Noncommutative Differential Geometry on Infinitesimal Spaces

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Outline

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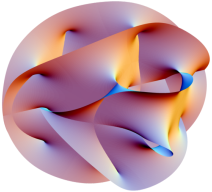
Introduction



(a) C^∞-manifold M

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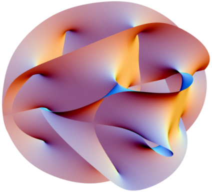

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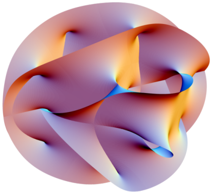

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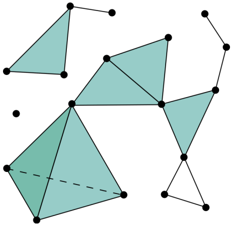

(b) Space X (posets)

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Spin Geometry

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$$\text{Dirac operator} \qquad D = \mathrm{c} \circ g^{-1} \circ \nabla$$

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$$\langle \psi, \phi \rangle = \int_M \langle \psi(x), \phi(x) \rangle_x \, dvol_g$$

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$\mathcal{C}^\infty(M)$ acting as bounded operators on $\mathcal{H}$.
For $f \in \mathcal{C}^\infty(M)$, we have the commutator $[D, f]\,\psi = -i\mathrm{c}(df)\psi$ as an operator in $B(\mathcal{H})$.

Consider the triple $\mathcal{A} = \mathcal{C}^\infty(M)$, $D = \partial\!\!\!/_M$, $\mathcal{H} = (L^2(M, \mathscr{S}), \pi)$.

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Consider the triple $\mathcal{A} = \mathcal{C}^\infty(M)$, $D = \not{\partial}_M$, $\mathcal{H} = (L^2(M, \mathscr{S}), \pi)$.

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Then one can define the *universal graded differential algebra* as follow:

$$\Omega^1(A) = \ker(\mu), \quad da := 1 \otimes a - a \otimes 1,$$

$$\Omega^k(A) = \{a_1 da_2 da_3 \cdots da_k, a_i \in A\}, \quad \Omega^*(A) = \bigoplus_k \Omega^k(A).$$

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If in addition, we require that A is equipped with an involution $*$

$$(da)^* = d(a^*).$$

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The representation in $B(\mathcal{H})$, $\pi(a_0 da_1 \cdots da_n) = a_0 [D, a_1] \cdots [D, a_n]$

$$\pi : \Omega^*_D \to \Omega_{dR}(M) \quad a_0 da_1 \cdots da_n \mapsto a_0 d_{dR} a_1 \cdot d_{dR} a_2 \cdots d_{dR} a_n$$

extends to a canonical isomorphism of GDA.

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Then $d = d_g$.

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Question: Can we extend this to the case where M is not a manifold ?

$$(\mathcal{A}, D, \mathcal{H}).$$

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Let $\mathcal{A}$ be a Banach algebra with an involution such that

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We also introduce the spectrum of a C^*-algebra:

$$Spec(\mathcal{A}) := \{[\pi] \mid \pi \text{ irreducible}\}.$$

Theorem (Gelfand-Naimark)

Let $\mathcal{A}$ be a commutative unital C^*-algebra, then there exists a compact Hausdorff topological space X such that:

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Duality space/algebra:

$$M \longleftrightarrow \mathcal{C}^\infty(M).$$

$$X \longleftrightarrow ?$$

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Lemma

The topological space M is homeomorphic to the subspace of all the maximal points of the inverse limit of the system (K_i, ϕ_{ij}).

The Behncke-Leptin construction

Let P be a partially ordered set (poset).

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This construction allows us the association

$$\mathcal{A}(P) \longleftrightarrow P$$

such that:

$$Spec(\mathcal{A}(P)) \simeq P.$$

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1) Associate a separable Hilbert space $H(X)$ and attach to every point $x \in X$ a subspace $H(x) \subseteq H(X)$ that decomposes into:

$$H(x) = H^{-}(x) \otimes H^{+}(x).$$

where $H^{-}(x) \simeq \ell^2(\mathbb{Z})$.

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2') If $\mathfrak{m}$ is the set of minimal points in X, then $x \in \mathfrak{m}$, one has:

$$H(x) = \mathbb{C} \otimes H^+(x) \simeq H^+(x).$$

The Behncke-Leptin construction

3) Associate to $x \in X$ an operator algebra $A(x)$ acting on $H(x)$ (extended by zero to the whole space $H(X)$) such that

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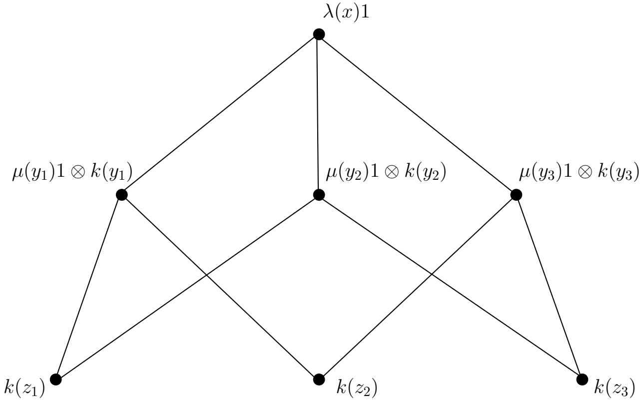
4) Build the C^*-algebra $A(X)$ associated to X as the algebra generated by the subalgebras $A(x)$ when x run over X:

$$A(X) = \bigoplus_{x \in X} A(x) \quad \text{acting on} \quad H(X) = \bigoplus_{x \in X} H(x).$$

The Behncke-Leptin construction: an example



Proposition

A continuous surjection $\phi : X' \to X$ between posets induces a unital *-homomorphism $\phi^* : A(X) \to A(X')$.

$$
\begin{array}{ccc}
A(X) & \xrightarrow{\phi^*} & A(X') \\
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$$A(X_1) \longrightarrow A(X_2) \longrightarrow \cdots \longrightarrow A(X_i) \longrightarrow \cdots$$

Direct limit of C^*-algebras

Define an inductive limit of C^*-algebra:

$$A' = \left\{ a = (a_n) \in \prod_n A_n : \exists N \in \mathbb{N},\ a_{n+1} = \psi_n(a_n)\ \forall n \geq N \right\}.$$

Let $(A_n, \psi_n)_n$ be an inductive sequence in the category of C^*-algebras. Then there exists an inductive limit $(A, \psi_{n,\infty})$ which satisfies the following:

(i) $A = \overline{\bigcup_{n\in\mathbb{N}} \psi_{n,\infty}(A_n)}$;

(ii) For any $n \in \mathbb{N}$ and $a \in A_n$, $\|\psi_{n,\infty}(a_n)\| = \lim_{p\to\infty} \|\psi_{n,p}(a)\|$.

(ii) For any $n \in \mathbb{N}$, $a \in \ker \psi_{n,\infty}$ if and only if $\lim_{p\to\infty} \|\psi_{n,p}(a)\| = 0$.

Take It to the Limit

We can draw the following commuting diagram:

$$
\begin{array}{ccccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_i & \longrightarrow & \cdots & \longrightarrow & A_\infty \\
\downarrow & & \downarrow & & & & \downarrow & & & & \vdots \\
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\end{array}
$$

Proposition (Functorial)

The spectrum $Spec(A_\infty)$ equipped with the hull-kernel topology is homeomorphic to the space X_∞ and

$$\varprojlim Spec(A_i) \simeq Spec(\varinjlim A_i).$$

Take it to the limit

Theorem A (D.T. and J-C. Nave)

The limit C^*-algebra A_∞ is isometrically *-isomorphic to C^*- algebra of the complex valued continuous sections $\Gamma(M, A_\infty)$ over the manifold M. The center $Z(A_\infty)$ is isomorphic to $C(M, \mathbb{C})$.

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Theorem B (T. and J-C. Nave)

The Hilbert space $L^2(M)$ of square integrable functions over the manifold M is a subspace of H_∞:

$$H_\infty = L^2(M) \oplus H.$$

Example of the lattice

Let Λ be the d-dimensional, we can write it as the direct product of d line lattices:

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Let $(A(L), \psi_L)$ be a C^*-algebra over L:

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Corollary A

The centre of the limit C^*-algebra A_∞, $Z(A_\infty)$ is isometrically *-isomorphic to $\mathcal{C}(\mathbb{R}^n)$ acting on $L^2(\mathbb{R}^n)$ as a subspace of H_∞.

Abelian subalgebra

The centre of A_i ($i < \infty$) will be denoted by $Z(A_I)$. We know by construction that A_i is generated by the algebras

$$A(x) = 1_{H^-(x)} \otimes \mathcal{K}(H^+(x))$$

for x running X. The centre $Z(A_i)$ of A_i is trivial.

We will also consider the commutative subalgebra $\mathfrak{A}$ generated by the projectors on $H(x)$ when $x \in \mathfrak{M}$ is a maximal point:

$$\mathfrak{A} = \oplus_{x \in \mathfrak{M}} 1_{H(x)}, \qquad a = \sum_i \lambda_i p_i.$$

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- We need to define a Dirac operator D.

If (M, g) is a compact spin manifold then data $(C^\infty(M), L^2(S), \partial_M)$ is enough to recover the geometric structure.

Finite spectral triple

A *spectral triple* is the data $(\mathcal{A}, \mathcal{H}, D)$ where:

(i) $\mathcal{A}$ is a real or complex $*$-algebra;

(ii) $\mathcal{H}$ is a Hilbert space and a left-representation $(\pi, \mathcal{H})$ of A in $\mathcal{B}(\mathcal{H})$;

(iii) D is a *Dirac operator*, which is a self-adjoint operator on $\mathcal{H}$.

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If in addition, $\mathcal{H}$ is equipped with a $\mathbb{Z}_2$-grading i.e. there exists a unitary self-adjoint operator $\gamma \in \mathcal{B}(\mathcal{H})$ such that

1) $[\gamma, \pi(a)] = 0$ for all $a \in \mathcal{A}$,

2) γ anticommutes with D,

then the spectral triple is said to be *even*. Otherwise, it is said to be *odd*. In the case where $\mathcal{H}$ is finite dimensional, then the triple $(\mathcal{A}, \mathcal{H}, D)$ is called a discrete spectral triple.

Finite spectral triple

We consider the spectral triple $(\mathfrak{A}, \mathfrak{H}(X), \rho)$, where

$$\mathfrak{h} = \bigoplus_{x \in \mathfrak{M}} \mathbb{C}, \quad \mathfrak{H}(X) = \mathfrak{h} \oplus \mathfrak{h}^*$$

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The triple $(\mathfrak{A}, \mathfrak{H}, \rho)$ embeds the commutative algebra $\mathfrak{A}$ into the *Cartan subalgebra* h of the Lie algebra $\mathfrak{gl}(2m, \mathbb{C})$.

$$M_{2m} = M_{2m}^{+} \oplus M_{2m}^{-}$$

Finite spectral triple

We define the parity element $\gamma \in M_{2m}(\mathbb{C})$ such that

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The graded commutator is then given by :

$$da = -[D, a] := Da - \epsilon_a a D,$$

where $\epsilon_a = 1$ if a is even and $\epsilon_a = -1$ if a is odd.

Finite spectral triple

For any element $a \in \mathfrak{A}$, we have

$$\rho(a) = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_m & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_m \end{pmatrix}.$$

Then,

$$D = \sum_{i<j} \omega_{ij} \hat{e}_{ij}, \qquad da = \sum_{i<j} \omega_{ij} \alpha_{ij}(a) \hat{e}_{ij}.$$

Dirac operator associated to X

Let $D \in M_{2m}(\mathbb{C})$ be an odd and hermitian matrix and let ω_{ij} be the coefficients of the block D^-. We say that D is an admissible Dirac operator associate to X if it satisfies:

$a)$ vertices i and j do not share an edge $\Leftrightarrow \omega_{ij} = 0, \ \forall i, j \in \mathfrak{M}$,

$b)$ the eigenvalues μ_n satisfy the asymptotic $\mu_n(D) = O(h^{-1})$.

The prototypical example is given by the *combinatorial Dirac operator*, for which:

$$\omega_{ij} := \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Case $n = 2$

Let $a = (a_1, a_2) \in M_2(\mathbb{C})$ and the Dirac operator:

$$D = \frac{i}{h}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad da = \frac{i}{h}\begin{pmatrix} 0 & a_2 - a_1 \\ a_1 - a_2 & 0 \end{pmatrix}.$$

If we define the following distance:

$$d(x,y) = \sup_{a \in A} \{|a(x) - a(y)| : \|[D,a]\| \leq 1\}$$

then one can show that for $X = \{x, y\}$

$$d(x,y) = h.$$

Without prior assumption, we see the emergence of a small parameter h in place of the usual distance Δx.

Case of a lattice

In the general case of a triangulation K_i, we define D_i as the block matrix

$$D_i = \frac{i}{h}\begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}$$

where D_i^- is the adjacency matrix associated to K_i.

$$D_i^- = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Case of a lattice

Then the limit operator D_∞ acts on A_∞ by the commutator:

$$[D_\infty, a] = ([D_0, a_0], [D_1, a_1], \cdots, [D_i, a_i], \cdots) \in \prod_{i \in I} M^-_{2m_i}(\mathbb{C}).$$

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Proposition

i) $\sigma_{A_\infty}([D_\infty, a]) = \overline{\cup_i \sigma_{A_i}([D_i, a_i])}$

ii) $\| [D_\infty, a] \| = \|d_c a\|_\infty$

Differential calculus

(Spectral Theorem)

Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there is a measure space (X, Σ, μ) and a function $L^\infty_\mu(X)$ and a unitary operator $U : \mathcal{H} \to L^2_\mu(X)$ such that

$$U^* T U = A,$$

where T is the multiplier operator:

$$T(\varphi)(x) = f(x)\varphi(x),$$

and $\|T\| = \|f\|_\infty$.

Theorem C (D.T. and J-C. Nave)

There exists a finite measure μ and a unitary operator

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}, d\mu)$$

such that,

$$U[D, a]U^{-1}\phi = \frac{da}{dx}\phi, \quad \forall \phi \in L^2(\mathbb{R}),$$

Moreover, the norm of $[D, a]$ is given by $\| \, [D, a] \, \| = \|d_c a\|_\infty$.

Differential calculus

Theorem C' (D.T. and J-C. Nave)

There exists a finite measure μ and a unitary operator

$$U : \otimes_{i=1}^{d} L^2(\mathbb{R}) \to \otimes_{i=1}^{d} L^2(\mathbb{R}, d\mu),$$

such that

$$U[D, a]U^{-1}\phi = \sum_{k=1}^{d} a_1\phi_1 \otimes \cdots \otimes \frac{\partial a_k}{\partial x_k}\phi_k \otimes \cdots \otimes a_d\phi_d,$$

for all $\phi = \phi_1 \otimes \cdots \phi_k \otimes \cdots \otimes \phi_d$ in $\otimes_{i=1}^{d} L^2(\mathbb{R})$.

Outline

Conclusion

We have the following results: given a compact spin manifold (M, g),

- associate to each K_i a C^*-algebra A_i with limit $C(M)$,
- define a differential structure $da = [D_i, a]$ on each A_i,
- for the lattice, (D_i) converges to the usual Dirac operator ∂_M.
- Using the same tools than the continuous case $(C^\infty, L^2(M), \partial_M)$

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- Beyond the lattice case (simplicial complex, fractals..),
- Application to noncommutative algebraic geometry and p-adic number theory.

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Stay tuned !

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