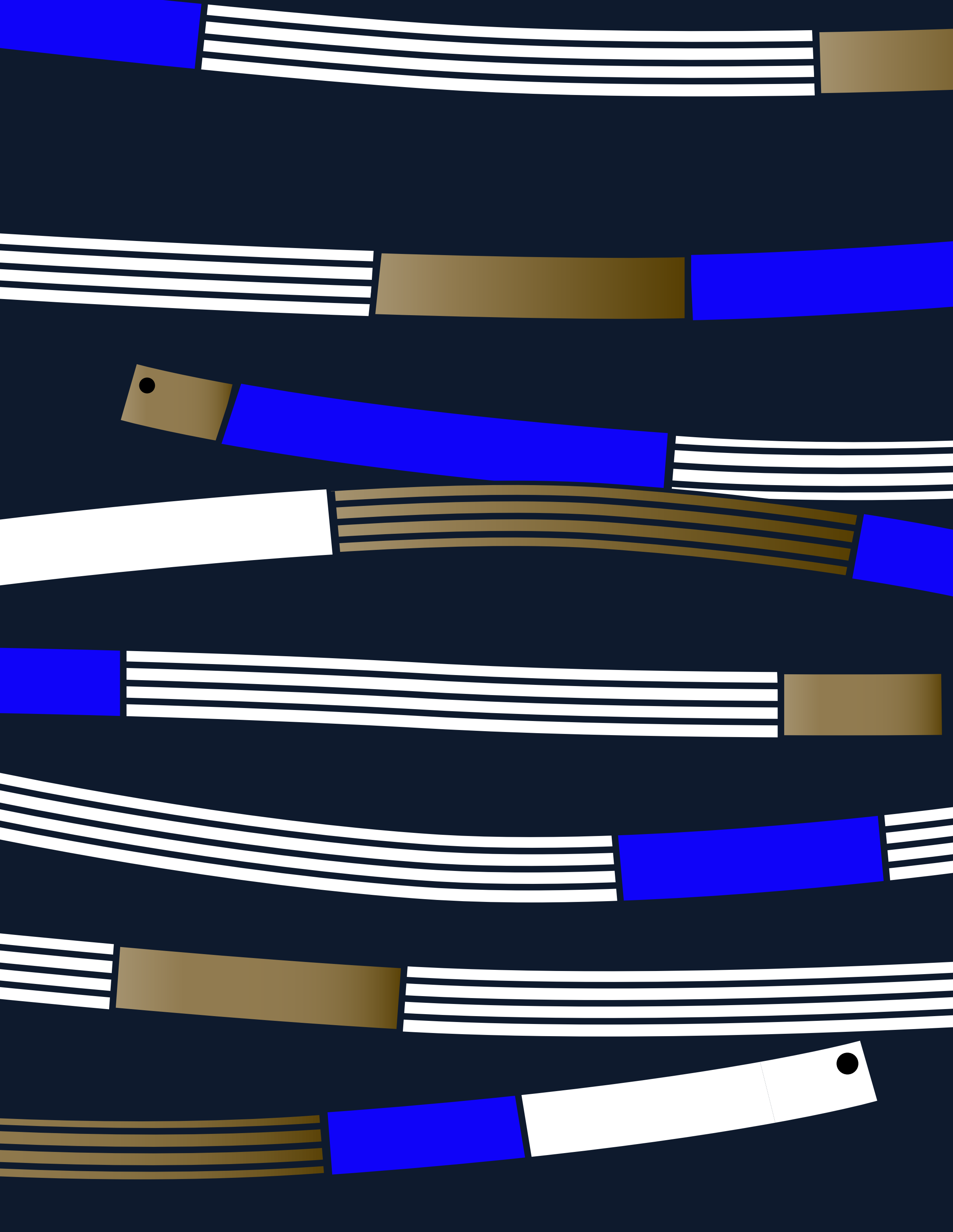

## $\boxed{\text{Yang Mills Functional / Equations :}}$

(Re)cap :

1) Fiber bundle $\pi : E \longrightarrow B$
local trivialization $\{ \Phi_u : \pi^{-1}(U) \xrightarrow{\cong} U \times F \}$

M : compact oriented Riemannian manifold
G : compact connected Lie group ( dim < $\infty$ )

→ Vector bundle $\pi : E \longrightarrow M$
$$\Phi_U : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^n$$

→ Principal bundle $\pi : P \longrightarrow M$
$$\Phi_U : \pi^{-1}(U) \xrightarrow{\cong} U \times G$$

$G \curvearrowright P$ smooth, free ; $\Phi_U$ G - equivariant .

→ Associated bundle : $\pi : P \to M$ $\quad \rho : G \to GL(V)$

$$E := P \times_\rho V = P \times V / \sim$$

$$(p, v) \sim (pg, g^{-1} \cdot v)$$

$$\Phi_U : (U \times G) \times_\rho V \xrightarrow{\cong} U \times V$$

$\therefore \quad ad(P) := P \times_{Ad} \mathfrak{g}$

2) <u>Connection</u> :

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) \quad \in T^*M \otimes \mathrm{End}(E)$$

i) $\nabla_X s$ $\quad \mathcal{F}$-linear in $X$ ; $\mathbb{R}$-linear in $s$

ii) $\nabla_X (f s) = (Xf) s + f \nabla_X s$

local op : $\quad \nabla^U : \mathfrak{X}(U) \times \Gamma(U, E) \to \Gamma(U, E)$

$e : U \to Fr(E)|_U \qquad \{ \omega_e , (U, e) \} \longleftrightarrow \nabla$

3) <u>Connection on $\pi : P \to M$</u> :

$\mathcal{H}$ : smooth right invariant horizontal distribution

$$0 \to \mathcal{H}_p \to T_p P \xrightarrow{\omega_p} V_p \simeq \mathfrak{g}$$

$$T_p P = V_p \oplus \mathcal{H}_p \; ; \qquad \mathcal{H}_p = \ker \omega_p .$$

i) $\omega \in \Omega^1(P, \mathfrak{g})$ smooth

ii) $R_g^* \omega = (\mathrm{Ad}\, g^{-1}) \omega \qquad$ ($G$-equivariant map)

iii) $\omega|_{V_p} = \mathbb{1}_{V_p} \qquad (\omega(\underline{X}) = X)$

$\{ \mathcal{H}$ invariant split $\} \longleftrightarrow \{$ connection $\omega$ on $P \}$

For $\nabla$ on $E \to M \quad \rightsquigarrow \quad \omega$ on $Fr(E) \to M$

$$\omega_e = e^* \omega \quad \text{for } e : U \to Fr(E)|_U .$$

## I) Structure of $\mathcal{A}(P)$:

$\omega, \omega'$ connection 1-forms associated to connections $\mathcal{H}, \mathcal{H}'$

Let $\eta = \omega - \omega'$

Then $r_g^* \eta = (\mathrm{Ad}\, g^{-1})\, \eta$

i.e $\eta$ is $G$ - equivariant

And $\eta(\mathcal{V}) = 0$ i.e $\eta$ is horizontal

Let $\Omega^k_{Ad}(P, \mathfrak{g}) = \{$ horizontal r-inv. $k$-forms $\}$ (tensorial of type Ad.)

The isomorphisms $f_p : \mathfrak{g} \xrightarrow{\simeq} (\mathrm{ad}\, P)_x$ induce an isomorphism:

$$\Omega^k_{Ad}(P, \mathfrak{g}) \xleftrightarrow{\simeq} \Omega^k(M, \mathrm{ad}\, P) \quad (*)$$

$\therefore \quad \eta \in \Omega^1(M; \mathrm{ad}\, P)$

If we denote by

$$\mathcal{A}(P) := \{ \text{connections on } \pi : P \to M \}$$

Then $\mathcal{A}(P)$ is an affine space

and $T_A \mathcal{A}(P) = \Omega^1(M; \operatorname{ad} P)$

at a point $A \in \mathcal{A}(P)$.

Curvature:

Let $h_p : T_p P \simeq V_p \oplus \mathcal{H}_p \longrightarrow \mathcal{H}_p \qquad p \in P$

Consider the dual map:

$$h^* : T^*P \longrightarrow T^*P \qquad (h^*\alpha = \alpha \circ h)$$

then $\Omega := h^* d\omega \in \Omega^2(P, \mathfrak{g})$

$$\Omega(X, Y) = d\omega(hX, hY) = -\omega[hX, hY]$$

1) $\Omega(X, Y) = 0 \qquad$ for $X, Y \in \mathcal{V}$

2) $r_g^* \Omega = (\operatorname{Ad} g^{-1}) \Omega$

i.e $\quad \Omega \in \Omega^2(M; \operatorname{ad} P) \quad$ by $(*)$

Notation:

$A \in \mathcal{A}(P)$ connection

$F(A)$ or $F_A$ associated curvature.

## II) Yang-Mills functional:

### 1) Structure of $\Omega^*(M; \mathrm{ad}\, P)$

$\Omega^*(M; \mathrm{ad}(P))$ as a graded Lie algebra

structure : $\Omega^p \otimes \Omega^q \longrightarrow \Omega^{p+q}$

$$[\omega^p, \omega^q] = (-1)^{pq+1} [\omega^q, \omega^p]$$

Since $G$ is compact it admits

a positive definite $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ Ad-invariant

$$\langle \mathrm{Ad}(g) u, \mathrm{Ad}(g) v \rangle = \langle u, v \rangle$$

i) $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$

ii) $[u, v] \wedge w = u \wedge [v, w]$

2) <u>Hodge operation:</u>

$(M, g)$ oriented Riemannian manifold

Let $dvol_g$ the volume forme induced by $g$

Hodge star op: unique linear map s.t

$$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$$

s.t $\qquad \alpha \wedge * \beta = \langle \alpha, \beta \rangle \, dvol_g$

where $\langle \cdot, \cdot \rangle$ is naturally induced by $g$.

on $\Omega^k(M) \otimes \Omega^k(M) \longrightarrow \Omega^0(M)$.

3) <u>Yang-Mills functional</u>

Combining 1) and 2) we have a natural inner product on $\Omega^*(M; ad(P))$

$$(\theta, \varphi) := \int_M \theta \wedge * \varphi$$

Now let $F(A) \in \Omega^2(M; ad(P))$

curvature

$$L(A) = \| F(A) \|^2_{L^2} = \int_M F(A) \wedge * F(A)$$

## III) Yang-Mills equations ($\pi : P \to M$ , $A$)

### 1) Covariant derivatives

$\varphi \in \Omega^k(P, \mathfrak{g})$

$D\varphi := h\, d\varphi$

In particular it restrict to $\Omega^k_{Ad}(P, \mathfrak{g})$

$$D : \Omega^k_{Ad}(P, \mathfrak{g}) \to \Omega^{k+1}_{Ad}(P, \mathfrak{g})$$

$$\therefore \quad d_A : \Omega^k(M, \mathrm{ad}\, P) \to \Omega^{k+1}(M, \mathrm{ad}\, P)$$

i) $D\varphi = d\varphi + [\omega, \varphi]$

ii) $D\Omega = 0$ (Bianchi)

### 2) Variations of $L(A)$:

Since $\mathcal{A}(P)$ is an affine space we vary on lines:

$$A_t = A + t\eta \qquad \eta \in \Omega^1(M; \mathrm{ad}\, P)$$

Lemma :

$$F(A_t) = F(A) + t\, d_A \eta + \frac{1}{2} t^2 [\eta, \eta]$$

proof : locally

$$F(A_t) = d(A + t\eta) + \frac{1}{2}[A + t\eta, A + t\eta]$$

$$= F(A) + t(d\eta + [A, \tau]) + \frac{1}{2} t^2 [\eta, \eta]$$

$$= F(A) + t\, d_A \eta + \frac{1}{2} t^2 [\eta, \eta]$$

$\boxtimes$

Prop: $A$ is stationary for $L(A) = \| F(A) \|^2$

iff $\quad d_A{}^* F(A) = 0$

proof : Using lemma

$$\| F_t \|^2 = \| F \|^2 + 2t (d_A \eta, F) +$$

$$t^2 \{ \| d_A \eta \|^2 + (F, [\eta, \eta]) \} + \cdots$$

At extremum $\quad (d_A \eta, F) = 0 \quad \forall \eta$

$\therefore \quad (\eta, d_A^* F) = 0 \quad \forall \eta$

$\therefore \quad d_A^* F = 0$

Recall: $\quad d_A^* = \pm * d_A *$

$\therefore \quad d_A * F = 0$ $\quad \square$

Recap: $A$ stationary:

$$\begin{cases} d_A * F(A) = 0 & \text{nonlinear!} \\ d_A F(A) = 0 & \text{(Bianchi} \end{cases}$$

i.e $A$ extremal iff $F(A)$ is harmonic.

## IV) Gauge invariance:

**Def:** (Gauge transformation)

$\Phi : P \to P$ $G$-equivariant diffeo.

$$\begin{tikzcd} P \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

$\mathcal{G}(P) = \{ \Phi : P \to P \}$ gauge group.

$\mathcal{G} \circlearrowright \mathcal{A}$ as follow:

$$H^{\Phi} := \Phi_* H$$

It turns out that $H^{\Phi} \in \mathcal{A}$.

Now our functional $\mathcal{L}$ is invariant under $\mathcal{G}_{\mathbb{R}}$!

Therefore if we define:

$$\mathcal{A}_{YM} = \{ \text{Yang Mills connection} \}$$

then $\mathcal{A}_{YM} / \mathcal{G}$ is the space of classical solutions.