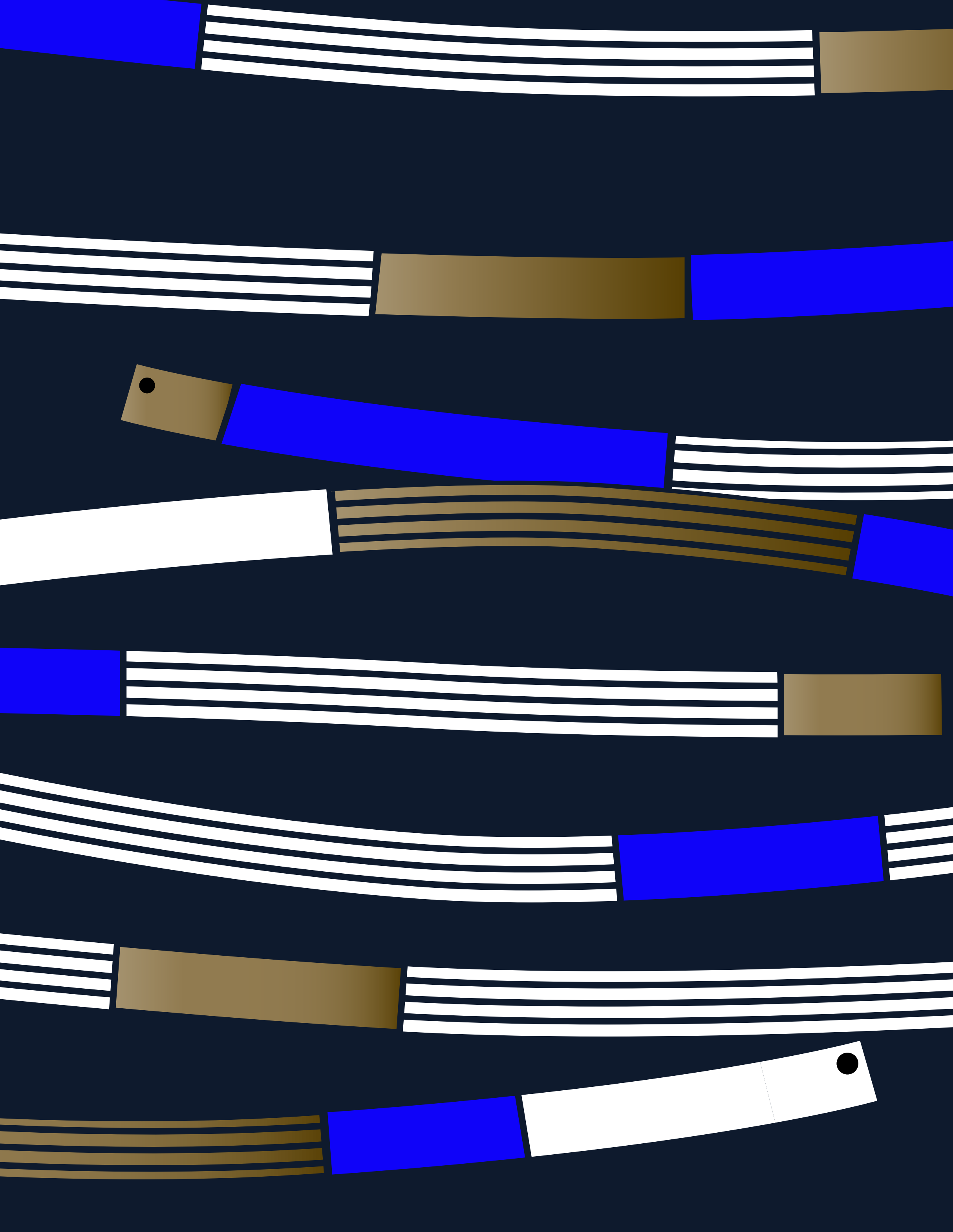


1) Avant-propos : Instantons

Yang-Mills Functional :

Let $F(A) \in \Omega^2(M, \mathrm{ad}\, P)$

$$L(A) = \| F_A \|_{L^2}^2 = \int_M F_A \wedge * F_A \, d\mathrm{vol}_g \qquad (*)$$

A is stationary point of $(*)$ iff

$$\begin{cases} d_A * F = 0 \\ d_A F = 0 \qquad \text{(Bianchi)} \end{cases}$$

Let (M, g) be dim 4 oriented Riemannian manifold. Then $*^2 = id$ and one has the decomposition

$$\Omega^2(M) = \Omega^2_+(M) \oplus_\perp \Omega^2_-(M)$$

eigenspace : $\omega \in \Omega^2_\pm$, $*\omega = \pm \omega$

$\Omega^2_+ = \{\text{Self dual space}\}$

$\Omega^2_- = \{ \text{anti self dual} \}$

$\omega \in \Omega^2(M)$ can be decompose

$$\omega = \omega_+ + \omega_-$$

By orth:

$$\langle \omega_+, \omega_- \rangle = \langle \omega_-, \omega_+ \rangle = 0$$

We can apply it to $F_A \in \Omega^2(M; \operatorname{ad} P)$

$$F_A = F_A^+ + F_A^-$$

Then $L(A) = \| F_A^+ \|^2 + \| F_A^- \|^2$

If $F_A = F_A^\pm$ we say that the $\underbrace{\text{gauge field}}_{\text{connection } A.}$ describes an (anti)- instantons

Notice that if $F_A = F_A^\pm$

Then $d_A * F_A = d_A F_A^\pm = 0$ (Bianchi identity).

Hence instantons are solutions to (YM)

Notice that $F_A^{\pm} = 0$ is a first order PDE, where $d_A * F_A = 0$ is 2^{nd} order.

ADHM construction:

Recall that the quaternionic Stiefel bundle

$$\pi^c : S_{\mathbb{H}}(1, k+1) \simeq S^{4k+3} \longrightarrow G_{\mathbb{H}}(1, k+1) \simeq \mathbb{H}P^k$$

is k-classifying for the principal $Sp(1)$-bundle $P \longrightarrow S^4 \simeq \mathbb{H}P^1$

Take the canonical $Sp(1)$-connection

$$\omega^c = q^+ dq$$

on $S_{\mathbb{H}}(1, k+1)$ and pull it back via a family of classifying mappings

$$f : S^4 \longrightarrow \mathbb{H}P^k$$

i.e s.t $P \cong f^* S_{\mathbb{H}}(1, k+1)$

$$\omega = f^* \omega^c$$

If this family is suitable then this yields to a family of (anti) self dual $Sp(1)$ connections

$$
\begin{tikzcd}
P \arrow[r, "F"] \arrow[d, "\pi"'] & S^{4+k} \arrow[d, "\pi^{C}"] \\
S^4 \arrow[r, "f"'] & \mathbb{H}P^k
\end{tikzcd}
$$

<u>Quaternionic ADHM data</u> : (1)

$$v : \mathbb{H}^2 \to \mathcal{L}(\mathbb{H}^k, \mathbb{H}^{k+1})$$

$$v(x_1, x_2) = C x_1 + D x_2$$

where $C, D \in M_{k+1,k}(\mathbb{H})$

satisfying :

a) $\operatorname{rank}_{\mathbb{H}} v(x_1, x_2) = k$, $\forall (x_1, x_2) \in \mathbb{H}^2 \setminus \{0\}$

b) $v^{+}(x_1, x_2)\, v(x_1, x_2)$ is real, $\forall (x_1, x_2) \in \mathbb{H}^2$

The mapping v defines a smooth classifying mapping

$$u : \begin{cases} S^4 \simeq \mathbb{HP}^1 \to G_{\mathbb{H}}(1, k+1) \\ u([x_1, x_2]) := \operatorname{im}(v([x_1, x_2]))^{\perp} \end{cases}$$

$$\omega = u^* \omega^c = u^{\dagger} du$$

<u>Complex ADHM data :</u> (2)

Identifying $\mathbb{H}^2 \simeq \mathbb{C}^4$

$$A : \begin{cases} \mathbb{C}^4 \to \mathcal{L}(W, V) \quad W = \mathbb{C}^k \; ; \; V = \mathbb{C}^{2k+2} \\ A(z) = \sum_{i=1}^{4} A_i z_i \end{cases}$$

V is endowed with its standard scalar product h and a skew form

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$h(z, w) = z^{\dagger} w \qquad J(z, w) = z^{t} J w$$

We use the identification

$$\mathbb{H}^k \simeq \mathbb{C}^k \oplus j\,\mathbb{C}^k \simeq \mathbb{C}^{2k}$$

$$z_1 + j z_2 = (z_1, z_2)$$

Let $\sigma : \mathbb{H}^k \to \mathbb{H}^k$ be the anti-linear isomorphism defined by right mult j.

$$\sigma^2 = -\,id$$

$$\sigma(z_1 + j z_2) = -\bar{z}_2 + j\bar{z}_1 = (-\bar{z}_2, \bar{z}_1)$$

Viewing $z \in \mathbb{C}^{2k}$:

$$\sigma(z) = \bar{z}\, J$$

Compatibility between (h, J, σ)

$$h(\sigma(z), w) = J(z, w).$$

2) The ADHM construction:

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\text{Quaternionic ADHM data} \arrow[rr, leftrightarrow, dashed, "\text{ADHM construction}"] \arrow[d, leftrightarrow, "\text{lemma}"] & & \text{$(8k-3)$-parameter family of instantons on } S^4 \arrow[d, leftrightarrow, "?"] \\
\text{Complex ADHM data} \arrow[d, "\text{Horrocks construction}"] & & \text{All instanton } S^4 \arrow[d, leftrightarrow, dashed, "\text{Atiyah Ward correspondence}"] \\
\text{Algebraic Vector bundle over } \mathbb{C}P^3 \text{ of a certain type.} \arrow[u, dashed, bend left] \arrow[rr, leftrightarrow, dashed, "\text{Serre correspondance}"'] & & \text{Holomorphic vector bundle over } \mathbb{C}P^3 \text{ of a certain type}
\end{tikzcd}
$$

Lemma :

(1) and (2) are one-to-one if

i) $\sigma(A(z)w) = A(\sigma(z))\bar{w}$ $\quad w \in W$

ii) $\dim_{\mathbb{C}}(\operatorname{im} A(z)) = k$ $\quad z \neq 0$

iii) $A(z)^t J A(z) = 0$

$\hookrightarrow$ isotropic $\quad \operatorname{im} A(z) \subset (\operatorname{im} A(z))^J$

Mapping A satisfying i) + ii) + iii) are called complex ADHM data.

<u>Prop:</u> (Horrocks construction) (instanton bundle)

Any linear map $A : W \to V$ satisfying (2) give rise to a holomorphic vector bundle $\pi : \mathcal{L} \to P_3(\mathbb{C})$ of rk 2 s.t

1) $\mathcal{L}$ is holomorphically trivial over each fiber

2) There exists a holomorphic symplectic involution on $\mathcal{L}$.

<u>Proof:</u>

Recall $(V, \underbrace{\sigma}_{\text{sump}}, \underbrace{J}_{\text{skew form}}, \underbrace{h}_{\text{Hermitian form}})$

$z \in \mathbb{C}^4$

$\mathcal{E}_z := \operatorname{im} A(z)$ $\qquad$ $\mathcal{E}_z^{\circ} := (\operatorname{im} A(z))^{J}$

$\mathcal{L}_z := \mathcal{E}_z^{\circ} / \mathcal{E}_z$

condition 1)

Since $\dim_{\mathbb{C}}(\operatorname{im} A(z)) = k$ then

$$\dim_{\mathbb{C}} (\operatorname{im} A(z))^{J} = k+2 \qquad (2k+2-k)$$

$\Rightarrow \quad \dim_{\mathbb{C}} \mathcal{L}_z = 2$

Since $\mathcal{E}_z, \mathcal{E}_z^{\circ} \hookrightarrow V$, $\mathcal{L}_z$ inherits of a skew form from J + non degenerate by quotient.

$\mathcal{E}_z, \mathcal{E}_z^{\circ}$ and $\mathcal{L}_z^{\circ}$ only depends on $[z] \in \mathbb{C}P^3$ by construction (invariant by homotetly)

Then $\mathcal{E} := \bigcup_{[z]\in\mathbb{C}P^3} \mathcal{E}_{[z]} \qquad \mathcal{E}' := \bigcup_{[z]\in\mathbb{C}P^3} \mathcal{E}'_{[z]}$

vertical subbundle of $\underline{V} = \mathbb{C}P^3 \times TV \simeq \mathbb{C}P^3 \times V$

endowed with h, J.

$$\mathcal{L} \hookrightarrow \mathbb{C}P^3 \times V =: \underline{V}$$

Then $\mathcal{L} := \bigcup \mathcal{L}_{[z]} = \mathcal{E}^{\circ}/\mathcal{E}$

vector bundle

$\mathcal{L}$ can be identified with $\mathcal{E}^{\perp}$ in $\mathcal{E}^{\circ}$.

$\mathcal{L}$ is an algebraic vector bundle thus carry a holomorphic structure.

Since $h(\sigma(z), w) = J(z, w)$

$z \in \mathcal{E}^{\perp}_{[z]}$ $\quad h(z, w) = 0 \ \forall w \in \mathcal{E}_{[z]}$

$\sigma^2 = -1$ $\quad h(z, w) = -h(z, \sigma^2(w))$

$= -J(z, \sigma(w))$

In addition by compatibility $(h, \sigma) = 0$ $\quad \sigma(w) \in \sigma(\mathcal{E}_{[z]})$

$z \in (\sigma(\mathcal{E}_{[z]}))^{\circ}$

$$\mathcal{E}^{\perp}_{[z]} = (\sigma(\mathcal{E}_{[z]}))^{\circ}$$

condition (ii)

and since $\sigma(A(z)w) = A(\sigma(z))\overline{w}$

$$\sigma(\mathcal{E}_{[z]}) = \mathcal{E}_{\sigma[z]}$$

$(\mathcal{E}_{\sigma[z]})^{\circ} = \mathcal{E}^{\perp}_{[z]}$

$\Rightarrow \quad \mathcal{E}^{\circ}_{[z]} = \mathcal{E}^{\perp}_{\sigma[z]}$

$\mathcal{E}^{\circ}_{[z]} \cap \mathcal{E}_{\sigma[z]}$ $\qquad J(v, \underset{\in \operatorname{im} A(\sigma(z))}{w}) = 0 \quad \forall z \in \mathcal{E}_{[z]}$

$A(\sigma(z))w'$

$J(v, A(\sigma(z))w')$

$= J(v, \sigma(A(z)w'))$

$= h(v, \underbrace{A(z)w'}_{\in \mathcal{E}_{[z]}}) = 0 \quad \forall v \in \mathcal{E}_{[z]}$

$\Rightarrow h(A(z)w', A(z)w') = 0$

$\Rightarrow w' = 0.$

(

$\downarrow$

By positive definiteness of h:

$$\mathcal{E}^{\circ}_{[z]} \cap \mathcal{E}_{\sigma[z]} = 0$$

Thus $V = \mathcal{E}_{\sigma([z])} \oplus \mathcal{E}^{\perp}_{\sigma([z])} = \mathcal{E}_{\sigma[z]} \oplus \mathcal{E}^{\circ}_{\sigma[z]}$

Now $\mathcal{E}^{\circ}_{\sigma[z]} = \mathcal{L}_{[z]} \oplus \mathcal{E}_{[z]}$

$V = \mathcal{E}_{[z]} \oplus \mathcal{L}_{[z]} \oplus \mathcal{E}_{\sigma([z])}$

and the corresponding splitting of $\underline{V} = \mathbb{C}P^{3} \times V$

$$\mathcal{L}_{[z]} = \{ v \in V : h(v, w) = 0, \ J(v, w) = 0 \ \forall u \in \operatorname{Im} A([z]) \}$$

i.e $\mathcal{L}_{[z]} = \mathcal{E}^{\perp}_{[z]} \cap \mathcal{E}^{\perp}_{\sigma([z])} = \mathcal{E}^{\circ}_{\sigma[z]} \cap \mathcal{E}^{\circ}_{[z]}$

Let us now recall that the fiber bundle

$$\pi : \begin{cases} \mathbb{C}P^3 \longrightarrow \mathbb{H}P^1 \\ \pi([z_1, z_2, z_3, z_4]) = [(z_1 + j z_3, z_2 + j z_4)] \end{cases}$$

$\mathbb{H}^2 \simeq \mathbb{C}^4$

$\mathbb{H}P^1 \simeq \mathbb{C}P^3$

$\mathbb{C} = \mathrm{span}\{1, i\} \subset \mathbb{H}$

$\mathbb{H}^2 \simeq \mathbb{C}^2 \oplus j\, \mathbb{C}^2$

↳ obtain from

$$\begin{array}{ccc} \mathbb{C}^4 & \simeq & \mathbb{H}^2 \\ \downarrow & & \downarrow \\ \mathbb{C}P^3 & \longrightarrow & \mathbb{H}P^1 \\ [z_1, z_2, z_3, z_4] & \longrightarrow & [(z_1 + j z_3, z_2 + j z_4)] \end{array}$$

Notice that under the identification

$$\mathbb{H} \simeq \mathbb{C}^2$$

$$\pi^{-1}(\underset{\mathbb{H}}{\underset{\uparrow}{[1, x]}}) \simeq \mathbb{C}P^1$$

$x = \zeta + j \xi$

$(\zeta, \xi) \in \mathbb{C}^2$

$= (\lambda \zeta, \lambda \xi)$

$\varphi : \mathbb{C}P^1 \xrightarrow{\simeq} S^2$ $\quad$ $\sigma(z) = \bar{z}\, J$

$(\zeta, \xi) \longmapsto (2 \bar{\zeta} \xi, |\xi|^2 - |\zeta|^2)$

$\pi : \mathbb{C}P^3 \longrightarrow \mathbb{H}P^1$

$\pi^{-1} \simeq \mathbb{C}P^1$

$-\overline{z'} + j \bar{z} = (-\bar{z}', \bar{z})$

$\sigma(\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4) = (-\bar{\lambda} \bar{z}_3, -\bar{\lambda} \bar{z}_4, \bar{\lambda} \bar{z}_1, \bar{\lambda} \bar{z}_2)$

$(= (-\bar{z}_3, \bar{z}_4, \bar{z}_1, z_2)$

$$\begin{tikzcd} \mathbb{C}P^3 \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & \mathbb{C}P^3 \arrow[dl, "\pi"] \\ & \mathbb{H}P^1 & \end{tikzcd}$$

$\rightsquigarrow$ $[z] \in \pi^{-1}([x]) \iff \sigma[z] \in \pi^{-1}([x])$

i.e $\quad \sigma(\pi^{-1}([x])) = \pi^{-1}([x])$

i.e the fibers are projective line preserved by σ.

Recall the isomorphism

$$\mathbb{C}P^1 \xleftrightarrow{\sim} S^2$$

$$(\zeta, \xi) \longmapsto (2\bar{\zeta}\xi, |\zeta|^2 - |\xi|^2)$$

$$\sigma(\zeta, \xi) = (-2\bar{\zeta}\xi, |\xi|^2 - |\zeta|^2)$$

antipodal map.

Projective line invariant by σ are

real lines

We show that

$\mathcal{L}_{[z]}$ depends only on

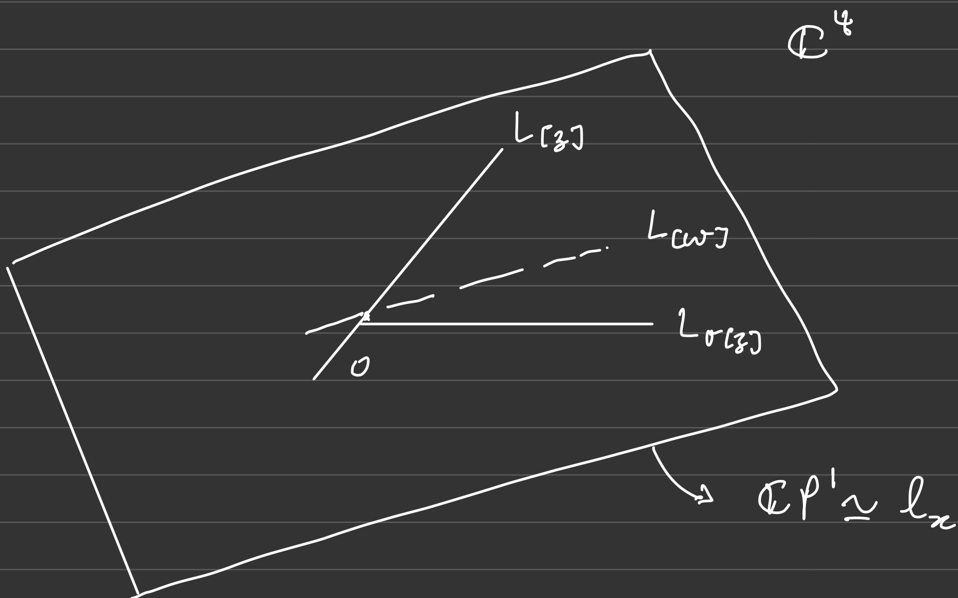
$x = \pi([z]) \in \mathbb{HP}^1$

i.e the fiber through $[z]$

this is a real line l_x through

$([z], \sigma[z])$.

Let $[w]$ be any point in l_x



$L_{[z]}$; $L_{[w]}$, $L_{\sigma[z]}$ are complex lines through 0 in $\mathbb{C}^4$ corresponding to

$[z]$, $[w]$, $\sigma[z]$.

We see that any vector $w \in L_{[w]}$

is a linear comb of $z_1 \in L_{[z]}$ $z_2 \in L_{\sigma[z]}$

$L_{[w]}$ rotates from $L_{[z]}$ to $L_{\sigma[z]}$

when $[w]$ varies from $[z]$ to $\sigma[z]$.

Since $A(w)$ depends linearly on w we get:

$$\mathcal{E}^\circ_{[w]} \cap \mathcal{E}^\circ_{[z]} = \mathcal{E}^\circ_{[w]} \cap \mathcal{E}^\circ_{\sigma[z]} = \mathcal{E}^\circ$$

$$\mathcal{L}_{[z]} \cap \mathcal{L}_{[w]} = \mathcal{E}^\circ_{[z]} \cap \mathcal{E}^\circ_{\sigma([z])} \cap \mathcal{E}^\circ_{[w]} \cap \mathcal{E}^\circ_{\sigma[w]}$$

$$= \mathcal{E}^\circ_{[z]} \cap \mathcal{E}^\circ_{\sigma[z]} \cap \mathcal{E}^\circ_{\sigma[w]}$$

$$= \mathcal{E}^\circ_{[z]} \cap \mathcal{E}^\circ_{\sigma[z]}$$

$$= \mathcal{L}_{[z]}$$

By dim. $\mathcal{L}_{[z]} = \mathcal{L}_{[w]}$.

i.e $\mathcal{L}|_{l_x}$ is trivial with

$$\mathcal{L}|_{\ell_x} = R_x = \mathcal{E}^{\circ}_{[z]} \cap \mathcal{E}^{\circ}_{\sigma[z]} .$$

Antiholomorphic structure σ :

$$\begin{tikzcd} \underline{V} \arrow[r, "\sigma"] \arrow[d] & \underline{V} \arrow[d] \\ \mathbb{C}P^3 \arrow[r, "\sigma"'] & \mathbb{C}P^3 \end{tikzcd}$$

$\sigma : \underline{V} \to \underline{V} \qquad \sigma([z], v) = (\sigma([z]), \sigma(v))$

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\sigma"] \arrow[d] & \mathcal{L} \arrow[d] \\ \underline{V} \arrow[r] & \underline{V} \end{tikzcd}$$

To get a holomorphic symplectic involution on $\mathcal{L}$ one combine with the anti-linear isomorphism $\sigma^* \overline{\mathcal{L}} \simeq \mathcal{L}$.

$\boxed{\diagup}$

Using $\mathcal{L} \hookrightarrow \underline{V}$ one can pullback the canonical connection ω on $\underline{V}$:

$$\tilde{\omega} := \pi^* \omega$$

is (anti) self dual.