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# Formation of singularities for equivariant (2 + 1)-dimensional wave maps into the 2-sphere

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**Abstract**
In this paper we report on numerical studies of the Cauchy problem for equivariant wave maps from (2 + 1)-dimensional Minkowski spacetime into the 2-sphere. Our results provide strong evidence for the conjecture that large-energy initial data develop singularities in finite time and that singularity formation has the universal form of adiabatic shrinking of the degree-one harmonic map from $\mathbb{R}^2$ into $S^2$.

Mathematics Subject Classification: 35L67, 35L70, 74H35

## 1. Introduction

A wave map is a function from the Minkowski spacetime $(\mathbb{R}^{n+1}, \eta)$ into a complete Riemannian manifold $(N, g)$, $U : \mathbb{R}^{n+1} \to N$, which is a critical point of the action

$$S(U) = \int g_{AB}\, \partial_a U^A \partial_b U^B \eta^{ab}\, \mathrm{d}^n x\, \mathrm{d}t. \tag{1}$$

The associated Euler–Lagrange equations constitute the system of semilinear wave equations

$$\Box U^A + \Gamma^A_{BC}(U)\partial_a U^B \partial^a U^C = 0 \tag{2}$$

where Γs are the Christoffel symbols of the target metric $g$. Wave maps are interesting both for mathematicians by providing a simple geometric setting for studying the problems of global existence and formation of singularities, and for physicists (who call them sigma models) as toy models of extended structures in field theory (see [1] for a recent review). In this paper we consider the case where the domain manifold is the $(2+1)$-dimensional Minkowski spacetime, $M = \mathbb{R}^{2+1}$, and the target manifold is the 2-sphere, $N = S^2$, with the standard metric

$$g = \mathrm{d}u^2 + \sin^2 u\, \mathrm{d}\theta^2. \tag{3}$$

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We restrict attention to equivariant maps of the form

$$u = u(t, r) \qquad \theta = \phi \tag{4}$$

where $(r, \phi)$ are the polar coordinates on $\mathbb{R}^2$. The wave map system (2) then reduces to the semilinear scalar wave equation,

$$u_{tt} = u_{rr} + \frac{1}{r}u_r - \frac{\sin(2u)}{2r^2}. \tag{5}$$

The main open question for equation (5) is the issue of global regularity; namely, do all solutions starting with smooth initial data

$$u(0, r) = u_0(r) \qquad u_t(0, r) = u_1(r) \tag{6}$$

remain smooth for all times, or do they lose regularity for some data? Our paper reports on numerical investigations of this problem.

Note that the conserved energy associated with solutions of (5)

$$E[u] = \pi \int_0^\infty \left( u_t^2 + u_r^2 + \frac{\sin^2 u}{r^2} \right) r \, \mathrm{d}r \tag{7}$$

is invariant under dilations: if $u_\lambda(t, r) = u(t/\lambda, r/\lambda)$, then $E[u_\lambda] = E[u]$. In this sense 2 + 1 is a critical dimension for wave maps. Note also that the requirement that energy be finite imposes a boundary condition at spatial infinity $u(t, \infty) = k\pi$ $(k = 0, 1, \ldots)$ which compactifies $\mathbb{R}^2$ into $S^2$ and thus breaks the Cauchy problem into infinitely many disconnected topological sectors labelled by the degree $k$ of the map $S^2 \to S^2$.

Let us recall what is known rigorously about the problem. Besides the routine local existence proof, the only global result is that there is a unique smooth solution for all times provided that the initial energy is sufficiently small [1]. In order to obtain global existence without the assumption of small energy it would be sufficient to show that the energy cannot concentrate at the hypothetical singularity. This kind of result was derived by Shatah and Tahvildar-Zadeh [2] for convex targets and then extended by Grillakis [3] for non-convex targets with an arbitrarily narrow neck. However, the Morawetz identity, which is the basic tool in proving non-concentration of energy, loses the desired positivity properties for the geometry of a 2-sphere. This raises the question: is the restriction on admissible targets assumed in [3] only of a technical nature or is it essential? In particular, is concentration of energy possible in the case of a 2-sphere as a target?

We present numerical evidence that for large-energy solutions the energy does, in fact, concentrate and consequently the solutions blow up in finite time. We show that the process of energy concentration proceeds via adiabatic evolution along the one-parameter family of dilations of the degree-one static solution. In this sense the shape of blow-up is universal. The rate of blow-up, determined by the asymptotic speed of adiabatic evolution, is slower than that predicted by the geodesic approximation and goes to zero as the singularity is approached.

The paper is organized as follows. In section 2 we derive static solutions and study their stability. As mentioned above these solutions play an essential role in the process of singularity formation. For completeness, in section 3 we discuss singular self-similar solutions and argue that they play no role in the Cauchy problem. The main body of the paper is contained in section 4 where we present the results of numerical investigations. On the basis of these results we formulate three conjectures about the nature of singularity formation in the model. Finally, in section 5 we comment on some earlier work on this problem and point out some open problems.

## 2. Static solutions

Static solutions of equation (5) can be interpreted as spherically symmetric harmonic maps from the Euclidean space $\mathbb{R}^2$ into $S^2$. They satisfy the ordinary differential equation

$$u'' + \frac{1}{r}u' - \frac{\sin(2u)}{2r^2} = 0 \tag{8}$$

where $' = \mathrm{d}/\mathrm{d}r$. The obvious constant solutions of (8) are $u = 0$ and $u = \pi$; geometrically these are maps into the north and the south pole of $S^2$, respectively. The energy of these maps attains the global minimum $E = 0$. Another constant solution is the equator map $u = \pi/2$ but this solution is singular and has infinite energy. The fact that equation (8) is scale invariant does not exclude non-trivial regular solutions with finite energy (Derrick's argument is not applicable) and, in fact, such solutions are well known both in the mathematical literature as harmonic maps from $\mathbb{R}^2$ into $S^2$ and in the physics literature as instantons in the two-dimensional Euclidean sigma model. They can be derived in many ways. One possibility is to use a Bogomol'nyi-type argument which goes as follows. Let $x = \ln r$ and $U(x) = u(r)$. Then, assuming staticity

$$\begin{aligned} E[u] &= \pi \int_0^\infty \left(u'^2 + \frac{\sin^2 u}{r^2}\right) r\,\mathrm{d}r = \pi \int_{-\infty}^{\infty} (U'^2 + \sin^2 U)\,\mathrm{d}x \\ &= \pi \int_{-\infty}^{\infty} (U' - \sin U)^2\,\mathrm{d}x - 2\pi \cos U\Big|_{-\infty}^{\infty}. \end{aligned} \tag{9}$$

Thus, in the topological sector $k = 1$ the energy attains the minimum, $E = 4\pi$, on the solution of the first-order equation $U' - \sin U = 0$, which is $U(x) = 2\arctan(\mathrm{e}^x)$ up to translations in $x$. Therefore,

$$u_S(r) = 2\arctan(r) \tag{10}$$

is the static degree-one solution (the problem has reflection symmetry so, of course, $-u_S(r)$ is also the solution). By dilation symmetry, the solution $u_S(r)$ generates the orbit of static solutions $u_S^\lambda(r) = u_S(r/\lambda)$. We remark in passing that the solution (10) can be alternatively obtained in an elegant geometric way by taking the identity map between 2-spheres and making the stereographic projection.

We now consider the linear stability of the static solution $u_S(r)$. Inserting $u(t, r) = u_S(r) + \mathrm{e}^{\mathrm{i}kt}v(r)$ into (5) and linearizing, we obtain the eigenvalue problem (the radial Schrödinger equation)

$$Lv = \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + V(r)\right) v = k^2 v \tag{11}$$

where

$$V(r) = \frac{\cos(2u_S)}{r^2} = \frac{1 - 6r^2 + r^4}{(1 + r^2)^2 r^2}. \tag{12}$$

This potential has no bound states as can be shown by the following standard argument. Consider the perturbation induced by dilation

$$v_0(r) = -\frac{\mathrm{d}}{\mathrm{d}\lambda} u_S^\lambda(r)\Big|_{\lambda=1} = r u_S'(r) = \frac{2r}{1 + r^2}. \tag{13}$$

This is the solution to $k^2 = 0$ (the so-called zero mode). The fact that the zero mode $v_0(r)$ has no nodes implies by the standard result from Sturm–Liouville theory that there are no negative eigenvalues, and *eo ipso* no unstable modes around $u_S(r)$. Note that the zero mode is not a genuine eigenfunction because it is not square integrable. Therefore, the operator $L$ has the purely continuous spectrum $k^2 \geqslant 0$.

## 3. Non-existence of self-similar solutions

Since equation (5) is scale invariant, it is natural to look for self-similar solutions of the form

$$u(t,r) = f\left(\frac{r}{T-t}\right) \tag{14}$$

where $T$ is a positive constant. Substituting this ansatz into (5) one obtains the ordinary differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\rho^2} + \left(\frac{1}{\rho} - \frac{\rho}{1-\rho^2}\right)\frac{\mathrm{d}f}{\mathrm{d}\rho} - \frac{\sin(2f)}{2\rho^2(1-\rho^2)} = 0. \tag{15}$$

Let us consider equation (15) inside the past lightcone of the point $(t = T, r = 0)$, that is for $\rho \in [0, 1]$. It is well known that there are no solutions which are analytic at both ends of this interval [1]. However, it is not well known that there are solutions which are less regular. They can be easily derived by setting $\rho = 1/\cosh(y)$, so that (15) simplifies to

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} - \frac{1}{2}\sin(2f) = 0. \tag{16}$$

This equation is solved (up to translations in $y$) by $f(y) = 2\arctan(\mathrm{e}^y)$, so in terms of $\rho$ we obtain a one-parameter family of self-similar solutions

$$f_\alpha(\rho) = 2\arctan\left(\frac{\alpha\rho}{1+\sqrt{1-\rho^2}}\right). \tag{17}$$

These solutions are analytic at $\rho = 0$ but they are not differentiable at $\rho = 1$ (and consequently have infinite energy). Since such solutions cannot develop from smooth initial data inside the whole past lightcone of the singularity, they are not expected to play any role in the Cauchy problem. The numerical results described below support this expectation. In this respect the wave maps in 2 + 1 dimensions are completely different from the wave maps in 3 + 1 dimensions where a stable analytic self-similar solution determines the process of singularity formation [4].

## 4. Numerical results

We have solved numerically the Cauchy problem (5) and (6) for various one-parameter families of initial data which interpolate between small and large energy. The details of our numerical methods are described in the appendix. The results described below are universal in the sense that they do not depend on the choice of the family of initial data (nor on the topological sector). For concreteness we present them for the degree-zero initial data of the form

$$u(0,r) = A\left(\frac{r}{R}\right)^3 \exp\left[-\left(\frac{r-R}{\delta}\right)^4\right] \qquad u_t(0,r) = 0 \tag{18}$$

where the amplitude $A$, the radius $R$ and the width $\delta$ are free parameters. Below we fix $R = 2$, $\delta = 0.4$ and vary $A$. We emphasize that the amplitude is by no means distinguished: any parameter which controls the energy of the initial data could be varied. Note that regularity at the centre requires that $u(t, 0) = 0$ for all $t < T$, where $T$ is the time when the first singularity (if there is any) develops at $r = 0$. Since the initial momentum is zero, the initial profile splits into two waves, ingoing and outgoing, travelling with approximately unit speed. The

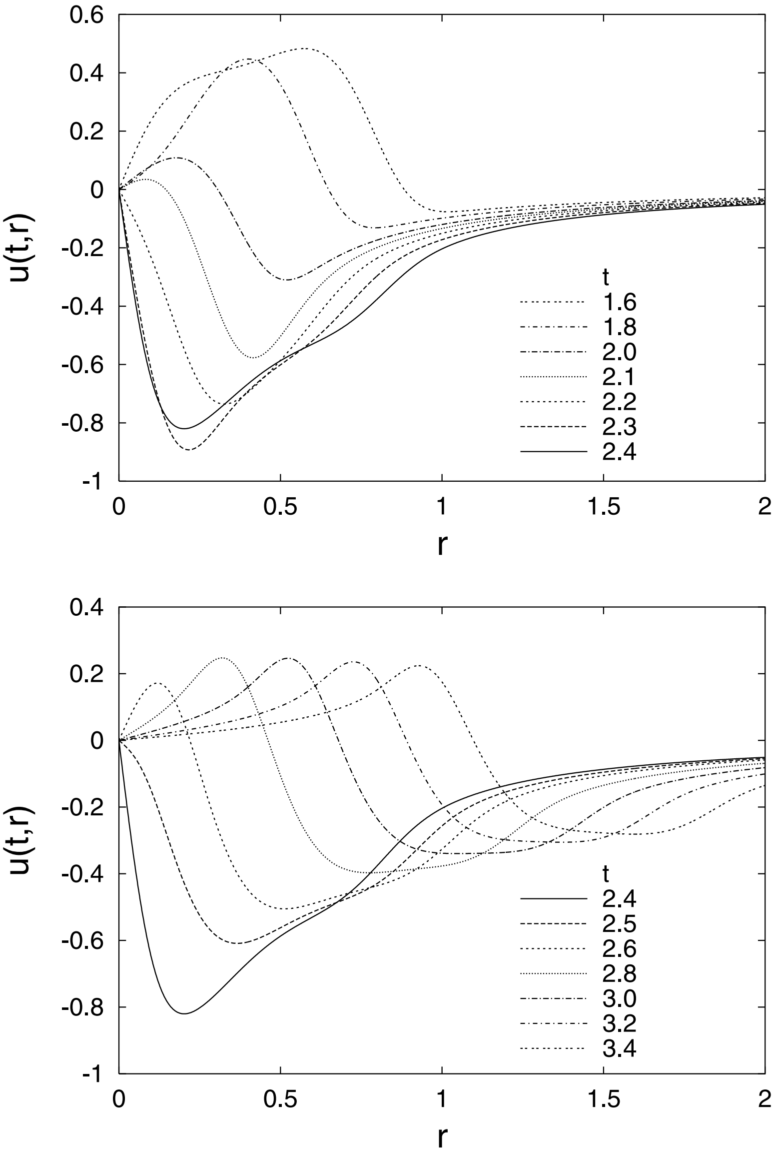


**Figure 1.** Snapshots of the evolution of initial data (6) with small amplitude $A = 0.5$. The ingoing wave bounces back and disperses. The minimal radius is attained at $t \approx 2.4$.

evolution of the outgoing wave has nothing to do with singularity formation so we shall ignore it in what follows. The behaviour of the ingoing wave depends on the amplitude $A$. For small amplitudes, the ingoing wave shrinks, reaches a minimal radius, and then expands to infinity leaving behind the zero-energy region (see figure 1). When $A$ increases, the minimal radius at which the wave bounces back decreases and seems to go to zero for some critical value of the amplitude $A^{\star}$. Finally, for the supercritical amplitudes $A > A^{\star}$, the wave does not bounce back and keeps shrinking to zero size in finite time. More precisely, we observe that evolution of

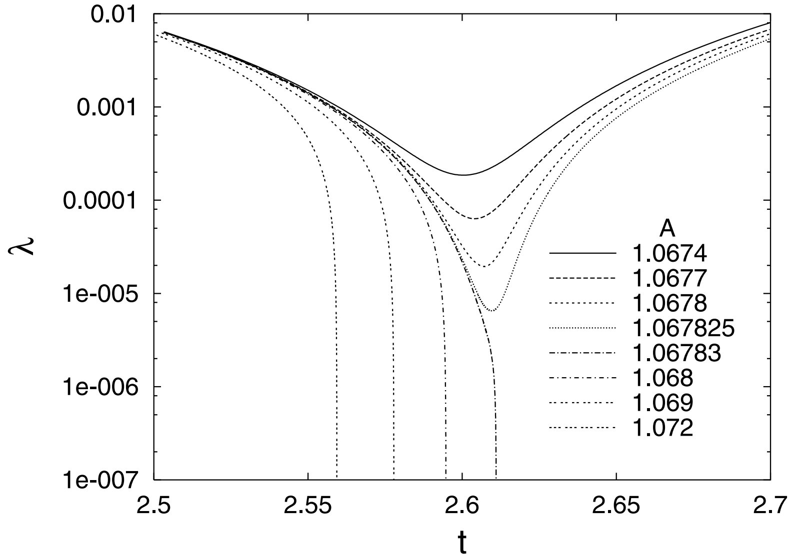


**Figure 2.** The scale factor $\lambda(t)$ for a sequence of solutions with nearly critical initial amplitudes. Numerically, $\lambda(t)$ was calculated from the formula $u_r(t, 0) = -2/\lambda(t)$. The critical amplitude was estimated to be $A^\star \approx 1.067\,8281$.

the wave near the centre (the so-called inner solution) is well approximated by the degree-one static solution $u_S$ with a time-dependent scale factor $\lambda$

$$u(t, r) \approx -2 \arctan\left(\frac{r}{\lambda(t)}\right). \tag{19}$$

We shall refer to this formula as the adiabatic approximation. Using the adiabatic approximation the evolution of the ingoing wave can be described as follows. For subcritical amplitudes $A < A^\star$ the scale factor $\lambda(t)$ decreases, attains a minimum $\lambda_{min}$, and then increases. When $A \to A^\star$, then $\lambda_{min} \to 0$. For supercritical amplitudes $A > A^\star$, the scale factor decreases monotonically to zero in finite time. As follows from (19), $u_r(t, 0) \sim \lambda^{-1}(t)$, hence for the supercritical solutions the gradient blows up at the centre in finite time. Various aspects of this behaviour and some numerical details are shown in figures 2–5.

We have not been able to develop a rigorous mathematical understanding of the adiabatic approximation (19) but we can make some heuristic arguments which help us to understand the observed behaviour. Let us define a similarity variable $\eta = r/\lambda(t)$. Substituting $u = u(t, \eta)$ into equation (5) gives

$$-\lambda^2 \ddot{u} + 2\eta\dot{\lambda}\lambda\dot{u}' + (1 - \dot{\lambda}^2\eta^2)u'' + \left[1 + (\lambda\ddot{\lambda} - 2\dot{\lambda}^2)\eta^2\right]\frac{u'}{\eta} - \frac{\sin(2u)}{2\eta^2} = 0 \tag{20}$$

where $\dot{} = \partial/\partial t$, $' = \partial/\partial\eta$. In order to 'explain' the observed behaviour we make two assumptions. The first assumption, which is the essence of adiabaticity, says that the dynamics of the solution near the centre is slaved in the varying scale $\lambda(t)$. This implies that we can neglect the first two terms in (20) which involve the *explicit* time derivatives of $u$. The second assumption concerns the rate of blow-up and says that

$$\frac{\lambda(t)}{T - t} \to 0 \qquad \text{as} \quad t \nearrow T. \tag{21}$$

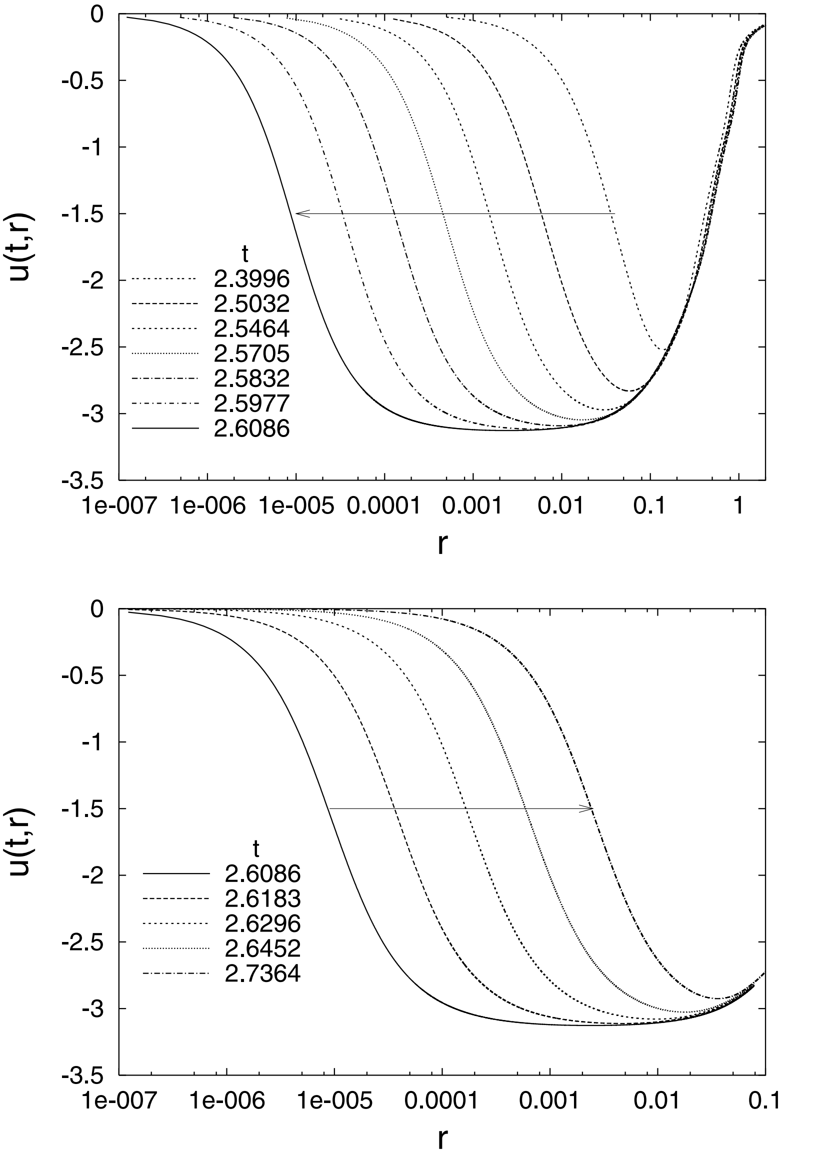


**Figure 3.** Snapshots of the evolution for the marginally subcritical amplitude $A = 1.067\,82$. The upper plot shows the shrinking phase, while the lower plot shows the expanding phase. The horizontal arrows indicate the direction of motion. For sufficiently small $r$ all profiles have the shape of the suitably rescaled static solution. The scale factor attains the minimum $\lambda_{min} \approx 0.9308 \times 10^{-5}$ for $t \approx 2.6086$. Note that $u(t, r) > -\pi$ for all times.

Equation (21) implies that the terms involving the time derivatives of $\lambda$ in equation (20) tend to zero as $t \nearrow T$. After dropping these terms and the first two terms, equation (20) becomes formally the same as equation (8) and therefore is solved by $u_S(\eta)$. Although this *ad hoc* explanation is certainly not satisfactory, it is fully consistent with the numerics. In particular, it explains why the 'amount' of the inner solution which is approximated by (19) increases as the wave shrinks (see figure 5).

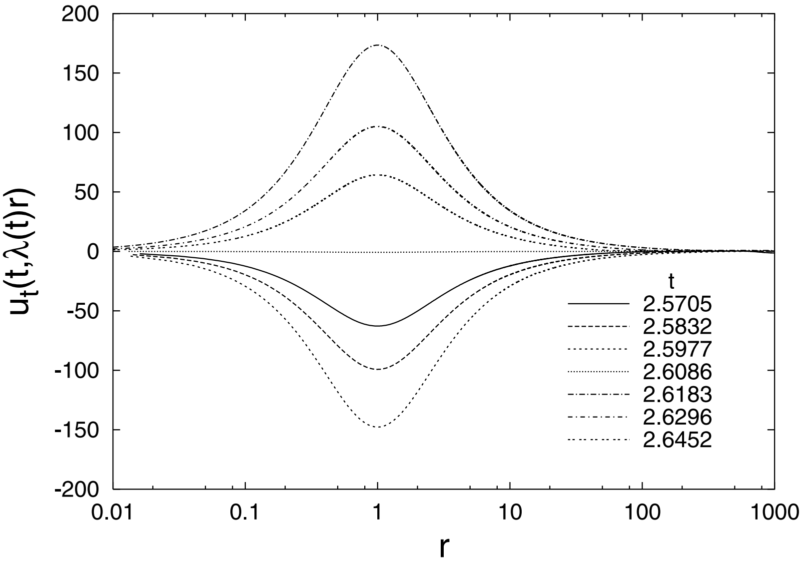


**Figure 4.** The momenta for the same data as in figure 3, rescaled by the factors $\lambda(t)$. In agreement with the adiabatic approximation (19) the profiles have the shape of the zero mode $v_0(r)$ with the amplitudes given by the logarithmic derivative of the scale factor.

Of course, it would be very interesting to find the exact rate of blow-up. We obtain a reasonable fit to the power-law behaviour

$$\lambda(t) \sim (T-t)^{\alpha} \qquad \text{as} \quad t \nearrow T \tag{22}$$

with the anomalous exponent $\alpha \approx 1.1 \pm 0.05$ (see figure 6). However, in view of the limited resolution of our numerics near the blow-up and the lack of theoretical arguments behind (22) we caution the reader not to take (22) as a serious prediction; in particular we cannot rule out logarithmic corrections to the power-law behaviour.

Now, on the basis of the numerical studies just described we would like to put forward three conjectures which summarize the main points of our findings.

**Conjecture 1 (On blow-up for large data).** *For initial data (6) with sufficiently large energy, the solutions of equation (5) blow up in finite time in the sense that the derivative* $u_r(t,0)$ *diverges as* $t \nearrow T$ *for some* $T > 0$.

**Conjecture 2 (On blow-up profile).** *Suppose that the solution* $u(t,r)$ *of the initial-value problem (5) and (6) blows up at some time* $T > 0$. *Then, there exists a positive function* $\lambda(t) \searrow 0$ *for* $t \nearrow T$ *such that*

$$\lim_{t \nearrow T} u(t, \lambda(t) r) = \pm u_S(r) \qquad \textit{for} \quad r > 0. \tag{23}$$

**Conjecture 3 (On energy concentration).** *Suppose that the solution* $u(t,r)$ *of the initial-value problem (5) and (6) blows up at some time* $T > 0$. *Define the kinetic and the potential energies at time* $t < T$ *inside the past lightcone of the singularity by*

$$E_K(t) = \pi \int_0^{T-t} u_t^2 r \, \mathrm{d}r \qquad \textit{and} \qquad E_P(t) = \pi \int_0^{T-t} \left( u_r^2 + \frac{\sin^2 u}{r^2} \right) r \, \mathrm{d}r. \tag{24}$$

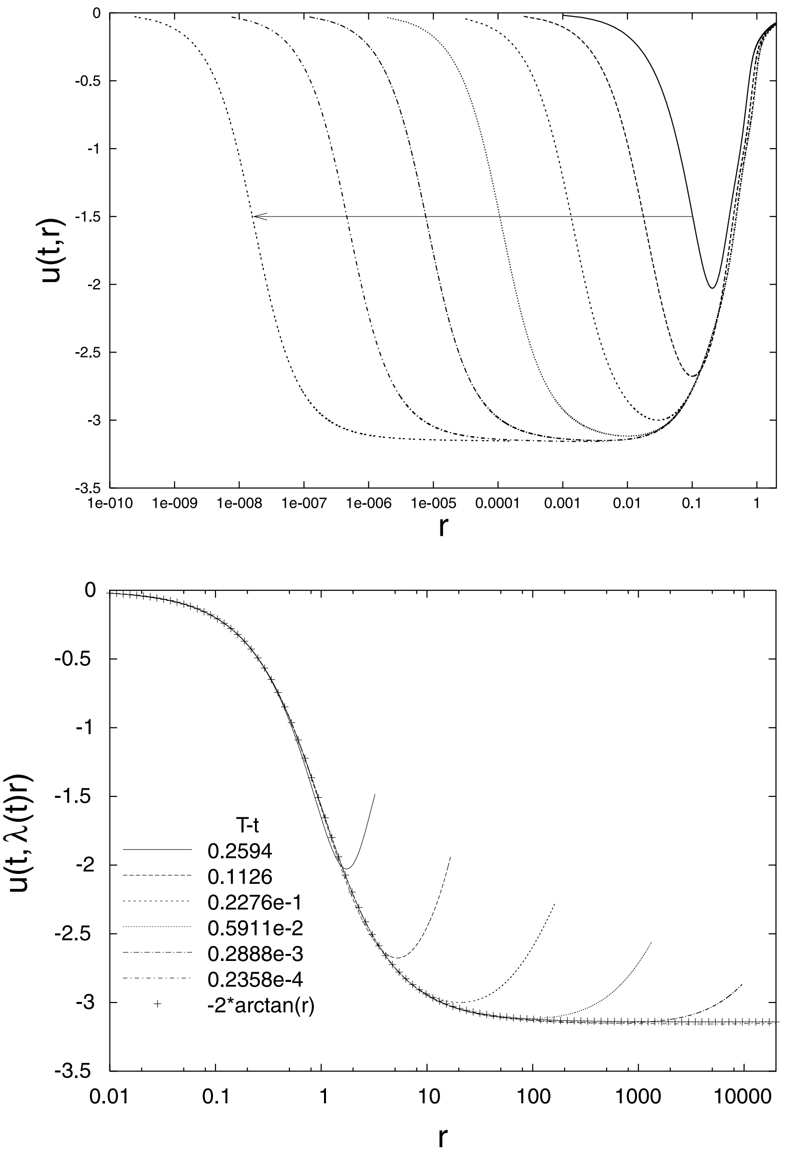


**Figure 5.** Snapshots of the supercritical evolution for the amplitude $A = 1.072$. The upper plot shows the ingoing wave shrinking indefinitely as $t$ approaches the blow-up time $T \approx 2.559\,3558$. In the lower plot the profiles from the upper plot, rescaled by the factors $\lambda(t)$, are shown to collapse to the static profile $-u_S(r)$. Note that the fifth profile, corresponding to $T - t = 0.000\,2888$, overshoots $-\pi$; this seems to be the necessary and sufficient condition for blow-up, which is reminiscent of a similar phenomenon for the heat flow for harmonic maps [9].

*Then:*

*(a) the kinetic energy tends to zero at the singularity*

$$\lim_{t \nearrow T} E_K(t) = 0 \tag{25}$$

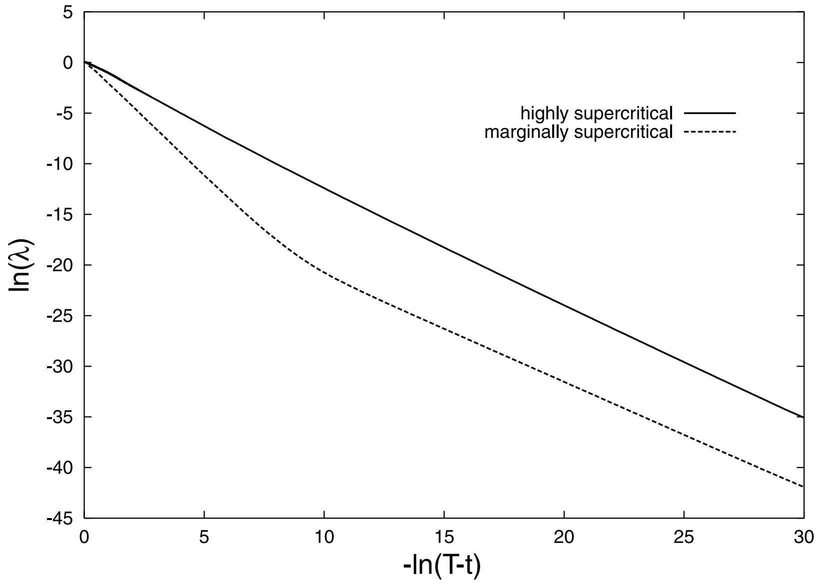


**Figure 6.** The scale factor from two supercritical evolutions. In both cases the asymptotic behaviour of $\lambda(t)$ is well approximated by the power law $\lambda \sim (T-t)^{\alpha}$ with the exponent $\alpha \approx 1.1$. However, in the marginally supercritical case the transient regime with $\alpha \approx 2.3$ in clearly seen before the true asymptotic state is reached. As $A \to A^{\star}$, the crossover between the transient and the asymptotic regimes occurs closer and closer to the blow-up, which suggests that solutions with exactly critical initial data blow up at the much slower 'transient-regime' rate.

*(b) the potential energy equal to the energy of the static solution $u_S$ concentrates at the singularity*

$$\lim_{t\nearrow T} E_P(t) = E[u_S] = 4\pi. \tag{26}$$

We have already discussed the evidence for conjectures 1 and 2. Conjecture 3 is basically a consequence of (19) and (21). To see this let us substitute $u_S(r/\lambda(t))$ into (24) to obtain

$$E_K(t) = \pi\dot{\lambda}^2 \int_0^{(T-t)/\lambda(t)} {u_S'}^2 r^3\,\mathrm{d}r \qquad E_P(t) = \pi \int_0^{(T-t)/\lambda(t)} \left({u_S'}^2 + \frac{\sin^2 u_S}{r^2}\right) r\,\mathrm{d}r. \tag{27}$$

Assuming (21), the upper limits in these integrals tend to infinity as $t \nearrow T$, so (25) and (26) follow (note that the integral in $E_K$ diverges logarithmically). Conjecture 3 means that as the blow-up is approached the excess energy above the energy of the static solution flows outward from the inner region. This is clearly seen in our simulations.

We note that an averaged weak version of (25)

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{T-\epsilon}^{T} E_K(t)\,\mathrm{d}t = 0 \tag{28}$$

was proved by Shatah and Tahviladar-Zadeh [5].

## 5. Final remarks

We would like to comment on two papers by Piette and Zakrzewski [6] and by Linhart [7] which were devoted to the adiabatic evolution in the degree-one topological sector. These authors considered initial data which have the shape of the static solution and a non-zero momentum directed inwards. They observed adiabatic shrinking (19) with the scale factor $\lambda$ changing approximately linearly in time. This was 'explained' by the geodesic approximation (an old idea due to Manton [8]) as follows. Substituting the ansatz (19) into the action one obtains the effective action for the scale factor $\lambda(t)$. The potential energy part does not depend on $\lambda$, so only the kinetic energy part contributes to the effective action,

$$S_{eff}[\lambda] = \int \mathrm{d}t \int_0^\infty u_t^2 r \, \mathrm{d}r \sim \int \mathrm{d}t \, \dot{\lambda}^2 \int_0^\infty \frac{r^3 \, \mathrm{d}r}{(\lambda^2 + r^2)^2} \sim c \int \dot{\lambda}^2 \, \mathrm{d}t \tag{29}$$

where the 'constant' $c$ is logarithmically divergent[4]. As long as the geodesic approximation is used to model blow-up, this divergence is irrelevant because it can be removed by truncating the action at some large radius. By finite speed of propagation, such a truncation cannot affect the blow-up. Thus, the effective action (29) yields the scale factor going to zero linearly $\lambda \sim T - t$ as $t \nearrow T$. To verify the accuracy of this approximation, Piette and Zakrzewski have solved the Cauchy problem numerically and obtained $\lambda \sim (T - t)^{1+\epsilon}$, where $\epsilon$ is a positive number of the order of 0.1. The smallness of $\epsilon$ was interpreted in favour of the geodesic approximation. Although we have confirmed these results numerically (see the discussion above and figure 6), we disagree with the authors of [6,7] regarding the accuracy of the geodesic approximation. As we wrote above, the assumption (21) is crucial for the validity of the adiabatic approximation. In contrast, the linear decay of $\lambda$, predicted by the geodesic approximation, is inconsistent with the observed adiabatic evolution, and, as follows from (27), gives the wrong prediction on energy concentration at the blow-up which contradicts conjecture 3 and theorem (28). The reason why the geodesic approximation fails to capture these crucial features of blow-up is easy to understand: this approximation completely neglects radiation which is essential in expelling the excess energy from the inner region.

Besides the obvious problem of proving conjectures 1–3, the research presented here raises a number of questions. Most interesting among them, in our opinion, are:

(a) What mechanism selects the time evolution of the scale factor $\lambda$? In answering this question the methods of centre manifold theory might be useful. In particular, using weakly nonlinear stability analysis it should be possible to derive the amplitude equations for the nonlinear evolution of the zero mode. The problem shares many features with the problem of blow-up for the nonlinear Schrödinger equation in two spatial dimension. It is feasible that the techniques of asymptotic matching used there [10] could also be applied to our problem.

(b) What is the evolution at the threshold for singularity formation? What does the fine-tuning accomplish dynamically? The model does not fit into the framework of 'standard' threshold behaviour where a codimension-one stable manifold of a certain critical solution separates blow-up from dispersion [11]. Unless more accurate fine-tuning were to reveal a new universal behaviour very close to the threshold (which is unlikely), it seems that solutions evolving from exactly critical initial data also blow up in the adiabatic manner but at a much slower rate (see figure 6).

(c) To what extent are the results specific to the equivariance ansatz? Is the blow-up stable under general perturbations?

We hope to be able to say more about these problems in future publications.

[4] In the language of the geodesic approximation this divergence means that the volume of the moduli space is infinite, which *nota bene* is equivalent to the fact that the zero mode is not square integrable. In the literature one can find statements that zero modes which are not square integrable are 'frozen' by infinite inertia. Papers [6, 7] demonstrate that these statements, based on the naive picture of (29) as the action for the free particle with infinite mass, are wrong.

**Acknowledgments**

PB thanks Shadi Tahvildar-Zadeh for helpful remarks. We also thank Jim Isenberg and Steve Liebling for discussions and for informing us that their independent studies of the problem are in agreement with our results. When this paper was almost finished, we received the preprint 'Equivariant wave maps in two space dimensions' by Michael Struwe in which a version of conjecture 2 is proved. We are grateful to Michael Struwe for letting us know about his result. This research was supported in part by the KBN grant 2 P03B 010 16.

## Appendix. Numerical methods

In order to solve equation (5) numerically we rewrite it as the first-order system in time:

$$u_t = v \tag{A1$a$}$$

$$v_t = u_{rr} + \frac{1}{r}u_r - \frac{\sin(2u)}{2r^2}. \tag{A1$b$}$$

We solve this system by finite differencing. To ensure regularity at the origin we require that $u(r,t) = \mathrm{O}(r)$ for $r \to 0$, from which the inner boundary conditions follow: $u(0,t) = 0$, $v(0,t) = 0$. As the outer boundary condition we impose an approximate outgoing wave condition. A naive centred-difference scheme applied to the right-hand side of (A1$b$) would trigger an instability near $r = 0$. To avoid this, we use a scheme which is natural for the operator

$$\mathcal{L}u = u_{rr} + \frac{1}{r}u_r = \frac{1}{r}\partial_r(r\partial_r u) \tag{A2}$$

and takes the form

$$\mathcal{L}u \approx \frac{1}{r}\left[\frac{1}{h}\left(\left(r+\frac{h}{2}\right)\frac{u(r+h)-u(r)}{h} - \left(r-\frac{h}{2}\right)\frac{u(r)-u(r-h)}{h}\right)\right] \tag{A3}$$

where $h$ denotes the spatial mesh size.

For time evolution we use a standard leapfrog scheme. In that way we obtain a scheme which is second-order accurate in space and time.

Preliminary results obtained on a uniform, fixed grid show that the most interesting inner solution is well approximated by the static solution $u_S$ with a time-dependent scale factor $\lambda$ (see equation (19)). Therefore, to follow this solution we have to change the scale of spatial resolution in time and keep it roughly proportional to $\lambda$, at least as long as $\lambda$ decreases. To this end we apply an adaptive algorithm in which both the mesh size and the time step are refined as the solution shrinks. We start with a uniform grid covering an interval $(0, R)$ with some initial resolution characterized by a mesh size $h = \Delta r_0$. We continue the evolution of the system on this grid with a time step $\Delta t_0$ as long as $u_r(0,t)h \leqslant C$, where $C$ is some small fixed constant (spatial tolerance factor). When this inequality is violated we refine the original grid on the interval $(0, R/2)$ by covering it with the resolution $\Delta r_1 = \Delta r_0/2$. The values of functions at the new points, not defined on the parent grid, are obtained by interpolation. From that time on we continue the evolution on the finer grid with the time step $\Delta t_1 = \Delta t_0/2$. Iterating this process several times we obtain the resolution adaptively adjusted to the solution.

In order to make sure that the numerical results are reliable, we have reproduced them using a different implicit finite-differencing scheme in which we have used $\ln(r)$ as the spatial variable.

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