). The number of pivot columns in A is the dimension of $\operatorname{Nul} A$.
e). If V is a vector space of dimension k, and $\mathbf{x}$ is in V, then the coordinate vector of $\mathbf{x}$ relative to any basis for V is in $\mathbb{R}^k$.

SOLUTION:

4a). FALSE. The inverse of AB is $B^{-1}A^{-1}$, and in general matrix multiplication is not commutative.
4b). TRUE. This is the basic idea of the algorithm for finding the inverse of a matrix using row reduction.
4c). FALSE. The determinant of a matrix in echelon form is the product of its (main) diagonal entries, but a pivot entry can be above the main diagonal (iff the matrix is singular).
4d). FALSE. The number of pivot columns in A is the rank of A, and in general, it is not equal to the dimension of $\operatorname{Nul} A$.
4e). TRUE. The coordinate vector of $\mathbf{x}$ is the list of weights when you write $\mathbf{x}$ as a linear combination of the basis vectors, and the number of these weights is equal to the dimension of the space.