Noncommutative differential geometry and infinitesimal spaces

Damien Tageddine

Department of Mathematics and Statistics
McGill University, Montreal

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Abstract

In this thesis dissertation, we introduce the language of noncommutative differential geometry to formalize discrete differential calculus.

In Chapter 2, we begin with a brief review of the inverse limit of posets as an approximation of topological spaces. We then show how to associate a C^*-algebra over a poset, giving it a piecewise-linear structure. Furthermore, we explain how dually the algebra of continuous function $C(M)$ over a manifold M can be approximated by a direct limit of C^*-algebras over posets. Finally, in the spirit of noncommutative differential geometry, we define a finite dimensional spectral triple on each poset. We show how the usual finite difference calculus is recovered as the eigenvalues of the commutator with the Dirac operator. We prove a convergence result in the case of the d-lattice in $\mathbb{R}^d$ and for the torus $\mathbb{T}^d$.

Chapter 3 presents a follow-up work on the noncommutative differential geometry on discrete spaces introduced in the previous chapter. On the one hand, we reformulate the definition of finite dimensional compatible Dirac operators using Clifford algebras. This definition also leads to a new construction of a Laplace operator. We then show that any sequence of compatible Dirac operators $(D_n)_{n\in\mathbb{N}}$ yields to a bounded operator. On the other hand, after a brief introduction of Green's function on manifolds, we show that when the Dirac operators are interpreted as transition matrices, the sequence $(D_n)_{n\in\mathbb{N}}$ converges in average to the usual Dirac operator on a spin manifold. The same conclusion can be drawn for the Laplace operator.

Résumé

Dans cette thèse, nous introduisons le langage de la géométrie différentielle noncommutative afin de formaliser le calcul différentiel discret.

Dans le Chapitre 2, nous commençons par une brève description de limites inverses d'ensembles partiellement ordonnés (parfois appelé poset d'après l'anglais *partially ordered set*) comme approximation d'espace topologique. Nous montrons ensuite comment une C^*-algèbre peut être associée à un poset. Cette C^*-algèbre induit, de fait, une structure linéaire par morceau sur l'espace en question. En outre, nous expliquons comment, de manière duale, l'algèbre des fonctions continue $C(M)$ sur une variété M peut être approximée par une limite directe de C^*-algèbres associées à des posets. Enfin, et dans l'esprit de la géométrie différentielle noncommutative, nous définissons un triplet spectral sur chaque poset. Nous montrons que les formules de différences finies usuelles se retrouvent comme valeurs propres du commutateur avec l'opérateur de Dirac. Nous prouvons la convergence de ces formules dans le cas de la d-lattice et du tore $\mathbb{T}^d$.

Le Chapitre 3 est une suite immédiate des travaux sur la géométrie différentielle noncommutative sur des espaces discrets développés dans le chapitre précédent. D'une part, nous reformulons la définition d'opérateur de Dirac de dimension finie en termes d'algèbre de Clifford. Cette définition conduit à une nouvelle construction du Laplacien. Nous montrons ensuite que n'importe quelle séquence d'opérateurs de Dirac $(D_n)_{n\in\mathbb{N}}$ définit un opérateur borné. D'autre part, suit à une brève introduction sur les fonctions de Green définies sur des variétés lisses, nous montrons que lorsque les opérateurs de Dirac sont interprétés comme des matrices de transition, la séquence d'opérateur $(D_n)_{n\in\mathbb{N}}$ converge en moyenne vers l'opérateur de Dirac classique sur une variété avec une structure spinorielle. Une conclusion semblable est démontrée pour l'opérateur de Laplace.

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Contribution to the original knowledge

Chapters 2 and 3 constitute the main body of this thesis and are considered original scholarship and distinct contributions to knowledge. Chapter 1 collects the background material necessary to understand the main body of the thesis: the covered material does not contain any new contributions to knowledge and does not constitute original scholarship. All sources of the included material are clearly cited and referenced. Chapter 4 outlines possible future directions of research of the author. The ideas and projects proposed are, to the best of the authors knowledge, new and unexplored.

Chapter 2 is a reformatted and slightly modified version of the preprint article [117] available on the author's website.

Chapter 3 is a reformatted and slightly modified version of a preprint article available on the author's website.

Contribution of Authors

The author, Damien Tageddine, has written this whole thesis alone. His supervisor, Jean-Christophe Nave, has proofread the thesis and helped with the editing.

Chapter 1 is written by the author alone and collects background material for the main body of the thesis: as such, it does not contain any original ideas by the author. The only originality lies in the way the material is organised.

Chapters 2 and 3 constitute the main body of this thesis and are considered original scholarship and distinct contributions to knowledge. Chapter 2 is a reformatted and slightly modified version of the preprint article [117] available on the author's website. In particular, all results presented in this chapter are joint work with Jean-Christophe Nave. The reformatting and modifications compared to [117] are due to the author of this thesis, and any errors introduced through this process are solely his responsibility.

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Chapter 4 is written by the author, and represents his ideas alone, unless otherwise stated.

Contents

Introduction

The approximation theory of partial differential equations (PDE) can take various aspects. Traditionally, numerical analysis proposes different strategies to discretize operators. Depending on the situation, finite differences, finite elements, finite volumes, or spectral methods may be utilized. In this process, the focus is usually analytical. That is, the aim is to control asymptotic convergence of the approximation error in a small parameter ($\varDelta t, \varDelta x, ...$). In fact, only a small subset of discretization techniques aim to preserve certain underlying structures (e.g. geometric, algebraic, etc. . .) of the continuous operator at the discrete level.

The general motivation for the present work is the discretization of partial differential equations (PDE). This thesis aims at laying down the foundation of a broad framework to study discrete differential calculus in a discretization-free fashion. Using the tools of noncommutative differential geometry, we establish a geometric formalism of finite difference calculus in order to tackle the problem of differential operators approximations.

0.1 Related approaches and background

The approximation theory of partial differential equations (PDE) can take several aspects. The various methods rely on the intuitive geometric idea that the fine structure of a space M (one can think of a domain in $\mathbb{R}^d$ or a smooth manifold) is discrete. The resulting discretized space, say X, is governed by a parameter — being a grid spacing, the size of a mesh or a time step for example — denoted by h, ε or $\varDelta x$, which plays the role of an infinitesimal. In the rest of this work, we will loosely call this type of discrete space *infinitesimal space*. Information extracted from the continuous space can be represented by a family of morphisms $(\chi_x)_{x\in X}$ with

$$\chi_x : C^\infty(M) \to \mathbb{C}, \quad \chi_x(f) = f(x),$$

which can be related to either sampling morphisms in finite difference (volume) language, or nodal basis in finite element denominations. These maps encapsulate the local data available from the algebra of functions over the continuous space M.

The geometric approach of discrete differential calculus has been pioneered by Whitney in his work on geometric integration theory [127]. The classical differential forms can be interpreted as cochains when restricted to a simplicial complex K by means of the de Rham map:

$$C : \Omega^p(M) \to C^p(K, \mathbb{Z}), \quad C(\omega) := \sigma \mapsto \langle \omega, \sigma \rangle \, .$$

Vice-versa, a cochain can be used to define a differential form using Whitney's interpolation map $\mathcal{W} : C^p(K, \mathbb{Z}) \to \Omega^p(M)$,

$$\mathcal{W}(x_0, \ldots, x_p) = p! \sum_{i=0}^{p} (-1)^i \lambda_i d\lambda_0 \wedge \cdots \wedge \widehat{d\lambda_i} \wedge \cdots \wedge d\lambda_p.$$

This viewpoint has then been successfully used in lattice (quantum) field theory in [128, 109, 1] and in computational electromagnetism [26, 114].
Moreover, the idea of deriving a discrete theory that parallels the continuous one has then been further explored by Hirani in the discrete exterior calculus (DEC) [72] and subsequently developed by Desbrun et al. [45]. In DEC, the point of view — which is also shared to some extent by our work — is that the discrete theory can, and indeed should, stand on its own right. The authors base their approach on simplicial complexes and its differential calculus on chains and cochains. In that setting, a differential form is an element in the dual of the space of chains. The basic data in the theory is given by the triple $(K, \Omega^*(K), d)$ where K is a simplicial complex, d is the coboundary map and $\Omega^*(K)$ the space of cochains. To this, one

adds a Hodge-star map:

$$(K, \Omega^*(K), d), \quad * : \Omega^k(K) \to \Omega^{d-k}(*K)$$

where $*K$ is the dual simplicial complex.

In the realm of finite element method, the pioneering work of Arnold et al. [5, 4] has also initiated a change of paradigm. The main idea behind is that geometrical and topological properties of differential operators are key points to understand how their discrete counterpart can be derived. The finite element exterior calculus (FEEC) is the result of this work and aims at studying approximations of PDEs that arise from Hilbert complexes. Let W_1, W_2 be Hilbert spaces along with a differential map $d : W_1 \to W_2$. The fundamental data of FEEC is then given by the polynomial subspaces W_1^h and W_2^h determined by projection maps π_1 and π_2 such that the following diagram commutes:

$$\begin{tikzcd} W_1 \arrow[r, "d"] \arrow[d, "\pi_1"'] & W_2 \arrow[d, "\pi_2"] \\ W_1^h \arrow[r, "d"] & W_2^h \end{tikzcd}$$

The discretization can be again summarized by the triple $(W, d, \Omega(W))$ where W is a polynomial algebra, d a derivation map generating the exterior algebra $\Omega(W)$ with coefficients in W.

One can also mention of Christiansen et al. [32] on compatible differential forms on simplicial complexes. Geometric integration and more generally structure preserving methods have applied this change of paradigm too [3, 103, 93, 68, 33]. Symmetries and conservation laws of discrete operators parallel their continuous counterparts [73, 122]. It has been shown that long-term stability can be obtained as a by-product [124]. Finally, for an application of Lie groups to construct invariant discretization schemes, one can refer to [20]. Overall, in the geometric discretization framework, the realization is that classical analysis of consistency and stability is no longer the main criteria to look for in a discretization. In this context, consistency and stability are a consequence of preserving geometrical properties.

Nevertheless, a fair amount of the theory of PDEs is developed on (subdomains of) $\mathbb{R}^n$, hence the later approaches are, at least in their original incarnations, focused on Euclidean spaces. However, examples of partial differential equations arise in a wide variety of applications. As such, extensions of some of the previously mentioned techniques to non-Euclidean domains remains a challenge. Hence, a crucial question in the theory of discretization, is the generalization of classical geometric approaches to smooth manifolds. Also, and still on the topic of convergence analysis of finite elements, [125] studies the cotangent discretization of the Laplace-Beltrami operator; the key result is that mean cruvature vectors converge in the sense of distributions, but fail to converge in L^2. Finally, there are the central research advances on diffusion maps in [85, 35] and the one on random point clouds in [17, 18]. As one can see, approaching the problem of compatible discretizations on manifolds rests heavily on the initial setup chosen to tackle it. The various results are therefore quite different, and perhaps appear disconnected from one another.

0.2 Present work

The main question that we would like to address in this work is the existence of a unifying framework to geometric discretizations. This question can be divided into three subsidiary questions.

The space: the existence of a sequence of approximating spaces, with topological structures and metric specified at an early stage, that converges — in a suitable sense — to a manifold.

The algebra: tied to the question of space is the question of the algebra of "functions" and local coordinates. One needs to identify an associative algebra playing the role of the algebra of continuous functions over a space that does not necessarily possess a manifold structure. It is a well established fact from the theory of Banach algebras [59] that C^*-algebras can be realized as the set of continuous sections over some topological space. In a very intuitive description, an element of a C^*-algebras can be thought as (noncommutative) functions over a space called the spectrum of a C^*-algebra [51, 24]. Hence, if one identifies the points of this spectrum one-to-one with the usual points of a topological space X, then a C^*-algebra appears as a good candidate for the set of continuous functions over X. Indeed, their normed space structure is a powerful tool to study boundedness and convergence of its elements. Thus, in the same fashion as for the space itself, can one construct a nested sequence of algebras such that the limit is essentially the space of continuous functions over the original manifold ?

The geometry: once the questions of space and algebra are addressed, it remains to define (if it exists and is it unique ?) a differential calculus — understood from an algebraic/geometric point of view in opposition to the usual analytic perspective — on such a space. What does such a differential structure on an infinitesimal space look like ? One can already notice that it will irremediably differ from its continuous counterpart since functions and forms do not commute anymore:

$$gdf \neq df\,g. \tag{1}$$

Moreover, the differential calculus is intimately tied to connections and distances between points parametrized by h. This fact is reminiscent of the continuous theory, where the line element ds — one can think of an infinitesimal displacement vector in a metric space — on a n-dimensional Riemannian manifold is a function of the metric tensor. Moreover, it is a well established fact in spin geometry [37, pp. 552-557] that this metric information can be summarized in a single operator $\not{D}$ called the Dirac operator [107, pp. 406-407] such that :

$$ds = \not{D}^{-1}. \tag{2}$$

Therefore, the metric can be deduced — in principle — from the data of the Dirac operator. Hence, one have a dual description of space : one purely topological given by an open cover and one purely algebraic given by the Dirac operator.

0.3 The noncommutative geometry framework

In this thesis we consider the formalism of Noncommutative Differential Geometry (NDG). NDG has been introduced by Connes in a series of papers [36] compiled in the red book [37] — and later extensively developed by Connes and his collaborators [39, 38]. This branch of mathematics is concerned with a geometric approach to noncommutative algebras [82, 120, 110, 111, 112]. In Connes' work, a noncommutative space is — heuristically speaking — the dual space of a C^*-algebra by analogy to the Gelfand theory where commutative C^*-algebras are dual objects to locally compact Hausdorff spaces. In fact, the notion of space becomes secondary and is replaced by the notion of a spectral triple (A, H, D) — where A is a C^*-algebra, H is a Hilbert space on which A is realized as an algebra of bounded linear operators, D is a Dirac operator. A new type of differential calculus using functional analysis is then derived; it is now referred to as quantized calculus. We also mention another type of noncommutative differential geometry over matrix algebras developed by Dubois-Violette et al. [54] and exposed in more detail in [91, 95, 94].

The idea of approximating a bounded region of space-time with finite topological spaces as been pushed by Sorkin [113]. Important examples of noncommutative spaces are provided by noncommutative lattices, which are a particular case of posets. This topic has been thoroughly studied by Bimonte et al. [22, 23] — summarized in Landi's book [86] — and techniques from noncommutative geometry have been used to construct models of gauge theory on these noncommutative lattices in Balachandran et al. [9, 10, 8]. It is also worth mentioning another approach to discrete noncommutative spaces and their differential calculus in the work of Dimakis et al. [47].

0.4 Objectives

The main objective of this work is to derive *ab initio* finite difference calculus using the language of noncommutative geometry. This leads us to define tools from differential geometry such as differential maps along with their differential complex, affine connections and a Laplace operator. It also allows us to study spectral convergence with respect to a parameter h. Indeed, the natural setting of C^*-algebras, through their representations into operator algebras, allows us to use the machinery of functional calculus. This main objective can be divided into three sub-objectives. First, we aim at establishing a proper notion of *discrete space* X, starting uniquely from the knowledge of a manifold M along with its algebra of functions. Secondly, we want to exhibit the algebra of continuous sections $\Gamma(X)$ over X. Following Gelfand-Naimark's theorem, this should be a C^*-algebra A. Thirdly, we define a so-called *Dirac operator* D governing the differential geometry over the space X. Once such an operator is defined, it provides an exterior algebra $\Omega(A)$ and some usual machinery from differential geometry.

In this work, we are able to give an intrinsic description of finite difference calculus in terms of noncommutative geometry and its quantized calculus. We recover some usual tools of differential geometry, such as an exterior derivative. Higher-order approximations are restated in terms of $\mathbb{Z}_2$-graded traces induced by positive operators. We also define and establish convergence of differential operators on infinitesimal spaces to their continuous counterpart. We further prove a generalized result on direct limits of C^*-algebras over posets. This extends the result of Bimonte et al. [23] proven in the special case of noncommutative lattice. Therefore, this work opens the door to a general framework to study approximation theory of PDEs.

0.5 Contributions of this thesis

In Chapter 2, we start by recalling general results on approximation of a compact Hausdorff space M by a sequence of ordered simplicial complexes (Proposition 2.1.1). We construct an inverse system of triangulations, (K_n) which become sufficiently fine for large n. We associate to each space K_n a C^*-algebra A_n such that the triangulation K_n is identified with its spectrum $Spec(A_n)$. The C^*-algebras give a piecewise-linear structure to the triangulations. We then form an inductive system

(A_n) with limit a C^*-algebra A_∞ and the functorial relation:

$$\lim_{\leftarrow} Spec(A_i) \simeq Spec(\lim_{\rightarrow} A_i).$$

In addition, the algebra A_∞ contains the space of continuous functions $C(M)$ in its center. Hence, the space $C(M)$ can be approximated by a sequence of C^*-algebras over each simplicial complex (Theorem 2.2.1 and Proposition 2.2.5):

$$C(M) \subset \overline{\bigcup_{n\in\mathbb{N}} \phi^*_{n,\infty}(\mathfrak{A}_n)}.$$

Furthermore, we define spectral triples $(\mathfrak{A}_n, \mathfrak{h}_n, D_n)$ on every triangulation X_n. We prove that the usual finite difference approximations are recovered as eigenvalues of the exterior derivative operator (Proposition 2.3.4). More precisely, we construct the following sequence:

$$[D, a] = ([D_0, a_0], [D_1, a_1], \cdots, [D_n, a_n], \cdots),$$

and we show convergence of this differential operator to the classical de Rham differential in the case of the d-dimensional lattice (Proposition 3) and the d-torus.

In Chapter 3, we fix a triangulation X, and associate a collection of Dirac operators $(D_t)_{t\in\mathbb{N}}$, where each matrix D_t can be seen as an irreducible matrix associated to the graph G obtained from X. The graph has n vertices labelled $1, \ldots, n$, and there is an edge from vertex i to a vertex j precisely when $\omega_{ij} \neq 0$. More precisely, in the probabilistic setting, a vertex i is connected to a vertex j with probability ω_{ij}. Then, if we let $a_t \in \mathrm{Dom}(D_t)$ and define the average operator,

$$S_N = \frac{1}{N}\sum_{t=1}^{N} e_t\,[D_t, a_t]\,e_t^*, \tag{3}$$

where $(e_t)_{t\in\mathbb{N}}$ is some family of projections. The key here is to choose the coefficients ω^t_{ij} associated to D_t in order for the average operator S_N to converge to $[\mathcal{D}, a]$ as $N \to \infty$, where $\mathcal{D}$ is the Dirac operator on the spin manifold M. The main result of this chapter goes as follows:

Let $\left\{x_{i_0}^k\right\}_{k=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on an open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. Let $\tilde{S}_n^{\hbar_n}$ be the associated operator given by:

$$\widehat{S}_n^{\hbar_n} : C^\infty(U_p) \to M_2(\mathbb{R}) \otimes U(\mathfrak{gl}_{2m_n}), \qquad \widehat{S}_n^{\hbar_n}(a) := \frac{1}{n} \sum_{k=1}^n e_k \left[D_X^k, a_k\right] e_k^*. \tag{4}$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then for $a \in C^\infty(U_p)$, in probability:

$$\lim_{n\to\infty} \Psi \circ \widehat{S}_n^{\hbar_n}(a) = [\mathcal{D}, a]\,(p).$$

Additionally, a similar result is proved for the case of the Laplace operator.

0.6 Outline

We end the introduction with an outline of the contents of the thesis. We have attempted to keep this document reasonably self-contained; where details are insufficient, we provide references for the interested reader.

Chapter 1 reviews the general theory that we believe is necessary to understand the results present in this thesis. We start with a brief overview of C^*-algebra, with a focus on the standard results of Gelfand and Naimark in the commutative case and the GNS construction in the noncommutative case. We then introduce some machinery of differential geometry. Vector and principal bundles are reviewed along with connections. Furthermore, we define the spin structure on a manifold which is crucial to construct the Dirac operator. Finally, we draw the general outlines of noncommutative differential geometry; from the definition of spectral triples to the construction of Dixmier's trace. The classical example of the two points space is also treated.

Chapter 2 starts by reviewing some technical preliminaries on inverse limits of triangulations and their associated posets. Then we construct a C^*-algebra over each poset and study the direct limit. Finally, the differential structure is presented in the form of spectral triples; we conclude with a discussion on the convergence results for the d-lattice in $\mathbb{R}^d$ and for the torus $\mathbb{T}^d$.

Chapter 3 starts with a presentation of the main results of Chapter 2. We then give a brief introduction to Clifford algebras. This introduction is used to then define Dirac operators on finite dimensional spaces in the Clifford algebra setting. In Section 3.2, we introduce a specific Hamilton-Jacobi equation with its associated Green function on $\mathbb{R}^d$ first, and their generalizations on a Riemannian manifold then. This is followed by some technical lemmas required to define the coefficients ω_{ij} necessary to prove the main theorems in Chapter 3. In Section 3.4, we prove our Theorem 3.4.1, and we obtain as a by-product a convergence result on the Laplacian.

Chapter 4 concludes this thesis with a brief overview of open questions and future projects that the author plans to address in the future.

Chapter 1

Preliminaries

1.1 C*-algebras and their spectra

In this section, we recall the theory of C^*-algebras and some fundamental results that will be useful in the present work.

1.1.1 Banach algebras

Let $(V, \|\cdot\|)$ be a normed vector space. The space $(V, \|\cdot\|)$ is called a *Banach space* if in addition V is complete with respect to the norm $\|\cdot\|$. If V is in addition an algebra, then we say that $\|\cdot\|$ is *multiplicative* if

$$\|xy\| \leq \|x\|\|y\|, \quad \forall x, y \in V. \tag{1.1}$$

Definition 1.1.1 (Banach algebra)**.** Let A be an associative algebra with an underlying normed vector space $(A, \|\cdot\|)$. Then $(A, \|\cdot\|)$ is a *Banach algebra* if in addition:

i) The norm $\|\cdot\|$ is multiplicative,

ii) $(A, \|\cdot\|)$ is a Banach space.

We will say that A is a *unital Banach algebra* if it possesses a unit 1_A for the multiplication. An element $a \in A$ is then called *invertible* if there is a $b \in A$ such that $ab = ba = 1_A$ and we write $b = a^{-1}$. The set of invertible in A is denoted by

$$\mathrm{Inv}(A) := \{a \in A \mid \text{there is } b \in A \text{ s.t. } ba = ab = 1_A\}.$$

Definition 1.1.2. The *spectrum* of an element a in a unital algebra A is the subset of complex numbers defined as

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} \mid (\lambda 1_A - a) \notin \mathrm{Inv}(A)\}\,. \tag{1.2}$$

In order to study the internal structure of Banach algebras, one focuses on their ideals; Let A be an algebra. A subalgebra $I \subset A$ is a right (left) ideal if $a \in A$ and $b \in I$ then $ab \in I$ $(ba \in I)$. We call $I \subset A$ an *algebraic* ideal if it is both a right and a left ideal. We also have the usual notions of *trivial ideals* $(I = 0, A)$ and *ideals generated by a set* $J \subset A$ (smallest ideal containing J). A *proper* (algebraic) ideal is one which is not equal to A (but may be zero) and a *maximal* (algberaic) ideal is a proper (algberaic) ideal not contained in any other proper (algebraic) ideal.

The ideals associated to *characters* are of particular importance.

Definition 1.1.3. Let A be a Banach algebra. A *character* on A is a nonzero algebra homomorphism $\varphi : A \to \mathbb{C}$. Let

$$\Omega(A) := \{\varphi : A \to \mathbb{C} \mid \varphi \text{ a character on } A\}\,. \tag{1.3}$$

We call $\Omega(A)$ the *character space* of A, also known as the spectrum of A.

1.1.2 *-algebras

For λ a complex number, we will denote by $\overline{\lambda}$ its conjugate.

Definition 1.1.4. Let A be an algebra. An *involution* on the algebra A is a map $* : A \to A, \quad x \mapsto x^*$, satisfying the following properties:

1) $(x + y)^* = x^* + y^*, \quad \forall x, y \in A,$

2) $(\lambda x)^* = \overline{\lambda} x^*, \quad \forall \lambda \in \mathbb{C},\ x \in A,$

3) $(xy)^* = y^* x^*, \quad \forall x, y \in A,$

4) $(x^*)^* = x, \quad \forall x \in A.$

An algebra A equipped with an involution is called an *involutive algebra* or simply a $*$-*algebra*. Given an element a in a $*$-algebra A, we call a^* the *adjoint* of a.

An element x in $*$-algebra A satisfying $x = x^*$ is called a *selfadjoint element*. In addition, a subalgebra B of A is called a *selfadjoint subalgebra*, whenever it is invariant under $*$; that is, $x^* \in B$, for every $x \in B$.
If A is a Banach algebra, then we call $I \subset A$ an *ideal* if it is a norm-closed algebraic ideal. In this case, A/I can be given the quotient norm

$$\|a + I\| = \inf_{b \in I} \|a + b\|, \quad a \in A \tag{1.4}$$

which make A/I into a Banach algebra. Then if A is in addition involutive, then the $*$ operation descend to A/I making it a Banach $*$-algebra.

1.1.3 C*-algebras

Definition 1.1.5. An abstract C^*-algebra is a Banach $*$-algebra $(A, \|\cdot\|)$ satisfying the C^*-*condition*:

$$\|a^* a\| = \|a\|^2 \text{ for every } a \in A. \tag{1.5}$$

We call a norm satisfying the C^*-condition a C^*-norm.

It follows immediately from the definition that every C^*-algebra is a Banach $*$-algebra but the converse need not hold. The seemingly simple requirement of the C^*-condition is in fact imposing nice structural properties the we don't see in arbitrary Banach algebra.

Definition 1.1.6. Let A be a $*$-algebra. Let $p : A \to [0, \infty)$ be a map. We say that p is a C^*-seminorm if for all $x, y \in A$:

1) $p(xy) \leq p(x)p(y),$

2) $p(x^*) = p(x)$,

3) $p(x^*x) = p(x)^2$.

Starting from a $*$-algebra A, it is however possible to produce a C^*-algebra, provided that A is equipped with suitable *C^*-seminorms*. Indeed, for $x \in A$, we can define:

$$\|x\| := \sup\{p(x) \ : \ p \text{ is a } C^*\text{-seminorm on } A\}. \tag{1.6}$$

It is possible that $\|x\|$ is infinite for some $x \in A$. Therefore, we have to assume that $\|x\| < \infty$ for every $x \in A$. Moreover, the set

$$I := \{x \in A \ : \ \|x\| = 0\} \tag{1.7}$$

is a two-sided *ideal* of A. The seminorm $\| \cdot \|$ descends to a C^*-norm on A/I. The completion of A/I with respect to this C^*-norm is called the enveloping C^*-algebra of A and denoted by $C^*(A)$.

Definition 1.1.7. A **-homomorphism* between two C^*-algebras A and B is an algebra homomorphism $\varphi : A \to B$ that involution preserving, i.e. $\varphi(a^*) = \varphi(a)^*$ for every $a \in A$. If A and B are unital, then φ is a *unital $*$-homomorphism* if $\varphi(1_A) = 1_B$.

We are now ready to introduce two archetypal example of unital C^*-algebras.

Example 1.1.1. 1) Let X be a compact Hausdorff topological space. Consider the space of functions:

$$C(X) := \{f : X \to \mathbb{C} \mid f \text{ is continuous}\} \tag{1.8}$$

equipped with pointwise operations. Then $C(X)$ is a C^*-algebra with involution given by $f(x)^* = \overline{f(x)}$ and the supremum norm.

2) Let H be a separable Hilbert space of dimension n with $1 \leq n \leq \infty$. We consider the algebra $\mathcal{L}(H)$ of linear operators $T : H \to H$. We equip $\mathcal{L}(H)$

with the operator norm:

$$\|T\| = \sup_{x\in H, \|x\|=1} \|Tx\|. \tag{1.9}$$

The algebra of bounded operators, denoted by $\mathcal{B}(H)$, is in fact a C^*-algebras for the operator norm and with the involution is given by the adjoint T^* of an operator T. In the case $n < \infty$, notice that the space $\mathcal{B}(H)$ is nothing else but the algebra of complex matrices $M_n(\mathbb{C})$.

Such C^*-algebras given as examples, are called *concrete C^*-algebra.* It is a striking fact in the theory that every abstract C^*-algebras can be realized as a C^*-subalgebra of a concrete C^*-algebra of one of the two previous types. In the case where A is a commutative C^*-algebra, this fact follows from the *Gel'fand representation.*

Let A be a unital C^*-algebra. Take $a \in A$ and define $\widehat{a} : A^* \to \mathbb{C}$ by $\widehat{a}(\varphi) = \varphi(a)$. Then $\widehat{a} \in \mathbb{C}(\Omega(A))$, equipped with the weak-* topology (the weak-* topology is the coarsest topology on A^*, making every $\hat{a}$, $a \in A$ continuous).

Theorem 1.1.1 (Gel'fand-Naimark)**.** *Let A be a commutative C^*-algebra with unit and let $X = \Omega(A)$ be its spectrum. The Gel'fand transform*

$$\widehat{a} : x \ni A \mapsto (\varphi \mapsto a(\varphi)) \in C(X) \tag{1.10}$$

is an isomorphism of A onto the C^*-algebra $C(X)$ of continuous complex function over X.

Remark 1.1.1. The Gel'fand transform is one instance of the duality of space; that is, a compact Hausdorff space X can either be seen as a topological space as it's stand or as the *spectrum* of a unital commutative C^*-algebra.

Theorem 1.1.2. *The correspondence between X and $C(X)$ is a categorical equivalence between the category of compact Hausdorff spaces and continuous maps to the category of unital C^*-algebras and unital *-homomorphisms.*

1.1.4 Element of representation theory

Historically, the study of C^*-algebras is motivated by the prototypical example of closed *-algebras of operators on Hilbert space coming from quantum mechanics. It is then natural to ask if a abstract C^*-algebra can be realized as operators acting on Hilbert space. Such a realization is called a *representation* of the C^*-algebra.

Definition 1.1.8. Let A be a $*$-algebra. A representation of A is a pair, (π, H), where H is a Hilbert space and $\pi : A \to \mathcal{B}(H)$ is a $*$-homomorphism. We also say that π is a representation of A on H.

Definition 1.1.9. A representation, (π, H), of a $*$-algebra A, is nondegenerate if the only vector ξ in H such that $\pi(a)\xi = 0$ for all a in A, is $\xi = 0$. Otherwise, the representation is degenerate.

In particular interest are the building blocks given by *irreducible representations* i.e. representations that cannot be decomposed into smaller ones.

Definition 1.1.10. A representation of a $*$-algebra is *irreducible* if the only closed invariant subspaces are 0 and H. It is *reducible* otherwise.

Definition 1.1.11. Let (π, H) be a representation of a $*$-algebra A. We say that a vector ξ in H is *cyclic* if the linear space $\pi(A)\xi$ is dense in H. We say that the representation is *cyclic* if it has a cyclic vector.

Of course, any representation is only considered up to a unitary equivalence.

Definition 1.1.12. Let A be a $*$-algebra. Two representations of A, (π_1, H_1) and (π_2, H_2), are unitary equivalent if there is a unitary operator $u : H_1 \to H_2$ such that $\pi_1(a) = u\pi_2(a)u^*$, for all in $a \in A$. In this case, we write $(\pi_1, H_1) \sim (\pi_2, H_2)$ or $\pi_1 \sim \pi_2$.

The fundamental tool to construct representations of an abstract C^*-algebras are given by the so-called *states.*

Definition 1.1.13 (State)**.** Let A be a C^*-algebra. A linear function ϕ on A is positive if $\phi(a^*a) \geq 0$, for all a in A. In case that A is unital, the linear functional ϕ

is a *state* if it is positive and, in addition, $\phi(1) = 1$. The set of *states* of the algebra A is denoted by $\mathcal{S}(A)$.

States can be viewed as noncommutative generalizations of *probability measures*. Indeed, in the commutative case, the C^*-algebra is of the form $C_0(X)$ for some locally compact Hausdorff space X and the set $\mathcal{S}(A)$ consists of *positive Radon measures* on X.

Hence, starting with a state ϕ, one can produce a representation (π_ϕ, H_ϕ) as follow. Define the zero set

$$N_\phi = \{a \in A \mid \phi(a^*a) = 0\}, \tag{1.11}$$

which turns out to be a closed left ideal in A. Then, define the bilinear form

$$\langle a + N_\phi, b + N_\phi \rangle = \phi(b^*a) \tag{1.12}$$

and gives a well defined inner product on the quotient space A/N_ϕ. Therefore, we consider the completion of A/N_ϕ is a Hilbert space, denoted H_ϕ.
Additionally, the map

$$\pi_\phi(a)(b + N_\phi) = ab + N_\phi, \tag{1.13}$$

for a, b in A extends to define $\pi_\phi(a)$ as a bounded linear operator on H_ϕ.

Definition 1.1.14. Let A be a unital C^*-algebra and let ϕ be a state on A. The triple $(H_\phi, \pi_\phi, \xi_\phi)$ is called the Gel'fand-Naimark-Segal (GNS) representation of ϕ.

Theorem 1.1.3 (Gel'fand-Naimark-Segal)**.** *Let A be a unital C^*-algebra and suppose that π is a representation of A on the Hilbert space H with cyclic vector, ξ, of norm one. Then*

$$\phi(a) = \langle \pi(a)\xi, \xi \rangle, \tag{1.14}$$

for all a in A, defines a state on A. Moreover, the GNS representation of ϕ is unitary equivalent to π in the sense that there is a unitary operator $u : H \to H_\phi$ satisfying $u\pi(a)u^* = \pi_\phi(a)$, for all a in A and $u\xi = \xi_\phi$.

Therefore, the GNS construction allows one to associate a representation of an abstract C^*-algebra A to a state in $\mathcal{S}(A)$ and vice-versa.

Now, let us notice that the set of states $\mathcal{S}(A)$ is a convex subset in the dual A^*.

Let X be a convex set in a vector space V. A point $x \in X$ is called an *extreme point* if, whenever $x = ty + (1-t)z$ for some $t \in (0,1)$, we have $x = y = z$. The subset of extreme points is denote by $\mathrm{Ext}(X)$. In addition, if S is any subset in V, then the smallest convex set containing S is called the *convex hull* of S and is denoted $\mathrm{co}(S)$. The *closed convex hull* of S, denoted $\overline{\mathrm{co}}(S)$, is the closure of of the convex hull and the smallest closed and convex set containing S.

Theorem 1.1.4 (Krein-Millman)**.**

1) *(Existence) Every non-empty convex subset X of a Hausdorff locally convex topological vector space has an extreme point; that is the set $Ext(X)$ is not empty.*

2) *Suppose X is a Hausdorff locally convex topological vector space and K is a compact and convex subset of X. Then K is equal to the closed convex hull of its extreme points:*
$$K = \overline{co}(Ext(K)).$$

Now, if we recall the Banach-Alaoglu theorem; that is, the unit ball in A^* is weak-* compact, then it implies, by inclusion, that $\mathcal{S}(A)$ is also compact. Hence, using the Krein-Millman theorem, the state space $\mathcal{S}(A)$ has extreme points.

Definition 1.1.15. The extreme points of the state of space are called *pure states*, and the set of pure states of A is denoted by $\mathcal{P}(A)$ and satisfies:

$$\mathcal{S}(A) = \overline{\mathrm{co}}(\mathcal{P}(A)).$$

One can then refine the GNS construction and exhibit the relation between irreducible representations and pure states.

Theorem 1.1.5. *Let ϕ be a state on the unital C^*-algebra A. The GNS representation (π_ϕ, H_ϕ) is irreducible if and only if ϕ is not a non-trivial convex combination of two other states. That is, if there are states ϕ_0 and ϕ_1 and $0 < t < 1$ such that $\phi = t\phi_0 + (1-t)\phi_1$, then $\phi_0 = \phi_1 = \phi$.*

In other words, we have the following correspondence:

$$\{\text{Pure states } \phi\} \longleftrightarrow \{\text{Irreducible representations } (\phi, H_\phi)\}.$$

1.1.5 Spectrum and primitive spectrum

We have seen that for a commutative C^*-algebra, the space of characters is the analogue of the topological space X; this is visible through the Gel'fand-Naimark theorem. For noncommutative C^*-algebras, there is more than one candidate for the analogue of the notion of space. We shall introduce some of them. We first define the *primitive spectrum.*

Definition 1.1.16 (Primitive spectrum)**.** The *primitive spectrum* $Prim(A)$ is the space of kernels of irreducible *-representations equipped with the *hull-kernel (Jacobson) topology.*

Definition 1.1.17 (Hull-kernel (Jacobson) topology)**.** Let A be a C^*-algebra and $Prim(A)$ its primitive spectrum with power set $2^{Prim(A)}$. One define the Jacobson's closure operator as follows, for any $W \in 2^{Prim(A)}$

$$\mathrm{Cl}(W) := \left\{ I \in Prim(A) : \bigcap_{J \in W} J \subseteq I \right\} \tag{1.15}$$

The closure operator Cl satisfies the Kuratowski axioms [51, pp.69-70] and therefore it defines a topology on $Prim(A)$ by means of its closed sets. In addition, $Prim(A)$ can be given the structure of a partially ordered set, by means of inclusions of ideals. A topology on the set $Prim(A)$ can be induced by the partial order, using the so-called *Alexandrov topology.*

Proposition 1.1.1. *Let W a subset of $Prim(A)$ and $I \in Prim(A)$. Then the following claims are true.*

i) The subset W is closed if and only if $I \in W$ and $I \subseteq J$ then $J \in W$.

ii) The space $Prim(A)$ is a T_0-space.

iii) The point $\{I\}$ is closed in $Prim(A)$ iff I is maximal among primitive ideals.

Let us now introduce the *spectrum* of a C^*-algebra.

Definition 1.1.18 (Spectrum)**.** The *spectrum* of a C^*-algebra is the set of all unitary equivalence classes of irreducible $*$-representations. It is denoted by $Spec(A)$. One can also encounter the term of *structure space*, designated by $\widehat{A}$, do refer to the spectrum.

Furthermore, there is a canonical surjection

$$Spec(A) \to Prim(A), \quad \pi \mapsto \ker(\pi) \tag{1.16}$$

The inverse image under thus map of the Jacobson topology on $Prim(A)$ is a topology for $Spec(A)$. We have the following topological properties of the spectrum.

Proposition 1.1.2. *Let A be a C^*-algebra. The following conditions are equivalent:*

i) $Spec(A)$ is a T_0-space.

ii) Two irreducible representations of $Spec(A)$ with the kernel are equivalent.

iii) The canonical map $Spec(A) \to Prim(A)$ is a homeomorphism.

1.1.6 AF-algebras

There is a special type of C^*-algebras that will focus our attention in this work. An approximately finite-dimensional C^*-algebra, or simply AF-algebra, that is the inductive limit of an increasing sequence of finite-dimensional C^*-algebras, all with the same unit. These algebras allow to produce a large number of examples of C^*-algebras for which the spectrum and the primitive spectrum coincides as topological spaces.

Definition 1.1.19 (AF-algebra)**.** A C^*-algebra A is said to be approximately finite dimensional (AF) if there exists an increasing sequence

$$A = \varinjlim \cdots \to A_i \xrightarrow{\alpha_i} A_{i+1} \to \cdots \tag{1.17}$$

where each A_i is a finite-dimensional C^*-algebra and the connecting maps α_i are unital *-homomorphisms. The inductive system specifying an AF-algebras is not unique; one can always drop to a subsequence. Suppressing the connecting maps, A can be written as $A = \overline{\cup_n A_n}$.

In addition, AF-algebras are interesting because the centra of AF-algebras exhaust all separable abelian C^*-algebras with unit. In other words, any separable abelian C^*-algebra is center of a C^*-algebra that is the inductive limit of an increasing sequence of finite dimensional C^*-algebras.

Lemma 1.1.1 ([31])**.** *Let $A = \overline{\cup_n A_n}$ be an AF-algebra. If $x \in Z(A)$, there exists a sequence $\{x_n\}_n$ such that $x_n \in Z(A_n)$ and $x_n \to x$. Conversely, if $x_n \in Z(A_n)$ and $x_n \to x$, then $x \in Z(A)$.*

Proof. Let $U(A_n)$ be the group of unitaries in A_n and $d\mu_n$ is a normalized Haar measure on $U(A_n)$, one may define $E_n : A \to A$ by:

$$E_n(x) = \int_{U(A_n)} uxu^* d\mu_n(u). \tag{1.18}$$

Let $a_n \in U(A_n)$, then using the invariance of $d\mu_n$ one has

$$\int_{U(A_n)} a_n uxu^* d\mu_n(u) = \int_{U(A_n)} vxv^* a_n d\mu_n(v). \tag{1.19}$$

Since A_n is generated by its unitary elements, E_n is then a projection of norm 1 form A onto A_n^c (= the relative commutant of A_n in A). In particular, E_n maps A_n onto $Z(A_n)$.

($\Rightarrow$) Let $x \in Z(A)$ and consider a sequence $\{y_n\}$ such that $y_n \in A_n$ and $y_n \to x$. Define $x_n = E_n(y_n)$. Since $x \in A_n^c$ for all n, $E_n(x) = x$. Thus,

$$\|x - x_n\| = \|x - E_n(y_n)\| = \|E_n(x - y_)\| \leq \|x - y_n\|. \tag{1.20}$$

Hence $x_n \to x$. □

($\Leftarrow$) For all $n > m$, one has that $Z(A_n) \subseteq A_n^c$; thus, the given x commutes with $\cup_n A_n$ and thus, with A.

Theorem 1.1.6 ([31]). *Let $\mathfrak{A}$ be an Abelian separable C^*-algebra with unit. Then there exists an approximately finite dimensional C^*-algebra A having $\mathfrak{A}$ as center.*

It is in that sense that the space of continuous functions $C(X)$ over a topological Hausdorff space X can be approximated by finite dimensional C^*-algebra i.e. matrix algebras.

1.1.7 Compact operators

Definition 1.1.20. An operator T on a hilbert space H is called *compact* if T takes the closed unit ball in H to a relatively compact subset of H.

The definition of a compact operator implies automatically that it is a bounded operator. The set of compact operators on H is denoted by $\mathcal{K}(H)$.
An equivalent definition of compactness is that the image under T of any bounded sequence in H has a convergent subsequence.

One can also characterize compact operators through the description of their eigenvalues. An operator $T : H \to H$ on an infinite-dimensional Hilbert space H is said to be *compact* if it can be written in the form

$$T = \sum_{n=1}^{\infty} \mu_n(T) \langle f_n, \cdot \rangle g_n, \tag{1.21}$$

where $\{f_1, f_2, \dots\}$ and $\{g_1, g_2, \dots\}$ are orthonormal sets (not necessarily complete), and $\mu_1, \mu_2, \dots$ is a sequence of positive numbers with limit zero, called the *singular values* of the operator. The singular values can accumulate only at zero.

Proposition 1.1.3. *Suppose the $T \in B(H)$ is normal and compact.*

(a) Every $\lambda \in \sigma(T) - \{0\}$ is an eigenvalue.

(b) If $\lambda \neq 0$, then the eigenspace $E_\lambda = \{x \in H : Tx = \lambda x\}$ is finite dimensional.

(c) $\sigma(T)$ is at most countable and has no nonzero accumulation points.

Hence, one can deduce a spectral theorem for normal and compact operator from functional analysis calculus.

There is a third definition of compact operators through operators of *finite rank*.

Definition 1.1.21. A bounded operator T on H is called a *finite rank* operator if it has finite-dimensional range $T(H)$. The collection of finite-rank operators on H is denoted by $B_f(H)$.

Then the set of compact operators can be defined as the norm closure of $B_f(H)$.

Corollary 1.1.1. *The collection $\mathcal{K}(H)$ of compact operators on H is a (norm closed two-sided) ideal in $B(H)$. In particular, $\mathcal{K}(H)$ is a C^*-algebra.*

1.1.8 The Behncke-Leptin Construction

In this section we are going to review the Behncke-Leptin construction that associate a C^*-algebra to a given poset P. The goal is to find a suitable C^*-algebra A such that its primitive spectrum $Prim(A)$ equipped with the hull-kernel topology is exactly the poset P with its order topology. This question has been first solved by Behncke and Leptin who give a complete classification of all separable C^*-algebras with finite spectrum. Later, Elliott in [69, pp.35-39] extended this classification using K-theory, to all separable C^*-algebras with a decreasing chains condition on their spectrum. It turns out that the association between a poset P and the spectrum of a C^*-algebra is unique up to a class of equivalence of $\mathbb{N}$-valued map d over P. The map d is called a *defector* and measures intuitively the degree of freedom in dimension between two separable C^*-algebras which have the same spectrum but are non-isomorphic.
This review will follow Behncke-Leptin's paper [16] — a detailed treatment of the construction in the finite case can be found in the work of Ercolessi et al. [60] — starting from a C^*-algebra over a finite *forest* F, then over a finite poset P, before extending to a countable case.

C*-algebra associated to a finite forest

Definition 1.1.22 (Forest)**.** A partial ordered set $(F, \leq)$ is a forest if it additionally satisfies for every $x, y, z \in F$:

$$x \leq y \quad \text{and} \quad y \leq z \quad \Rightarrow \quad x \leq y \quad \text{or} \quad y \leq x.$$

In what will follow, every partially ordered set will be thought as a topological space equipped with the *Alexandrov topology* (see remark **??**) induced by its partial order.

Definition 1.1.23 (Defector)**.** A defector d associated to the poset P is an $\overline{\mathbb{N}}$-valued function on P such that $d(x) > 0$ for each maximal element of P. Two defectors d and d' are said to be equivalent if there exists an automorphism φ of P such that $d' = d \circ \varphi$ and denoted by $d \sim d'$.

For $x \in F$ we denote by F_x the closure of the point x i.e. the partially ordered set $\{y \in F \mid y \geq x\}$. The restriction of the defector d to a set F_x is denoted d_x. To construct the C^*- algebra $A(F, d)$ associated to F subjected to the defect d, we proceed as follows :

1) Define the separable Hilbert space $H(F, d)$ over the forest F and attached to each point $x \in F$ a subspace $H(x) \subseteq H(F, d)$.

2) Define a subset of operators $R(x)$ acting on $H(x)$ and extended by zero to the whole space $H(F, d)$.

3) Build the C^*-algebra A associated to the forest F as the algebra generated by the $R(x)$ when x run over F.

Proposition 1.1.4. *Denote by $\ell^2(\mathbb{Z})$ the Hilbert space of square-summable sequences. To each point $x \in F$, we associate the Hilbert space $H(x)$ defined as the product:*

$$H(x) = H^-(x) \otimes H^+(x) \tag{1.22}$$

with $H^-(x) = \bigotimes_{x_i < x} \ell^2_{x_i}(\mathbb{Z})$ and $H^+(x) = H(F_x, d_x)$ and where $\ell^2_x(\mathbb{Z}) \simeq \ell^2(\mathbb{Z})$ for each $x \in F$.

Proof. F is a forest. □

We are now ready to define the C^*-algebra $A(F,d)$, which have F as a structure space. Let us define $R(x)$ the algebra of operators acting on the Hilbert space $H(x)$ by

$$R(x) = 1_{H^-(x)} \otimes K(H^+(x)), \tag{1.23}$$

where 1_{H^-} is the identity algebra acting on the subspace $H^-(x)$ and $K(H^+)$ is the algebra of compact operators on the Hilbert subspace $H^+(x)$. We then extend $R(x)$ to an algebra of operators over the whole Hilbert space $H(F,d)$ by requiring that $R(x) \cdot H(x)^\perp = 0$. By construction, we have the following properties

$$R(x) \cdot R(y) \subset R(x) \quad \text{if} \quad x \leq y \quad \text{and} \quad R(x) \cdot R(y) = 0 \quad \text{if } x \not\lesseqgtr y. \tag{1.24}$$

Finally $A(F,d)$ is the algebra of operators generated by the $R(x)$ when x runs over F i.e.

$$A(F,d) := \bigoplus_{x \in F} R(x). \tag{1.25}$$

Theorem 1.1.7 (Behncke-Leptin [16]). *Let F be a finite forest equipped with a defector map, and $A(F,d)$ be the C^*-algebra acting on the Hilbert space $H(F,d)$ as defined previously. Then the following are true:*

(i) if E is a closed set of F with complement the open set U, then $I(E) := \oplus_{x\in U} R(x)$ is a closed two-sided ideal of $A(F,d)$, $A(E) := \oplus_{x \in E} R(x)$ is a closed subalgebra of $A(F,d)$ and

$$A(F,d) = I(E) \oplus A(E).$$

(ii) Every two-sided ideal is of the form $I(E)$ for some closed set $E \subset F$ and $I(E)$ is primitive if $E = \overline{\{x\}}$. In particular $\hat{A}(F,d) \simeq F$.

In order ton unify the notation, we will denote by $A(U)$ the closed to sided ideal $I(E)$ associated to a closed set E with complement the open set U.

Proposition 1.1.5. *Let E_1 and E_2 be closed sets of F with complement the open sets U_1 and U_2. Then we have the following identities:*

(i) $A(U_1 \cup U_2) = \sum_{x \in U_1 \cup U_2} \oplus R(x)$;

(ii) $A(U_1 \cap U_2) = A(U_1) \cap A(U_2) = \sum_{x \in U_1 \cap U_2} \oplus R(x)$;

(iii) $A(E_1 \cap E_2) = A(E_1) \cap A(E_2) = \sum_{x \in E_1 \cap E_2} \oplus R(x)$.

Proof. Identity (i) follows by definition, since the complement $(E_1 \cap E_2)^c$ is exactly the union $(U_1 \cup U_2)$.
Identities (ii) and (iii) follow from the fact that by construction

$$\sum_{x \in X} R(x) \cap \sum_{y \in Y} R(y) = \{0\}$$

unless $X \cap Y \neq \emptyset$. We start with identity (ii), the first result in Theorem 1.1.7 together with the preceding claim allows us to write

$$\begin{aligned} I(E_1) \cap I(E_2) &= \sum_{x \in U_1} \oplus R(x) \cap \sum_{y \in U_2} \oplus R(y), \\ &= \sum_{z \in U_1 \cap U_2} \oplus R(z) \oplus \left(\sum_{x \in U_1 - U_2} \oplus R(x) \cap \sum_{y \in U_2 - U_1} \oplus R(y) \right), \\ &= \sum_{z \in U_1 \cap U_2} \oplus R(z), \\ &= I(E_1 \cup E_2) =: A(U_1 \cap U_2). \end{aligned}$$

The proof of identity (iii) follows the same steps. □

We immediately deduce the generalization to arbitrary unions and finite number of intersections.

Corollary 1.1.2. *Let S, respectively V, be an arbitrary collection of open sets, respectively finite collection, in the forest F then the following holds:*

$$A\left(\bigcup_{U \in S} U\right) = \overline{\sum_{U \in S} A(U)}, \quad A\left(\bigcap_{U \in F} U\right) = \bigcap_{U \in F} A(U)$$

Proof. This follows from Proposition 1.1.5. □

C*-algebra associated to a finite poset

The case of a finite poset P can be treated now. This is done by noticing that P can be covered by a forest $\overline{P}$ together with a surjection $\varphi : \overline{P} \to P$. Then the problem of finding a C^*-algebra over a poset P can be deduced from the one over a forest.

Definition 1.1.24 (Rope)**.** A rope r of P is a (not necessarily maximal) chain in P connecting a minimal element of P with another element of P, which we will call the upper endpoint of r

Definition 1.1.25 (Covering forest)**.** We call $\overline{P}$ the set of ropes of P ordered by inclusion. We let $\varphi : \overline{P} \to P$ the surjection map that assigns to each rope $r \in \overline{P}$ its end point $\varphi(r) \in P$. We call the pair $(\overline{P}, \varphi)$ the covering forest of P.

Proposition 1.1.6. *Let $(\overline{P}, \varphi)$ be the covering forest of a poset P. Then the following hold*

(i) $\overline{P}$ can be written as a disjoint union of forests.

(ii) If $r, s \in \overline{P}$ are in the inverse image $\varphi^{-1}(x)$ of some $x \in P$, then the subforests $\overline{P}_r$ and $\overline{P}_s$ are isomorphic.

Proof. Follows by construction... □

The defector map defined on P can be pulled back to a defector map $\bar{d} := d \circ \varphi$ on $\overline{P}$. Then, since $\overline{P}$ is a forest, we can realised it as the structure space of the C^*-algebra $A(\overline{P}, \bar{d})$ acting on $H(\overline{P}, \bar{d})$. The C^*-algebra $A(P, d)$ associated to the poset P is then identified as a subalgebra of $A(\overline{P}, \bar{d})$. Starting from the first property in proposition 1.1.6 we can compute the Hilbert space $H(\overline{P}, \bar{d})$ using the previous construction for a forest. The space $H(P, d)$ will be then built out of it. We first define the space $H(x)$ for an element $x \in P$. Using the second property in proposition 1.1.6, for any $r, s \in \varphi^{-1}(x)$ we have

$$H^+(r) := H(P_r, d_r) = H(P_s, d_s) =: H^+(s), \tag{1.26}$$

we then identify $H^+(x)$ as $H^+(r)$ for some $r \in \varphi^{-1}(x)$. Then adapting the factorizing property of proposition 1.22, we define the Hilbert space $H(x)$ as

$$H(x) = \bigoplus_{r\in\varphi^{-1}(x)} H^-(r) \otimes H^+(x). \tag{1.27}$$

Finally, if we denote by S the finite set of all minimal points of P, then we define the global Hilbert space $H(P,d)$ associated to P by

$$H(P,d) := \bigoplus_{x\in S} H(x) \tag{1.28}$$

We can now turn back to the construction of the C^*-algebra $A(P,d)$ by noticing again that

$$K(H(P_r,d_r)) = K(H(P_s,d_s)) =: K(H^+(x)) \tag{1.29}$$

for any ropes r and s with the same endpoints $x \in P$. Therefore, the generating algebra $R(r)$ and $R(s)$ of $A(\overline{P},\bar{d})$ are given by

$$R(r) = 1_{H^-(r)} \otimes K(H^+(x)) \quad \text{and} \quad R(s) = 1_{H^-(s)} \otimes K(H^+(x))$$

For each point $x \in P$ we define the algebra $R(x)$

$$R(x) = \bigoplus_{r\in\varphi^{-1}(x)} 1_{H^-(r)} \otimes K(H^+(x)) \tag{1.30}$$

and the algebra $A(P,d)$ is generated by the subalgebra $R(x)$ for x running in all of P:

$$A(P,d) := \bigoplus_{x\in P} R(x). \tag{1.31}$$

Theorem 1.1.8 (Behncke-Leptin [16])**.** *The algebra $A(P,d)$ satisfies conditions (i) and (ii) of theorem 1.1.7 with F replaced by P.*

Generalization to a countable space

The Behncke-Leptin construction admits a straightforward generalization to the case of countable poset P. Even though this construction is done without a defector map d, it follows the same general steps of the finite case. Consider I to be the system of all non-trivial $\mathbb{N}$-valued functions on P, which have their support on finite subchains of P. For $x \in P$, one define $I(x) = \{\alpha \in I : \alpha(x) > 0\}$. Then one may write $I(x) = I^-(x) \times I^+(x)$ with $I^-(x)$, respectively $I^+(x)$, be the set of all restrictions of elements in $I(x)$ to $\{y : y < x\}$ respectively $\{y : y \geq x\}$. Therefore, if we let $H(x) = \ell^2(I(x))$ then the previous decomposition of $I(x)$ induces the splitting $H(x) = H^-(x) \otimes H^+(x)$, with $H^{\pm}(x) = \ell^2(I^{\pm}(x))$. Therefore, one may define the algebra $R(x) = 1_{H^-} \otimes K(H^+)$ acting on the whole space H. For any subset $Q \subset P$ we associate the algebra

$$A(Q) = \langle R(x), x \in Q \rangle \,. \tag{1.32}$$

Again similarly to the finite case the algebra $R(x)$ satisfy property (1.24).

Proposition 1.1.7 (Behncke-Bös [16, 11])**.** *Let $Q \subset P$ be a finite subset, then the following are true:*

(i) $A(Q) = \bigoplus_{x \in Q} R(x)$,

(ii) $Prim(A(Q)) \simeq Q$,

(iii) $A(P) = \varinjlim A(Q)$.

We say that an open set $Q \subset P$ is a *p-open set* if for any $x, y \in Q$ there exists a $z \in Q$ such that $z \leq x, y$. Moreover, for an ideal $J \in A$ we define $P(J) = \{x \in P : R(x) \subset J\}$.

Proposition 1.1.8 (Behncke-Leptin [16])**.** *Every ideal J of A satisfies $J = A(P(J))$. Therefore, the map $J \to P(J)$ defines a bijective order-preserving map of the set of all ideals of A onto the system of all open sets of P. Moreover, primitive ideals are mapped on to p-open sets.*

1.2 Differential Geometry

In this section, we recall some fundamental machineries of differential geometry; mainly vector bundles, principal bundles and connections.

1.2.1 Vector bundles and Principal bundles

We start with the definition of vector bundle.

Definition 1.2.1 (Vector bundle). Let M be a manifold and k be $\mathbb{R}$ or $\mathbb{C}$. A k-vector bundle of rank n over M is a pair (E, π) where E is a manifold and $\pi : E \to M$ is a surjective submersion, such that there is a cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M satisfying

1) The cover $\mathfrak{U}$ trivializes E, that is, for every $\alpha \in A$ there exists a diffeomorphism $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times k^n$ such that the digram

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\psi_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times k^n \arrow[dl, "\pi"] \\
U_\alpha &
\end{tikzcd}$$

commutes.

2) For all $\alpha \in A$ and $p \in U_\alpha$, $\pi^{-1}(p)$ is a k-vector space and the map

$$\psi_\alpha|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \{p\} \times k^n \tag{1.33}$$

is an isomorphism of vector spaces.

The pair (U_α, ψ_α) is called a local trivialization. Vector bundles are equivalently characterized by transition maps between two local trivializations. For $\alpha, \beta \in A$, let $U_{\alpha\beta} = U_\alpha \cap U_\beta$. The maps

$$\widetilde{g}_{\alpha\beta} = \psi_\beta \circ \psi_\alpha^{-1} : U_{\alpha\beta} \times k^n \to U_{\alpha\beta} \times k^n \tag{1.34}$$

satisfy $\widetilde{g}_{\alpha\beta}(p, v) = (p, g_{\alpha\beta}(p)v)$, where $g_{\alpha\beta}(p) \in GL_n(k)$. The corresponding maps

$$g_{\alpha\beta} : U_{\alpha\beta} \to GL_n(k) \tag{1.35}$$

are called the *gluing maps*. These maps satisfy the following conditions, for all

$\alpha, \beta, \gamma \in A$ and

$$g_{\alpha\alpha}(p) = id_n, \quad \forall p \in U_\alpha, \tag{1.36}$$

$$g_{\beta\alpha}(p) = g_{\alpha\beta}(p)^{-1}, \quad \forall p \in U_\alpha, \tag{1.37}$$

$$g_{\beta\gamma}(p)g_{\alpha\beta}(p) = g_{\alpha\gamma}(p), \quad \forall p \in U_{\alpha\beta\gamma}. \tag{1.38}$$

Proposition 1.2.1. *Given a cover $\mathfrak{U} = \{U_\alpha\}_\alpha$ of a manifold M and a family of smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to GL_n(k)$ satisfying (1.36), (1.37) and (1.38), there exists a unique (up to isomorphism) vector bundle $\pi : E \to M$ with trivializing cover $\mathfrak{U}$ and gluing maps $g_{\alpha\beta}$.*

Furthermore, the usual operator on vector spaces, such as duality, direct sum, tensor product, exterior product, etc... extends to operations on vector bundles through the gluing maps.

Example 1.2.1. Let $E = (\mathfrak{U}, g^E_{\alpha\beta})$ and $F = (\mathfrak{U}, g^F_{\alpha\beta})$ be two vector bundles over the same trivializing cover. Then we define the following bundles:

i) The dual bundle E^* by
$$E^* = (\mathfrak{U}, (g^*_{\alpha\beta})^{-1}) \tag{1.39}$$

ii) The direct sum $E \oplus F$ by
$$E \oplus F = (\mathfrak{U}, g^E_{\alpha\beta} \oplus g^F_{\alpha\beta}) \tag{1.40}$$

iii) The tensor product $E \otimes F$ by
$$E \otimes F = (\mathfrak{U}, g^E_{\alpha\beta} \otimes g^F_{\alpha\beta}) \tag{1.41}$$

iv) The symmetric and exterior powers $\mathrm{Sym}^k E$ and $\bigwedge^k E$ by
$$\mathrm{Sym}^k E = (\mathfrak{U}, \mathrm{Sym}^k g_{\alpha\beta}) \quad \bigwedge^k E = (\mathfrak{U}, \wedge^k g_{\alpha\beta}) \tag{1.42}$$

v) The determinant line bundle $\det(E)$ by

$$\det(E) = \bigwedge^{rk(E)} E. \tag{1.43}$$

We now introduce principal bundle. Briefly these are fiber bundles where the fibers are Lie groups.

Definition 1.2.2 (Principal bundle)**.** Let G be a Lie group and M be a manifold. A principal G-bundle is a triple (P, π, M) such that

1) $\pi : P \to M$ is a smooth submersion.

2) There is a free and transitive right action $P \times G \to P$ such that π is G-invariant that is, $\pi(pg) = \pi(p)$.

3) There exists a trivialization cover $\mathfrak{U}$ that is, a cover of M such that for every $\alpha \in A$ there exists a diffeomorphism $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times k^n$ such that the digram

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\psi_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times G \arrow[dl, "\pi"] \\ U_\alpha & \end{tikzcd}$$

commutes and $\Psi_\alpha(pg) = \Psi_\alpha(p)g$.

The archetype of a principal bundle is given by the example of the *frame bundle.* For any real vector space V, let $\mathrm{Fr}(V)$ be the set of all ordered bases in V. Suppose that V has dimension n. We represent an ordered basis by $e = (e_1, \dots, e_n)$. Let $\pi : E \to M$ be a rank n vector bundle. Then, the *frame bundle* $\mathrm{Fr}(E)$ of E is the smooth manifold obtained as:

$$\mathrm{Fr}(E) = \sqcup_{x \in M} \mathrm{Fr}(E_x). \tag{1.44}$$

There is a natural projection map $\pi : \mathrm{Fr}(E) \to M$ that maps $\mathrm{Fr}(E_x)$ to $\{x\}$. One can define a right $GL_n(k)$-action on $\mathrm{Fr}(E)$ as follows. Let $\{U_\alpha\}_\alpha$ be a trivialization cover for E. On a trivialization chart $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times k^n$, the right action is given by:

$$e \cdot g = \psi_\alpha^{-1}\left(g^{-1}\psi_\alpha(e_1), \dots, g^{-1}\psi_\alpha(e_n)\right). \tag{1.45}$$

This gives $\mathrm{Fr}(E)$ the structure of a principal $GL_n(k)$-bundle.

In some sense, vector bundles and principal bundles are two sides of the same coin. On one hand if $E = (\mathfrak{U}, g_{\alpha\beta})$ is a vector bundle, then we see that its frame bundle is a principal $Gl_n(k)$-bundle $P = (\mathfrak{U}, g_{\alpha\beta})$. On the other hand, one can construct a vector bundle from the data of a principal bundle.

Definition 1.2.3 (Associated vector bundle)**.** Given a principal G-bundle $P = (\mathfrak{U}, g_{\alpha\beta})$, and a representation (ρ, V) in $End(V)$, we define the *associated vector bundle* $P \times_\rho V$ by

$$E = (\mathfrak{U}, \rho(g_{\alpha\beta})). \tag{1.46}$$

1.2.2 Connections

Heuristically, a *connection* on a fiber bundle is a tool that allows to move in a consistent way from one fiber in the bundle to the other. The concept of connections exists over both vector bundles and principal bundles.

Definition 1.2.4 (Connection on vector bundle)**.** Let $\pi : E \to M$. Then a *connection on* E is a linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) \tag{1.47}$$

such that, for $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$, the Leibniz rule

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma. \tag{1.48}$$

Example 1.2.2. On the trivial bundle $M \times k^n$ we have a connection given by the de Rham differential $(f_1, \dots, f_n) \mapsto (df_1, \dots, df_n)$. This connection is called the *trivial connection*.

Proposition 1.2.2. *If it is not empty, the space of connections $\mathcal{A}_E$ is an affine space modeled on the vector space $\Omega^1(End(E))$.*

In particular, every connection on the trivial bundle $M \times k^n$ is of the form $\nabla = d + A$, where $A \in \Omega^1(\mathrm{End}(k^n))$. In a basis of k^n, one writes $A(e_j) = A^i_j e_i$, then

$$\nabla(f_1, \ldots, f_n) = (df_1, \ldots, df_n) + (A^i_1 f, \ldots, A^i_n f_i). \tag{1.49}$$

Thus one can think A as a 1-form with values in matrices of matrices of 1-form.

Moreover, if one consider a local trivialization $\Psi_\alpha : E|_{U_\alpha} \to U_\alpha \times k^n$. Then a connection ∇ on E induces a connection on $U_\alpha \times k^n$, hence an element $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(k^n))$ and one has:

$$(\nabla\sigma)_\alpha = d\sigma_\alpha + A_\alpha \sigma_\alpha. \tag{1.50}$$

This is called the connection 1-form of ∇ in U_α. There is a relation between two 1-forms A_α and A_β of ∇.

One can also define a connection on a principal bundle. There is in addition a strong relation between connections on vector bundles and connections on principal bundles.

Definition 1.2.5. Let $P = (\mathfrak{U}, g_{\alpha\beta})$ be a principal bundle with structure group $G \subset GL_n(k)$, i.e. G is a matrix group. Then a connection on P is equivalent to a collection of 1-forms $A_\alpha \in \Omega^1(M, \mathfrak{g})$ such that

$$A_\beta = g_{\alpha\beta} A_\alpha g_{\alpha\beta}^{-1} - dg_{\alpha\beta} g_{\alpha\beta}^{-1}. \tag{1.51}$$

Proposition 1.2.3. *A connection ∇ on a vector bundle E induces a connection Ω on the bundle of frames $Fr(E)$ and vice versa.*

Hence, the crucial idea here is that one can study connections on vector bundles by studying connections on principal bundles.

1.3 Spin Geometry

We are now ready to define the notion of Spin structure on a Riemannian manifold (M, g) which is the natural setting of the Dirac operator.

1.3.1 Clifford algebras

Definition

Let (V, q) be an n-dimensional vector space endowed with a quadratic from q and provided with an orthonormal basis $\{e_i\}_{i=1}^n$. We denote by η the bilinear form induced by polarizing q.

Definition 1.3.1. A *Clifford algebra* is an associative algebra $Cl(V, q)$ over the field $\mathbb{R}$ with generators $1, e_1, \ldots, e_n$ satisfying the relations

$$e_i^2 = -1, \quad e_i e_j + e_j e_i = 0 \quad \text{for } i \neq j. \tag{1.52}$$

We shall also denote it by $Cl(V)$ or by $Cl(n)$, dropping the quadratic form q to simplify the notation.

It follows from the given definition that $V \subset Cl(V, q)$ and

$$uv + vu = -2\eta(u, v), \quad u, v \in V. \tag{1.53}$$

As a vector space, $Cl(V, q)$ has dimension 2^n and can be provided with the basis given by 1 and elements of the form

$$e_I := e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_k} \tag{1.54}$$

where $I = \{i_1, i_2, \ldots, i_k\}$ is a strictly increasing subset of indices with $|I| := k$ elements taken from the set $\{1, 2, \ldots, n\}$. In particular, any element $x \in Cl(V)$ can be written in the form

$$x = \sum_I x_I e_I \tag{1.55}$$

where we add to the collection $\{I\}$ of sets of indices the subset $I = 0$ and put $e_0 := 1$.

Denote by $Cl_k(V)$ the subset $Cl(V)$ consisting of elements of degree k which are linear combinations of basis elements e_I with $|I| = k$. We introduce also the following subsets of $Cl(V)$:

$$Cl^{ev}(V) := \bigoplus_{k \text{ even}} Cl_k(V), \quad Cl^{od}(V) := \bigoplus_{k \text{ odd}} Cl_k(V). \tag{1.56}$$

Then $Cl^{ev}(V)$ will be unital subalgebra in $Cl(V)$ and

$$Cl(V) = Cl^{ev}(V) \oplus Cl^{od}(V) \tag{1.57}$$

which provided $Cl(V)$ with the structure of a *superalgebra*. In fact the Clifford algebra $Cl(V)$ does not depend on the choice of orthonormal basis $\{e_i\}$ and can be in fact defined form the following universal property.

Definition 1.3.2. The Clifford algebra $Cl(V, q)$ is a unique associative $\mathbb{R}$-algebra with unit which contains the quadratic space V and has the following property: for any associative $\mathbb{R}$-algebra A with unit 1_A and any linear map $f : V \to A$, satisfying the condition

$$f(v) \cdot f(v) = -q(v)1_A, \tag{1.58}$$

there exists a unique extension of f to an algebra homomorphism $\widetilde{f} : Cl(V, q) \to A$ such that the following diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "i"'] & Cl(V,q) \\ A \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

is commutative.

One can extend the previous definition to complex vector space provided with a non-degenerate bilinear form.

We introduce also the complexified Clifford algebra $Cl^c(V)$ an n-dimensional real vector space V by setting

$$Cl^c(V) := Cl(V) \otimes_{\mathbb{R}} \mathbb{C}. \tag{1.59}$$

This construction is particularly useful to determine the Clifford representations.

We denote by $Cl^{\times}(V)$ the *group of invertible elements* of the Clifford algebra $Cl(V)$. It is a Lie group which contains $V - \{0\}$.

Definition 1.3.3. The *Clifford group* $\mathit{\Gamma}(V)$ is the subgroup of multiplicative group $Cl^{\times}(V)$ generated by the elements $v \in V - \{0\}$.

Since every element of the group $\mathit{\Gamma}(V)$ generates a non-degenerate linear transform of the space V so we have a homomorphism

$$\pi : \mathit{\Gamma}(V) \to Gl(V). \tag{1.60}$$

This homomorphism takes values in the orthogonal group $O(V)$ and can be included into the exact sequence of group homomorphisms of the form

$$1 \to \mathbb{R}^{\times} \to \mathit{\Gamma}(V) \xrightarrow{\pi} O(V) \to 1. \tag{1.61}$$

Spinor Groups

We can now define the Pin and Spin group as subgroup of the invertible elements in the Clifford algebras.

Definition 1.3.4. The group $Pin(V)$ is defined as the subgroup of the Clifford group $\mathit{\Gamma}(V)$ generated by the unit vectors from V, i.e. by vectors $v \in V$ with $q(v) = 1$.

As in the case of the Clifford group, we have a homomorphism

$$\pi : Pin(V) \to O(V) \tag{1.62}$$

which is included into the exact sequence of group homomorphisms

$$1 \to \mathbb{Z}_2 \to Pin(V) \xrightarrow{\pi} O(V) \to 1. \tag{1.63}$$

Definition 1.3.5. The group $Spin(V)$ is the identity connected component of the group $Pin(V)$. It can be also defined as

$$Spin(V) = Pin(V) \cap Cl^{ev}(V). \tag{1.64}$$

As in the case of the Clifford group, there is an exact sequence of group homomorphisms

$$1 \to \mathbb{Z}_2 \to Spin(V) \xrightarrow{\pi} SO(V) \to 1. \tag{1.65}$$

For $n > 2$, the group $Spin(n)$ is a simply connected covering group of the group $SO(V)$.

Spinor representations

Definition 1.3.6. Let (V, q) be a quadratic k-vector space, where k is $\mathbb{R}$ or $\mathbb{C}$. A *Clifford representation* is a homomorphism

$$c : Cl(V) \to End_{\mathbb{C}}(S) \tag{1.66}$$

from the Clifford algebra $Cl(V)$ into the algebra of linear operators acting in a complex vector space S called the *Clifford module* over $Cl(V)$ or the *spinor space* for the algebra $Cl(V)$. We shall assume that S is provided with an Hermitian inner product.

The standard definitions and properties from the representation theory of associative algebras apply also to Clifford representations.

The action of the representation c on the space S is often denoted by

$$c(x)s := x \cdot s \tag{1.67}$$

for $x \in Cl(V)$, $s \in S$, and called the *Clifford multiplication.*

It turns out that the complex Clifford algebra $Cl^c(n)$ is either a matrix algebra (for n even) or two copies of a matrix algebra (for n odd).

Proposition 1.3.1. *Let $n \in \mathbb{N}$. Then we have*

$$Cl^c(n) \simeq \begin{cases} M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}) & n = 2k+1 \ \textit{odd} \\ M_{2^k}(\mathbb{C}) & n = 2k \ \textit{even} \end{cases} \tag{1.68}$$

Hence, there is exactly one irreducible representation of $Cl^c(n)$ for n even, and two irreducible representations for n odd. The irreducible representations of $Cl^c(n)$ are known as the *complex spinors* and denoted by (c, Δ_n).

Given the spinor representations of the complex Clifford algebra, we can now easily define the Spin representations of the spin groups.

Definition 1.3.7 (Complex spin representation)**.** Let $Spin(n) \subset Cl_(n) \subset Cl^c(n)$. Then the *complex spin representation* of $Spin(n)$ is the restriction of the complex spin representation of $Cl^c(n)$. We denote the complex spin representation by Δ_n.

Proposition 1.3.2. *When n is even, the representation Δ_n is irreducible. When n is odd, the representation $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ splits into the direct sum of two irreducible representations.*

Spin Structures

Let (M, g) be an oriented n-dimensional Riemannian manifold. Then, we can define the bundle $SO(M)$ of oriented orthonormal frames of TM: the fiber over $x \in M$ is the collection of all orientation-preserving isometries. This is a principal $SO(n)$-bundle and we have

$$TM \simeq SO(M) \times \iota\mathbb{R}^n \tag{1.69}$$

where $\iota : SO(n) \to Gl_n(k)$ denotes the inclusion of representations. Conversely, an $SO(n)$-structure on TM i.e. a principal $SO(n)$-bundle P such that $TM \simeq P \times_\iota \mathbb{R}^n$, defines an orientation and a Riemannian metric on M, by declaring the fiber over x to consist of oriented orthonormal frames. Now, recall that we have the short exact sequence

$$1 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\pi} SO(n) \to 1. \tag{1.70}$$

Definition 1.3.8. A *spin structure* on M is a principal $Spin(n)$ bundle P together with an isomorphism

$$TM \simeq P \times_\pi \mathbb{R}^n \tag{1.71}$$

The existence of a spin structure P implies the existence of an $SO(n)$-structure $\pi(P)$ on TM, hence gives M the structure of an oriented Riemannian manifold. Conversely, given an oriented Riemannian manifold M with tangent bundle $TM = (\mathfrak{U}, g_{\alpha\beta})$, a spin structure is a collection of lifts $\widetilde{g}_{\alpha\beta}$ such that the following diagram commutes:

$$\begin{tikzcd} & Spin(n) \arrow[d, "\pi"] \\ U_{\alpha\beta} \arrow[ur, dashed, "\widetilde{g}_{\alpha\beta}"] \arrow[r, "\widetilde{g}_{\alpha\beta}"] & SO(n) \end{tikzcd}$$

and that satisfy the gluing relations for a principal $Spin(n)$-bundle:

$$\begin{aligned} \widetilde{g}_{\alpha\alpha} &= 1, \\ \widetilde{g}_{\alpha\beta} &= (\widetilde{g}_{\beta\alpha})^{-1}, \\ \widetilde{g}_{\alpha\gamma} &= \widetilde{g}_{\beta\gamma}\widetilde{g}_{\alpha\beta}. \end{aligned}$$

The only non-trivial question is whether one can find lift satisfying the cocyle condition. The existent of such structure is then intimately tied to algebraic topology properties. One can use Čech cohomology machinery to investigate existence, however this falls out of the scope of this thesis work, one can refer to [107] for more details on this topic.

Example 1.3.1. We should now give some examples and non-examples of spin manifolds.

1 A genus g Riemann surface admits 2^{2g} inequivalent spin structures.

2 If $H^2(M, \mathbb{Z}_2)$ vanishes then, M is spin. For example, S^n is spin for all $n \neq 2$. (Note that S^2 is also spin, but for different reasons).

3 All even -dimensional complex projective spaces $\mathbb{C}P^{2n}$ are not spin.

4 All odd-dimensional complex projective spaces $\mathbb{C}P^{2n+1}$ are spin.

5 All compact, orientable manifolds of dimension 3 or less are spin.

1.3.2 Clifford Modules and Dirac Operators

Let (M, g) be an n-dimensional Riemannian manifold with a spin structure P.

Definition 1.3.9. The *spinor bundle* S_n associated to P is the associated bundle to the $Spin(n)$-bundle P via the complex spin representation:

$$S_n = P \times_c \Delta_n \tag{1.72}$$

Now, we define a bundle of Clifford algebras over M acting on the spinor bundle.

Definition 1.3.10. The *Clifford bundle* is the vector bundle over M with typical fiber the Clifford algebra $Cl(M)_x := Cl(T_x^*M, g_x)$.

In order to see how the Clifford bundle acts on the spinor bundle, one can also define the Clifford bundle as an associate bundle.

Definition 1.3.11. The Clifford bundle is a bundle associated with the bundle of oriented orthonormal frames:

$$Cl(M) = SO(M) \times_\rho Cl(n) \tag{1.73}$$

where $\rho : SO(n) \to Aut(Cl_n)$ is an embedding of $SO(n)$ which acts then by automorphisms.

Proposition 1.3.3. *The Clifford multiplication* $\mathbb{R}^n \times \Delta_n \to \Delta_n$*,* $(v, s) \to c(v)s$*, extends to a map of sections*

$$c : \Gamma(T^*M) \times S_n \to S_n \quad (\theta, \psi) \to c(\theta)\psi. \tag{1.74}$$

Definition 1.3.12 (Spin connection)**.** Let ∇ be a connection on T^*M with the 1-forms $A_\alpha \in \Omega^1(U_\alpha, so(n))$. The connection $\widetilde{\nabla}$ on the vector bundle S_n defined by the 1-forms $B_\alpha = c \circ \rho_*^{-1}(A_\alpha) \in \Omega^1(U_\alpha, \mathrm{End}(\Delta_n))$ is called the *spin connection* associated to ∇.

We have now all the ingredients to define Dirac operators. Let (M, g) be a Riemannian manifold with spin structure P. This structure induces a bundle of spinors S_n. The lift of the Levi-Civita connection ∇_g to the spin bundle is a connection $\widetilde{\nabla}_g$ on S_n.

Definition 1.3.13. The *Dirac operator* $D : \Gamma(S_n) \to \Gamma(S_n)$ associated to P is given by the composition

$$\Gamma(S_n) \xrightarrow{\widetilde{\nabla}_g} \Gamma(T^*M \otimes S_n) \xrightarrow{c} \Gamma(S_n) \tag{1.75}$$

In a local orthonormal frame, the Dirac operator is given by $D = \sum_i c(e_i)\widetilde{\nabla}_{e_i}$. In particular, if $M = \mathbb{R}^n$ with the euclidean metric, we have $D = c(e_i)\frac{\partial}{\partial x^i}$ (the spinor bundle and spin connection are trivial) and hence

$$D^2 f = c(e_i)\frac{\partial}{\partial x^i}\left(c(e_j)\frac{\partial}{\partial x^j}\right) = -\sum \frac{\partial^2}{\partial (x^i)^2} f. \tag{1.76}$$

Remark 1.3.1. There is relation between the exterior derivative df and the Dirac operator given by the commutator with $f \in C^\infty(M)$:

$$[D, f] = c(df). \tag{1.77}$$

If $\dim M = 2k$ is even, the bundle $S_n = S_n^+ \oplus S_n^-$ and we have

$$D^+ := D|_{S_n^+} : S_n^+ \to S_n^-$$
$$D^- := D|_{S_n^-} : S_n^- \to S_n^+.$$

1.4 Noncommutative Geometry

We present a brief description of the main ideas of noncommutative differential geometry. We put the emphasis on the notion of spectral triples as the central idea in the field and one of the main tools used in this thesis.

1.4.1 Infinitesimals and the Dixmier Trace

Let us recall that if T is a compact (not necessarily self-adjoint) operator on a Hilbert space H, then there exists a decreasing sequence $\{\mu_n(T)\}$ of non-negative real numbers and an orthonormal basis $\{e_n\}$ for $(\ker T)^\perp$ such that for all $h \in H$

$$Th = \sum_{n=1}^{\infty} \mu_n(T)\langle h, e_n\rangle e_n. \tag{1.78}$$

In the noncommutative geometry formalism, compact operators play the role of *infinitesimals*. The size of an infinitesimal $T \in \mathcal{K}(H)$ is measured by the decay of the sequence of its singular values $\{\mu_n(T)\}$ with respect to n.

Definition 1.4.1 (Infinitesimals)**.** Let α be a non-negative real number. An infinitesimal of order α is a compact operator $T \in \mathcal{K}(H)$ such that

$$\mu_n(T) = O(n^{-\alpha}) \quad \text{as } n \to \infty.$$

Moreover, that this definition is sensible is confirmed by the following facts:

a) if a_i is an infinitesimal of order α_i, for $i = 1, 2$, then the product $a_1 a_2$ is an infinitesimal of order $\alpha_1 + \alpha_2$.

b) if a is an infinitesimal of order α, and b is any bounded operator, then ab and ba are infinitesimals of order α: one can think of the heuristic infinitesimal $f(x)dx$, where $f(x)$ is a function and dx is an infinitesimal of order 1.

c) One can define a noncommutative integral such that order 1 infinitesimals can be integrated, and higher order infinitesimals have a null integral.

We discuss briefly on this noncommutative integral which relies on the Dixmier trace. Heuristically, the usual trace of an infinitesimal of order 1 is at most logarithmically divergent. The Dixmier trace is a tool that extracts the coefficient of the logarithmic divergence, even though the partial sums $\frac{1}{\ln N}\sum_{n=0}^{N-1}\mu_n(a)$ are not necessarily convergent.

The space $L^{1,\infty}(H)$ is the space of compact linear operators T on H such that the norm

$$\|T\|_{1,\infty} = \sup_N \frac{\sum_{i=1}^N \mu_i(T)}{\log(N)} \tag{1.79}$$

is finite. Then, let

$$a_N = \frac{\sum_{i=1}^N \mu_i(T)}{\log N}. \tag{1.80}$$

The Dixmier trace $\mathrm{Tr}_\omega(T)$ of T is defined for positive operators T of $L^{1,\infty}(H)$ to be

$$\mathrm{Tr}_\omega(T) = \lim_\omega a_N \tag{1.81}$$

where $\lim_\omega$ is a scale-invariant positive extension of the usual limit, to all bounded sequences. Therefore, the Dixmier trace of an operator depends, a priori, on the choice of extension ω. When this is not the case i.e. when the value of Dixmier trace is independent of ω, then the operator is called *measurable*.

One of the fundamental results on the Dixmier trace is obtained on a manifold M, where Connes showed that Wodzicki's noncommutative residue (we refer to [37] for a definition) of a pseudodifferential operator on a manifold M of order $-\dim(M)$ is equal to its Dixmier trace.

1.4.2 Spectral triples

Definition 1.4.2 (Spectral triple)**.** A *spectral triple* is the data $(\mathcal{A}, \mathcal{H}, D)$ where:

(i) $\mathcal{A}$ is a real or complex $*$-algebra;

(ii) $\mathcal{H}$ is a Hilbert space and a left-representation $(\pi, \mathcal{H})$ of A in $\mathcal{B}(\mathcal{H})$;

(iii) D is a *Dirac operator*, which is a self-adjoint operator on $\mathcal{H}$.

We require in addition that the Dirac operator satisfies the following conditions

a) The resolvent $(D - \lambda)^{-1}$, $\lambda \notin \mathbb{R}$, is a compact operator on H.

b) $[D, a] \in B(H)$, for any $a \in A$.

If in addition, $\mathcal{H}$ is equipped with a $\mathbb{Z}_2$-grading i.e. there exists a unitary self-adjoint operator $\gamma \in \mathcal{B}(\mathcal{H})$ such that

1) $[\gamma, \pi(a)] = 0$ for all $a \in \mathcal{A}$,

2) γ anticommutes with D,

then the spectral triple is said to be *even*. Otherwise, it is said to be *odd*. In the case where $\mathcal{H}$ is finite dimensional, then the triple $(\mathcal{A}, \mathcal{H}, D)$ is called a discrete spectral triple.

1.4.3 Distance and integration on a spectral triple

Assume that $\mathcal{A}$ is a C^*-algebra. Then, a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ as defined above, also induces a distance on the space of states $S(A)$ defined by:

$$d(\phi, \psi) := \sup_{a \in \mathcal{A}} \{ |\phi(a) - \psi(a)| \ : \ \| [D, a] \| \leq 1 \} \,, \ \forall \phi, \psi \in S(A). \tag{1.82}$$

To complete the general picture, one needs an analogue of measure theory. For this purpose, one needs first to define the additional notion of dimension of a spectral triple.

Definition 1.4.3. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be of dimension $n > 0$ if $|D|^{-1}$ is an infinitesimal of order n^{-1} or, equivalently, $|D|^{-n}$ is an infinitesimal of order 1.

Having a spectral triple of dimension n, one can define the *integral* of any $a \in \mathcal{A}$ using the Dixmier trace of the Dirac operator as follows:

$$⨍ a := \frac{1}{\Lambda} \mathrm{Tr}_\omega a |D|^{-n}. \tag{1.83}$$

where Λ is a constant determined by the behaviour of the characteristic values of $|D|^{-n}$, namely one asks that $\mu_j \leq \Lambda j^{-1}$ as $j \to \infty$.

1.4.4 The canonical triple over a manifold

In order to understand the spectral triple philosophy, it is enlightening to study the canonical one on an n-dimensional Riemannian spin manifold (M, g). Indeed, one can show that part of the geometric information contained in the Riemannian data (M, g) can be instead described by the triple $(\mathcal{A}, \mathcal{H}, D)$ defined as follows.

1. $\mathcal{A} = \mathcal{F}(M)$ is the algebra of complex valued smooth functions on M.

2. $\mathcal{H} = L^2(M, S)$ is the Hilbert space of square integrable sections of the irreducible spinor bundle over M with rank equal to $2^{[n/2]}$. The scalar product in $L^2(M, S)$ is obtained for the measure associated to the metric g and is given by
$$(\psi, \phi) := \int_M \overline{\psi(x)}\phi(x)d\mu(g). \tag{1.84}$$

3. D is the Dirac operator associated to the Levi-Civita connection ω of the metric g.

If the dimension n of M is even, the spectral triple is even by taking for grading operator the volume form,
$$\gamma = i^{n/2} e_1 \cdots e_n \tag{1.85}$$
which anticommutes with the Dirac operator,
$$\gamma D + D\gamma = 0. \tag{1.86}$$
Furthermore, the factor $i^{n/2}$ ensures that,
$$\gamma^2 = 1, \quad \gamma^* = \gamma. \tag{1.87}$$

Proposition 1.4.1. *Let $(\mathcal{A}, \mathcal{H}, D)$ be the canonical triple over the manifold M as defined above. Then:*

a) The space M is the structure space of the algebra $\overline{\mathcal{A}}$ of continuous functions on M, which is the norm closure of $\mathcal{A}$.

b) The geodesic distance between any two points on M is given by
$$d(p, q) = \sup_{f \in \mathcal{A}} \{|f(p) - f(q)| \ : \ \| [D, f] \| \leq 1\}, \ \forall p, q \in M. \tag{1.88}$$

c) The Riemann integration on M is given by

$$\int_M f d\mu(g) = c(n)\, Tr_\omega(f|D|^{-n}),\ \forall f \in \mathcal{A}, \tag{1.89}$$

where the coefficients $c(n)$ depends on the Gamma function and is given by

$$c(n) = 2^{n-[n/2]-1}\pi^{n/2} n\Gamma\left(\frac{n}{2}\right). \tag{1.90}$$

1.4.5 A two point space example

Consider the space of two points $Y = \{1, 2\}$. The space Y is identified with the spectrum of the algebra $A = \mathbb{C} \oplus \mathbb{C}$ and any element $a \in A$ is a couple of complex numbers (a_1, a_2) with $a_i = a(i)$ for $i = 1, 2$. Aeven spectral triple (A, H, D, γ) can be defined on Y as follows. Consider a finite dimensional Hilbert space H decomposable as a direct sum $H = H_1 \oplus H_2$, defining a representation of A as diagonal matrices

$$A \ni a \mapsto \begin{pmatrix} a_1 1_{H_1} & 0 \\ 0 & a_2 1_{H_2} \end{pmatrix} \in B(H) \tag{1.91}$$

One then identifies any element of A with its matrix representation.
The operator D is defined as a 2-by-2 off-diagonal block matrix:

$$D = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix} \tag{1.92}$$

where $M : H_1 \to H_2$ is some linear operator. The parity element γ is given by:

$$\gamma = \begin{pmatrix} 1_{H_1} & 0 \\ 0 & 1_{H_2} \end{pmatrix} \tag{1.93}$$

Taking $a \in A$, one defines the derivation da using the commutator with D:

$$da = \frac{i}{\hbar}[D, a] = (a_2 - a_1)\begin{pmatrix} 0 & M^* \\ -M & 0 \end{pmatrix} \tag{1.94}$$

If one takes $H_1 = H_2$ and M the identity element, then in turns, the norm of da is given by $\|da\| = \hbar^{-1}|a_2 - a_1|$. Therefore, the commutative distance between the two point of Y is given by

$$d(1,2) = \sup_{a \in A} \{|a_2 - a_1| \ : \ \|da\| \leq 1\} = \hbar. \tag{1.95}$$

Chapter 2

Noncommutative geometry on triangulations

2.1 Preliminaries

Unless stated otherwise, we will consider M to be a smooth compact connected manifold M of dimension d.

2.1.1 Triangulation and posets

Let K be an *abstract simplicial complex* with elements σ and $|K|$ its geometric realization. The dimension of a simplex $\sigma \in |K|$, denoted $\dim(\sigma)$, is the dimension of the smallest affine space containing σ. The set K can be written as a union of subsets $K(n)$, where $\sigma^n \in K(n)$ is a simplex of dimension n. The subset $K(0)$, also denoted V, is the set of vertices; the subset $K(1)$, also denoted E, is the set of edges. A manifold M admits a *triangulation* $\mathcal{T}(K)$ if there exists a simplicial complex K and homeomorphism $\varphi : |K| \to M$ between M and the geometric realization $|K|$. We recall the following theorem due to Whitney on the existence of a triangulation.

Theorem 2.1.1 ([127, pp.124-135]). *Every k-smooth manifold M admits a triangulation, for $k \geq 1$.*

To every simplicial complex K, one can associate a partially ordered set (poset) $P(K)$ which is defined to be the poset of nonempty faces ordered by inclusion. We will denote by $\leq$ the partial order on $P(K)$. The preorder $\leq$ induces a topology $P(K)$ called the *Alexandrov topology* and generated by the bases of open sets $\mathcal{B} := \{U_x := \{y \in X : y \leq x\} : x \in X\}$. For instance, Figure 2.2 show the poset associate to a triangulation of the circle S^1.

Conversely, to every poset X, one can associate an abstract simplicial complex $K(X)$, where the simplices are nonempty chains in X.

A map $f : X \to Y$ between posets is continuous if and only if it is order preserving with respect to the orders associated with the order topologies on X and Y. The map f induces a simplicial map $K(f) : K(X) \to K(Y)$; vice-versa to every simplicial map $f : K \to L$, one can associate a continuous map $P(f) : P(K) \to P(L)$ between posets.

One can reverse the order $\leq$ on a poset X and define the space X^{op}. These spaces have the same underlying set. Open sets in X correspond to closed sets in X^{op} and vice-versa. Moreover, a continuous map $f : X \to Y$ induces a continuous map $f^{op} : X^{op} \to Y^{op}$ and vice-versa.

Finally, the space $K(P(K))$ is called *the barycentric subdivision* of the simplicial complex K and is denoted K'. In addition, K' is a simplicial complex and there exists a continuous embedding $i : K' \to K$. It identifies K' as a subspace of K. Furthermore, the map i also induces a continuous embedding on the posets:

$$P(i) : P(K') \to P(K),$$

where the elements of $P(K')$ are nonempty chains of $P(K)$. If the complex K is in a metric space, then one can define the diameter $\mathrm{diam}(\sigma)$ of a simplex σ; the largest of these is the mesh of K. We can then inductively form the n-th barycentric subdivision $K_n = (K_{n-1})'$; the sequence (K_n) can be constructed such that $mesh(K_n) \to 0$. We will denote by h_n the mesh length of K_n.

In the rest of this work, we will consider the space $X_n = P(K_n)^{op}$ where the elements are the simplices of K_n and the ordering is by reversed inclusion. The poset X_n is equipped with the Alexandrov topology induced by the inclusion order. Starting from a triangulation $\mathcal{T}(K)$ of M and a homeomorphism

$$\varphi : |K| \to M,$$

we construct a sequence of posets (X_n) associated to the successive barycentric subdivisions (K_n) of K. The maps $\phi_{n,m} : X_m \to X_n$ for $m \geq n$ sending an element from X_m to its carrier in K_n form a sequence $\{X_n, \mathbb{N}, \phi_{n,m}\}$:

$$X_0 \xleftarrow{\phi_{12}} X_1 \xleftarrow{\phi_{23}} X_2 \xleftarrow{\phi_{34}} X_3 \xleftarrow{\phi_{45}} \cdots$$

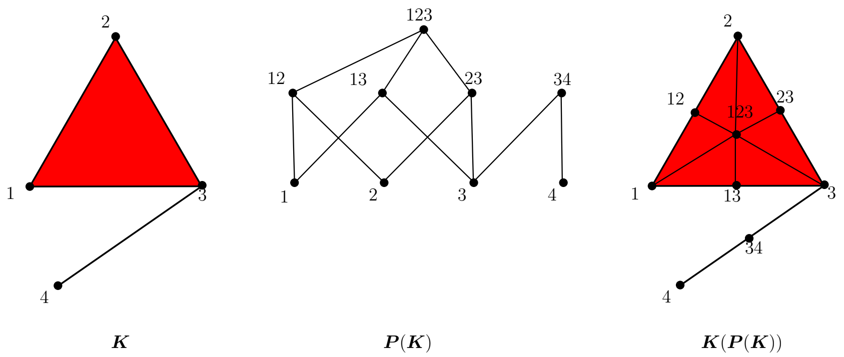

Figure 2.1: Simplicial complex, poset and barycentric subdivision.

2.1.2 The inverse limit construction

We have the system $\{X_n, \mathbb{N}, \phi_{n,m}\}$ where the maps $\phi_{n,m}$ satisfy by construction the coherence properties, for $\leq n \leq m$:

$$\phi_{l,n} \circ \phi_{n,m} = \phi_{l,m}, \quad \phi_{n,n} = id. \tag{2.1}$$

Therefore, the system $\{X_n, \mathbb{N}, \phi_{n,m}\}$ defines an inverse system of topological spaces. We define its inverse limit

$$X_\infty := \lim_{\leftarrow} X_i \tag{2.2}$$

which is a subset of the product space $\prod_{i\in I} X_i$ and we topologize it with the subspace topology. An element $x \in X_\infty$ is then a *coherent* sequence i.e a sequence:

$$x = (x_1, x_2, \cdots, x_i, \cdots x_j, \cdots) \in \prod_{i\in I} X_i, \quad x_i = \phi_{i,j}(x_j) \ \forall i \leq j. \tag{2.3}$$

Equivalently, recalling the definition of X_n from a simplicial complex K_n, one can see an element of X_∞ as a coherent sequence of nested simplices. The inverse limit X_∞ also comes equipped with natural projection maps $\phi_i : X_\infty \to X_i$ which pick out the i-th coordinate for every $i \in \mathbb{N}$.

The space X_∞ is a poset; the partial order on the sets X_n give a partial order $\leq$ on the set X_∞, where $y \leq x$ provided that $y_n \leq x_n$ for every $n \in \mathbb{N}$. Moreover, using the homeomorphism between M and $|K|$, we see that there is a natural map $p_n : M \to X_n$ for each n, since every point in K is contained in the interior of exactly one face of the n-th barycentric subdivision of K. We have the following commuting diagram:

$$\begin{tikzcd} & M \arrow[dl, "p_{n-1}"'] \arrow[dr, "p_n"] & \\ X_{n-1} & & X_n \arrow[ll, "\phi_{n-1,n}"] \end{tikzcd}$$

In addition, using the correspondence between points in X_n and faces of simplices in K_n, we can denote the simplex corresponding to $x_n \in X_n$ by $\sigma_n(x)$. We then immediately have that for every $n \geq 0$:

$$p_n^{-1}(U_x) = \mathrm{st}(\sigma_n(x)),$$

where st is the open star map. This implies that the maps p_n are continuous. We can then define a continuous map

$$p : M \to X_\infty, \quad p(a) = (p_0(a), p_1(a), \cdots). \tag{2.4}$$

The next claim allows us to create a map from X_∞ to M which acts as an inverse to p.

Lemma 2.1.1. *Given $x = (x_0, x_1, \cdots) \in X_\infty$, pick $a_n \in p_n^{-1}(x_n)$ for each $n \geq 0$.*

Then the sequence (a_n) *converges to* $a \in M$ *and the map*

$$G : X_\infty \to M, \quad x \mapsto a_x$$

is well-defined and continuous.

Proof. The points $a_n \in K_n$ lie in nested simplices of increasingly fine barycentric subdivisions of K. Any sequence obtained this way converges to the same point since they are obtained by intersection of nested closed sets with vanishing diameters. The proof of continuity of G can be found in [118, Prop. 2.4.16]. $\square$

Lemma 2.1.2. *Let* $x \in X_\infty$ *such that* $G(x) = a_x$*, then* $p(a_x) \geq x$*.*

Proof. Recall that the order in X_∞ is given by: $x \leq y$ in X_∞ if and only if $x_n \leq y_n$ in X_n for every n.
Now suppose that $p(a_x) \geq x$ is not true, then there exists n such that $p(a_n) \geq x_n$ is not true. This means that $p(a_n)$ is not contained in the simplex corresponding to $x_n \in X_n$. Thus, it contradicts the fact that $a_n \in p_n^{-1}(x_n)$. $\square$

Lemma 2.1.3. *The set* $p(M)$ *is precisely the subspace* $\mathfrak{M}$ *of all maximal elements in* X_∞*.*

Proof. Let y be a maximal element in X_∞. Then by Lemma 2.1.2, $p(a_y) \geq y$ and therefore $p(a_y) = y$. Conversely, if there exists $a \in M$ and $y \in X_\infty$ such that $y \geq p(a)$, then by definition, $y_n \geq p_n(a)$ for every n. Now, let $G(p_n(a)) = a_n$ and $G(y) = y_n$ for every n. Because $y_n \geq p_n(a)$, we have that $y_n \in p_n^{-1}(p_n(a))$ for every n. Hence, the sequences (a_n) and (y_n) have the same limit $a_y = a$. Thus, $p(a_y) = p(a)$ and $p(a) \geq y$ again by Lemma 2.1.2. We conclude that $p(a) = y$. $\square$

Proposition 2.1.1. *The space* M *is homeomorphic to the subspace* $\mathfrak{M}$ *of all maximal points of the inverse limit of the system* $\{X_n, \mathbb{N}, \phi_{n,m}\}$*.*

Proof. We need to prove that $G : p(M) \to M$ is a homeomorphism. By construction, we have that $G \circ p = id$, then, G is a bijection. By Lemma 2.1.1, G is continuous. Since $p(M)$ is equipped with the subspace topology, an open set U pf $p(M)$ can be written as $U = V \cap p(M)$ where V is an open set in X_∞. Now $G(U) = p^{-1}(V)$, thus $G(U)$ is open. Hence, G is a continuous and open bijective map and thus a homeomorphism. $\square$

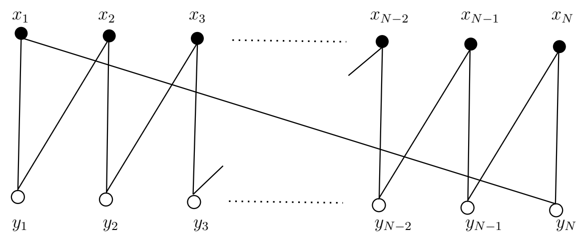

Figure 2.2: Poset associated to a triangulation of S^1.

2.1.3 C*-algebras and their spectra

We conclude this section by introducing some of the fundamental concepts on C^*-algebras that will be useful in the rest of this thesis dissertation; more complete details can be found in the literature [51, 24, 99].

A C^*- algebra A is a Banach algebra over $\mathbb{C}$ together with an involution $x \mapsto x^*$ such that:

$$(xy)^* = y^*x^* \quad \text{and} \quad \|x^*x\| = \|x\|^2 \quad \text{for} \quad x, y \in A. \tag{2.5}$$

The two archetypes of C^*-algebras are given by the space of continuous complex-valued functions that vanish at infinity $(C_b(X), \|\cdot\|_\infty)$ over a locally compact Hausdorff space X — in the commutative setting — and the space of bounded operators $(B(H), \|\cdot\|_{\text{op}})$ over a Hilbert space H — in the noncommutative case.
A central tool in the study of C^*-algebras is through their representations.

Definition 2.1.1 (Representations)**.** Let A be a $*$-algebra. A representation of A is a pair $(\pi, \mathcal{H})$ where $\mathcal{H}$ is a Hilbert space and $\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism. We also say that π is a representation of A on $\mathcal{H}$.

Another crucial tool to study C^*-algebras, and related to their representations, is the primitive spectrum.

Definition 2.1.2 (Primitive spectrum)**.** The *primitive spectrum* $Prim(A)$ is the space of kernels of irreducible *-representations equipped with the *hull-kernel (Jacobson) topology.*

The primitive spectrum becomes central to describe the internal algebraic structure of A. It can be turned into a topological space using the *hull-kernel (Jacobson) topology.* Let $W \in 2^{Prim(A)}$ an element of the power set, then the closure operator is given by

$$Cl(W) := \left\{ I \in Prim(A) : \bigcap \ker(\pi) \subseteq I \right\}.$$

A related and equally important notion, is the *spectrum* $Spec(A)$ of a C^*-algebra i.e. the set of non-zero unitary equivalence classes of irreducible *-representations. There is an immediate surjection map

$$Spec(A) \to Prim(A), \quad (\mathcal{H}, \pi) \mapsto \ker \pi, \tag{2.6}$$

which endows $Spec(A)$ with the pull-back of the Jacobson topology.

Remark 2.1.1. When the primitive spectrum $Prim(A)$ is a T_0-space, then the map (2.6) is a homeomorphism. This will always be the case in this work, therefore we will indistinguishably refer to the primitive spectrum or to the spectrum.

In the commutative case, the spectrum of A plays the role of a *space*. Indeed, any element $a \in A$ can be interpreted as a function over the space of characters through the *Gel'fand map*:

$$a \ni A \mapsto (\chi \mapsto \hat{a}(\chi)) \ (\chi \in Spec(A)). \tag{2.7}$$

If we let $X = Spec(A)$, then the Gel'fand transform is an isomorphism of A onto the C^*-algebra $C(X)$ of continuous complex functions over X.

2.2 C*-algebras over a triangulation

In this section, we show how to associate a C^*-algebra A_n to the space X_n defined in the previous section. The construction follows the works of Behncke and Leptin

[13, 14, 15, 11]. In order to give a more comprehensive presentation, we state the procedure as a sequence of axioms in the subsection 2.2.1. For more details, we refer to [60].

In the rest of this work, A will designate a C^*-algebra (eventually infinite dimensional) and H a representation of A. The letters $\mathfrak{A}$ and $\mathfrak{H}$ will be used in the commutative case.

2.2.1 C*-algebras over a topological space

We let X be a topological space. A C^*-algebra over X is a pair (A, ψ) consisting of a C^*-algebra A and a continuous surjection

$$\psi : Prim(A) \to X.$$

Let $\mathcal{O}_X$ be the set of open subsets of X, partially ordered by inclusion. For a C^*-algebra A, we let $I(A)$ be the set of all closed *-ideals in A partially ordered by inclusion. There is an isomorphism (see [98]) between $I(A)$ and the set of open subsets $\mathcal{O}_{Prim(A)}$ in $Prim(A)$. We will always identify $\mathcal{O}_{Prim(A)}$ and $I(A)$ through the isomorphism:

$$\mathcal{O}_{Prim(A)} \simeq I(A) \qquad U \mapsto \bigcap_{\pi \in Prim(A) \setminus U} \pi. \tag{2.8}$$

Then for (A, ψ) a C^*-algebra over X, we get a map

$$\psi^* : \mathcal{O}_X \to \mathcal{O}_{Prim(A)} \simeq I(A) \qquad U \mapsto \{\pi \in Prim(A) | \psi(\pi) \in U\} \simeq A(U).$$

We will denote by $A(U) \in I(A)$ the ideal associated to the open subset U. We can now identify the open sets in X with closed *-ideals of A, and points in X with irreducible representations of A.

The Behncke-Leptin construction

The Behncke-Leptin construction allows us to associate a C^*-algebra (A, ψ) over a partially ordered space X such that $\psi = id$ is the identity map. Hence, the spaces $Prim(A)$ and X can be identified.

The axioms of the Behncke-Leptin construction go as follows:

1) Associate a separable Hilbert space $H(X)$ to the space X and attach to every point $x \in X$ a subspace $H(x) \subseteq H(X)$ that decomposes into:

$$H(x) = H^-(x) \otimes H^+(x). \tag{2.9}$$

where $H^-(x) \simeq \ell^2(\mathbb{Z})$.

2) Let $\mathfrak{M}$ be the set of maximal points in X. Then for every $x \in \mathfrak{M}$, one has

$$H(x) = H^-(x) \otimes \mathbb{C} \simeq H^-(x). \tag{2.10}$$

2') If $\mathfrak{m}$ is the set of minimal points in X, then for every $x \in \mathfrak{m}$, one has

$$H(x) = \mathbb{C} \otimes H^+(x) \simeq H^+(x). \tag{2.11}$$

3) Associate to each point $x \in X$ an operator algebra $A(x)$ acting on $H(x)$ (extended by zero to the whole space $H(X)$) such that

$$A(x) = 1_{H^-(x)} \otimes \mathcal{K}(H^+(x)). \tag{2.12}$$

where $\mathcal{K}(H^+(x))$ is the set of compact operators over $H^+(x)$.

4) Build the C^*-algebra $A(X)$ associated to the space X as the algebra generated by the subalgebras $A(x)$ when x runs over X:

$$A(X) = \bigoplus_{x \in X} A(x) \quad \text{acting on} \quad H(X) = \bigoplus_{x \in X} H(x). \tag{2.13}$$

As mentioned already, using the isomorphism

$$\psi : X \to Spec(A), \quad \psi(x) = \pi_x \tag{2.14}$$

one can identify a point $x \in X$ with an irreducible representation $(\mathcal{H}_x, \pi_x)$:

$$\pi_x : A(X) \to B(\mathcal{H}_x), \quad a \mapsto \pi_x(a). \tag{2.15}$$

The irreducible representation $\mathcal{H}_x \subset H(x)$ is obtained as a subspace of $H(x)$. We define the following total space:

$$\mathcal{H}_X = \bigoplus_{x \in X} \mathcal{H}_x. \tag{2.16}$$

An element $a \in A$ then uniquely defines a *map* on X:

$$\hat{a} : X \to A, \quad \hat{a}(x) := \pi_x(a) = \sum_{i \in I_x} \lambda_i(x) 1 \otimes k_i(x) \tag{2.17}$$

where $\lambda_i(x) \in \mathbb{C}$ and $k_i(x)$ is a compact operator. In particular, if we identify the Hilbert space $H^-(x)$ with $\ell^2(\mathbb{Z})$, then we see that $\lambda(x)1$ is nothing else than a multiplication operator:

$$T_{\lambda(x)}(u) = \lambda(x) \cdot u. \tag{2.18}$$

for $u \in \ell^2(\mathbb{Z})$. This leads us to the fifth axiom.

5) For every $x \in \mathfrak{M}$, the representation $(\mathcal{H}_x, \pi_x)$ is one-dimensional:

$$\pi_x : A(X) \to \mathbb{C}, \quad a \mapsto \pi_x(a) = \lambda(x). \tag{2.19}$$

Example 2.2.1. Let σ be a 2-simplex and consider X to be the poset associated to σ with the opposite order. Then, Figure 2.3 shows a generic element $a_x \in A(x)$ for every vertex x of X. The full algebra $A(X)$ is obtained as a direct sum of $A(x)$.

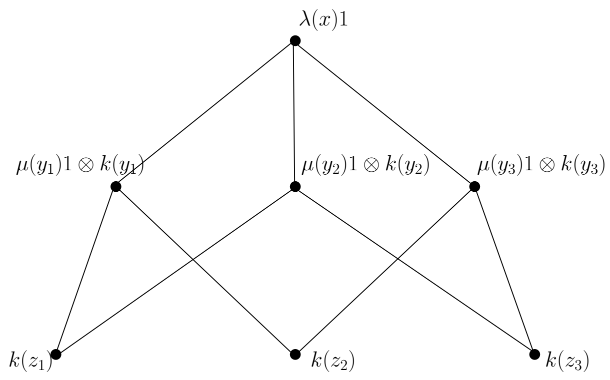

Figure 2.3: C^*-algebra associated to a poset.

Commutative subalgebras

Let (A, id) be the C^*-algebra associated to a finite connected poset X through the Behncke-Leptin construction. Among the subalgebras of A, those of particular interest are commutative ones. The centre of A will be denoted by $Z(A)$. We know by construction that A is generated by the algebras

$$A(x) = 1_{H^-(x)} \otimes \mathcal{K}(H^+(x))$$

for x running X. Moreover, we recall that the algebra of compact operators $\mathcal{K}(H)$ over an infinite dimensional Hilbert space H has a trivial centre. We deduce that, for a given $x \in X$, $A(x)$ has a trivial centre.

Proposition 2.2.1. *The centre $Z(A)$ of A is trivial.*

Proof. This is a direct consequence of the fact that the centre of $K(H)$ is trivial and the definition of the generating subalgebras $A(x)$ in the Behncke-Leptin construction. □

We will also consider the commutative subalgebra $\mathfrak{A}$ generated by the projectors on $H(x)$ when $x \in \mathfrak{M}$ is a maximal point:

$$\mathfrak{A} = \oplus_{x \in \mathfrak{M}} 1_{H(x)}. \tag{2.20}$$

2.2.2 C*-algebra over a simplicial complex

We go back now to a simplicial complex K and its associated poset $P(K)^{op}$ that we will denote X (seen as a topological space). Using the Behncke-Leptin construction, we can associate a C^*-algebra $(A(X), id)$ over X such that $Prim(A)$ is identified with X.
Now, let K and K' be simplicial complex such that K' is a barycentric subdivision of K. We denote by X and X' the associated posets. We then have a continuous surjection :

$$\phi : X' \to X.$$

Consider in addition that $(A(X), id)$, respectively $(A(X'), id')$, is a C^*-algebra over X, respectively X'. We would like to show that for the given map ϕ, there exists a pullback map ϕ^* such that the following diagram commutes:

$$\begin{tikzcd} A(X) \arrow[r, "\phi^*"] \arrow[d, "id"'] & A(X') \arrow[d, "id'"] \\ X & X' \arrow[l, "\phi"] \end{tikzcd}$$

i.e. such that the following proposition is satisfied:

$$\pi_x(a) = \pi_y(\phi^*(a)), \quad \forall x \in X, \ \forall y \in \phi^{-1}(x) : \dim(\sigma'_y) = \dim(\sigma_x). \tag{2.21}$$

Here, σ_x and σ'_y are the simplex associated to y and x in the identification of X and X' with K and K'. We are also using the isomorphism (2.14) to identify a point $x \in X$ with an irreducible representation $(H_x, \pi_x) \in Spec(A)$; then $\pi_x(a)$ is an operator acting on H_y and $\pi_y(\phi^*(a))$ an operator on H'_y. We are assuming here that H_x and H'_y can be identified as Hilbert spaces; the identification is constructed in Equation (2.48).

Proposition 2.2.2. *A continuous surjection $\phi : X' \to X$ between posets induces a unital *-homomorphism $\phi^* : A(X) \to A(X')$ satisfying* (2.21).

Proof. We recall that the algebra $A(X)$ is generated by the subalgebras $A(x)$ defined by (2.12) for x running in X. Then, it is enough to define ϕ^* on the algebras $A(x)$

and extend the map by linearity.

Therefore, if we start with the following decomposition:

$$A(X_i) = \oplus_{x\in X_i} A(x), \quad a = \sum_{x\in X} a_x, \quad \text{i=1,2} \tag{2.22}$$

with $X_1 = X$ and $X_2 = X'$, we define ϕ^* such that:

$$\phi^*(a) = \sum_{y\in X'} a_y, \tag{2.23}$$

where

$$a_y = \begin{cases} a_\phi(y) & \text{if } \dim(\sigma_y) = \dim(\sigma_{\phi(y)}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.24}$$

Thus, if we let $x \in X$ and consider the set:

$$\Phi^{-1}(x) = \{y \in \phi^{-1}(x) : \dim(\sigma'_y) = \dim(\sigma_x)\}. \tag{2.25}$$

then, we have defined ϕ^* such that it satisfies 2.21 i.e. for any $a \in A(X)$:

$$\pi_y(\phi^*(a)) = \pi_x(a), \quad \forall y \in \Phi^{-1}(x). \tag{2.26}$$

Furthermore, ϕ^* is a *-homomorphism by construction. In addition, the identity element on $A(X)$ is given by

$$1_{A(X)} = \sum_{x\in\mathfrak{M}} 1_{H(x)} \tag{2.27}$$

and since $\phi(\mathfrak{M}') = \mathfrak{M}$, then $\phi^*(1_{A(X)}) = 1_{A(X')}$ i.e. ϕ^* is unital. $\square$

2.2.3 The direct limit construction

We now recall the definition of a direct limit of C^*-algebras. Consider a direct sequence (A_n, ψ_n) of separable C^*-algebras with *-homomorphism $\psi_n : A_n \to A_{n+1}$. The product $\prod_n A_n$ equipped with the pointwise addition, multiplication, scalar multiplication and involution is a C^*-algebra [102]. We denote by A' the following set

$$A' = \left\{ a = (a_n) \in \prod_n A_n : \exists N \in \mathbb{N},\ a_{n+1} = \psi_n(a_n)\ \forall n \geq N \right\}. \tag{2.28}$$

Since (ψ_n) are contractions, then $(\|a_n\|)_n$ converges. One can then check that the map

$$p : A' \to \mathbb{R}^+,\ a \mapsto p(a) := \lim_{n\to\infty} \|a_n\|, \tag{2.29}$$

is a C^*-seminorm on A'. The *direct (inductive) limit* of the sequence $(A_n, \psi_n)_n$ is then defined as the enveloping C^*-algebra of (A', p). It is important to notice that the direct limit is not unique, in the sense that it highly depends on the choices of maps ψ_n. We now state the following proposition that characterizes the inductive limit A in terms of the algebras A_n.

Proposition 2.2.3 ([99])**.** *Let $(A_n, \psi_n)_n$ be an inductive sequence in the category of C^*-algebras. Then there exists an inductive limit $(A, \psi_{n,\infty})$ which satisfies the following:*

(i) $A = \overline{\bigcup_{n\in\mathbb{N}} \psi_{n,\infty}(A_n)}$;

(ii) For any $n \in \mathbb{N}$ and $a \in A_n$, $\|\psi_{n,\infty}(a_n)\| = \lim_{p\to\infty} \|\psi_{n,p}(a)\|$.

(ii) For any $n \in \mathbb{N}$, $a \in \ker \psi_{n,\infty}$ if and only if $\lim_{p\to\infty} \|\psi_{n,p}(a)\| = 0$.

We consider now the inverse system $\{X_n, \mathbb{N}, \phi_{m,n}\}$ defined in Section 2.1.2. To each poset X_n, we associate a C^*-algebra (A_n, id_n) through the Behncke-Leptin construction. We have then the following identification:

$$Spec(A_n) \simeq X_n \quad \forall n \in \mathbb{N}.$$

Moreover, using Proposition 2.2.2, the map $\phi_{n,n+1} : X_{n+1} \to X_n$ induces a pullback map $\phi^*_{n,n+1} : A(X_n) \to A(X_{n+1})$ for all $n \in \mathbb{N}$. We then have the following diagram in Figure 2.4.

Proposition 2.2.4. *The system $\{A_n, \mathbb{N}, \phi^*_{m,n}\}$ forms a direct system.*

$$\begin{tikzcd}
A_1 \arrow[r, "\phi_{12}^*"] \arrow[d, "id_1"] & A_2 \arrow[r, "\phi_{23}^*"] \arrow[d, "id_2"] & A_3 \arrow[r] \arrow[d, "id_3"] & \cdots \arrow[r] & A_\infty \arrow[d, dashed, "id"] \\
X_1 & X_2 \arrow[l, "\phi_{12}"'] & X_3 \arrow[l, "\phi_{23}"'] & \cdots \arrow[l] & X_\infty \arrow[l]
\end{tikzcd}$$

Figure 2.4: Direct system of C^*-algebras

Proof. We start by recalling that the maps $\phi_{m,n}$ satisfy the coherence properties:

$$\phi_{l,m} \circ \phi_{m,n} = \phi_{l,n}, \quad l \leq m \leq n, \quad \phi_{n,n} = id_n \quad \forall n \in \mathbb{N}.$$

From this, it follows that for any $l \leq m \leq n$, the following equalities hold:

$$\begin{aligned}
(\mathit{\Phi}_{l,m} \circ \mathit{\Phi}_{m,n})^{-1} &:= \{y \in (\phi_{l,m} \circ \phi_{m,n})^{-1}(x) : \dim(\sigma'_y) = \dim(\sigma_x)\}, \\
&= \{y \in \phi_{m,n}^{-1} \circ \phi_{l,m}^{-1}(x) : \dim(\sigma'_y) = \dim(\sigma_x)\}, \\
&= \mathit{\Phi}_{m,n}^{-1} \circ \mathit{\Phi}_{l,m}^{-1},
\end{aligned}$$

on one hand; and on the other hand

$$\begin{aligned}
(\mathit{\Phi}_{l,m} \circ \mathit{\Phi}_{m,n})^{-1} &= \{y \in \phi_{l,n}^{-1}(x) : \dim(\sigma'_y) = \dim(\sigma_x)\}, \\
&= \mathit{\Phi}_{l,n}^{-1}.
\end{aligned}$$

This implies by construction that the pullback maps $\phi_{m,n}^*$ also satisfy the coherence properties:

$$\phi_{m,n}^* \circ \phi_{l,m}^* = \phi_{l,n}^*, \quad l \leq m \leq n, \quad \phi_{n,n}^* = id_n \quad \forall n \in \mathbb{N}. \tag{2.30}$$

and thus $\{A_n, \mathbb{N}, \phi_{m,n}^*\}$ forms a direct system. $\square$

We can now write the direct limit as

$$A_\infty := \varinjlim (A_n, \phi_{n,n+1}^*)_{n \in \mathbb{N}}. \tag{2.31}$$

Let $Z(A_\infty)$ be the center of A_∞; consider the space MZ_A being the space of maximal ideals in $Z(A_\infty)$ equipped with the hull-kernel topology. From the Gel'fand-Naimark theorem [24, Thm 2.2.4 p.60], we deduce immediately that $Z(A_\infty)$ is *-isomorphic to the space of continuous functions $C(MZ_A, \mathbb{C})$. Therefore, to prove that $Z(A_\infty)$ is isomorphic to the space of functions $C(M, \mathbb{C})$ over the manifold M, we only need to prove that the spaces $\mathfrak{M}$ and MZ_A are homeomorphic. In fact, we can prove a stronger result:

Theorem 2.2.1. *The spectrum $Spec(A_\infty)$ equipped with the hull-kernel topology is homeomorphic to the space X_∞ and*

$$\varprojlim Spec(A_i) \simeq Spec(\varinjlim A_i). \tag{2.32}$$

Before proving this result, we recall the definition of a *state* and the interplay with representations. A state φ is a positive linear functional with $\varphi(1) = \|\varphi\| = 1$. We denote by $S(A)$ the space of states over the C^*-algebra A equipped with the weak* topology. In addition, the set $S(A)$ is convex; an *extreme point* of $S(A)$ is called a *pure state* and the set of pure states is denoted by $P(A)$. We will denote the set of extreme points of a convex set C by $\mathrm{ext}(C)$.
The GNS construction (see for instance [24, pp.114-115]) gives a one-to-one correspondence between positive linear functionals φ and (cyclic) representations $(H_\varphi, \pi_\varphi, \xi_\varphi)$.

Now let $x \in X_\infty$, then identifying X_i with $Spec(A_i)$, the corresponding representation π_x defines a coherent sequence

$$\pi_x = (\pi_1, \pi_2, \cdots) \in \prod_{i\in\mathbb{N}} Spec(A_i), \quad \text{such that } \pi_m = \phi_{m,n}(\pi_n),\ \forall m \leq n.$$

Moreover, according to the GNS construction, we can associate a pure state φ to any irreducible representation π. Therefore, we have the following coherent sequence of pure states:

$$\varphi_x = (\varphi_1, \varphi_2, \cdots) \in \prod_{i\in\mathbb{N}} S(A_i), \tag{2.33}$$

such that,

$$\varphi_m = \phi_{n,m}(\varphi_n), \tag{2.34}$$

$$\phi_{l,m} = \phi_{l,n} \circ \phi_{n,m}, \quad \text{if } l \leq m \leq n \ . \tag{2.35}$$

Hence, the inverse system of posets $\{X_n, \mathbb{N}, \phi_{m,n}\}$ induces an inverse system of states $\{S(A_n), \mathbb{N}, \phi_{m,n}\}$.

Lemma 2.2.1. *The inverse limit system $\{S(A_n), \mathbb{N}, \phi_{m,n}\}$ is homeomorphic to $S(A_\infty)$.*

Proof. For $x \in X_\infty$, the map φ_x defines a bounded linear functional on the algebraic inductive limit A' and uniquely extend over A_∞ such that $\|\varphi_x\| = 1$. Hence, $\varphi_x \in S(A_\infty)$.
Conversely, any state $\phi \in S(A_\infty)$ define a state $\varphi_n \in S(A_n)$ defined as follows

$$\varphi_n := \varphi \circ \phi^*_{n,\infty}(a) \tag{2.36}$$

for any $n \in \mathbb{N}$. In addition, the sequence (φ_n) is a coherent sequence satisfying (2.34) and (2.35). Thus, there is a bijection between $\lim_{\leftarrow} S(A_i)$ and $S(A_\infty)$.

Finally, the weak*-topology on $S(A_\infty)$ is equivalent to the subspace topology on $\lim_{\leftarrow} S(A_i)$ induced by the product topology on $\prod_{i\in\mathbb{N}} S(A_i)$. This gives us the expected homeomorphism. ☐

Lemma 2.2.2. *The inverse limit system $\{P(A_n), \mathbb{N}, \phi_{m,n}\}$ is homeomorphic to $P(A_\infty)$.*

Proof. We start by recalling that the inverse limit of convex spaces is convex (this follows from the fact that an arbitrary Cartesian product of convex sets is convex). Therefore, the set $\lim_{\leftarrow} S(A_i)$ is convex. In addition, the set of extreme points of $S(A_i)$ is exactly the set of pure states $P(A_i)$. Using a classical result in convex analysis [77, Thm.3 p.502], the set of extreme points in the product is given by:

$$\text{ext}\left(\prod_{i\in\mathbb{N}} S(A_i)\right) = \prod_{i\in\mathbb{N}} P(A_i) \tag{2.37}$$

Hence, the pure states of $\lim_{\leftarrow} S(A_i)$ are given by the coherent sequences in $\prod_{i\in\mathbb{N}} S(A_i)$ i.e. $\lim_{\leftarrow} P(A_i)$. Similarly, the set of pure states on A_∞ is denoted by $P(A_\infty)$. Consequently, using Lemma (2.2.1) we deduce that

$$\operatorname{ext}\left(\lim_{\leftarrow} S(A_i)\right) = \operatorname{ext}\left(S(A_\infty)\right) = P(A_\infty). \tag{2.38}$$

Finally, we recall that a sequence of states (φ_n) on A_∞ converges to a state φ in the usual weak topology if and only if the coordinate sequence (φ_n^i) on A_i converges for every $i \in \mathbb{N}$. Therefore, the space $P(A_\infty)$ is homeomorphic to the closed subspace of all systems satisfying (2.34) in the product space $\prod_{i\in\mathbb{N}} P(A_i)$ i.e

$$\lim_{\leftarrow} P(A_i) \simeq P(A_\infty). \tag{2.39}$$

$\square$

Proof of Theorem 2.32. Let $\pi \in Spec(A_\infty)$, then by the GNS construction, we can associate to it a pure state $\varphi \in P(A_\infty)$. Using Lemma 2.2.2, φ in turn correspond to a sequence of pure states (φ_i) in $\lim_{\leftarrow} P(A_i)$. Again by Lemma 2.2.2 and the GNS construction, we associate to (φ_i) a coherent sequence in X_∞.
Reciprocally, a coherent sequence of irreducible representations in X_∞ correspond to an element in $\lim_{\leftarrow} P(A_i)$ through the GNS construction.
Therefore, we can identify X_∞ with $Spec(A_\infty)$ as posets. The homeomorphism follows from the fact that the order topology on X_∞ is equivalent to the hull-kernel topology using the isomorphism (2.8). $\square$

Corollary 2.2.1. *The sets $\mathfrak{M}$ and MZ_A are homeomorphic.*

Proof. This follows again from the isomorphism (2.8) where the maximal points in X_∞ correspond to maximal ideals in $Spec(A_\infty)$. Then $\mathfrak{M}$ and MZ_A are homeomorphic with the subspace topology. $\square$

We have then proven that the C^*-algebra A_∞ contains the algebra of continuous functions $C(M, \mathbb{C})$ as its centre. In fact, one can go further in the characterization of the inductive limit using the following result.

Theorem 2.2.2 (Dauns-Hofmann [53, p.272],[61])**.** *Let A be a unital C^*-algebra with centre $Z(A)$. Let MZ_A be the space of maximal ideals of the center $Z(A)$ equipped with the hull-kernel topology. Then A is isometrically *-isomorphic to the C^*-algebra of all continuous sections $\Gamma(MZ_A, A)$ of the C^*-bundle (A, Ψ, MZ_A) over MZ_A. The fibre (stalk) above $x \in MZ_A$ is given by the quotient $\mathrm{A}_x \simeq {}^{A}/_{xA}$, the isometric *-isomorphism is Gel'fand's representation $a \mapsto \hat{a}$:*

$$\begin{cases} A & \rightarrow & \Gamma(MZ_A, A) \\ a & \mapsto & x \mapsto \hat{a}(x) = a + xA \end{cases}$$

with $\|\hat{a}\| = \sup_{x \in MZ_A} \|\hat{a}(x)\|$.

According to the Dauns-Hofmann theorem, the algebra A_∞ is isomorphic to the C^*-algebra of continuous sections $\Gamma(M, A_\infty)$ of a C^*-bundle (A, Ψ, M) over the manifold M. From the Behncke-Leptin construction, we get the following general form for a section at a point $x \in M$.

$$\hat{a}(x) = \sum_{i \in I_x} \lambda_i(x) \otimes k_i(x) + xA_\infty \tag{2.40}$$

where I_x is a finite indexing set. We see that the central elements are then given by functions $x \mapsto \lambda(x)$ on M.

We go back now to the commutative subalgebra $\mathfrak{A}$ defined in Equation (2.20) and show how it can be used to approximate $C(M)$. In the rest of this work, we will identify $C(M)$ with the centre $Z(A_\infty)$ and denote by $\mathfrak{A}_n$ the commutative subalgebra in $A(X_n)$.

Proposition 2.2.5. *The space of continuous functions $C(M)$ is approximated by the system of commutative subalgebras $(\mathfrak{A}_n, \phi^*_{n,\infty})$ in the following sense:*

$$C(M) = \overline{\bigcup_{n \in \mathbb{N}} \phi^*_{n,\infty}(\mathfrak{A}_n)} \cap C(M). \tag{2.41}$$

Proof. First, let us recall that, by Axiom 5), an element $a_n \in \mathfrak{A}_n$ is determined a map $\hat{a}_n : X_n \to A_n$ such

$$\hat{a}_n(x) = \sum_{i \in I_x} \lambda_i(x) 1_{H(x)}. \tag{2.42}$$

When restricted to the set of maximal points $\mathfrak{M}_n$, a_n acts as a scalar:

$$\pi_x(a_n) = \lambda_x \tag{2.43}$$

where $\lambda_x \in \mathbb{C}$. Then, using the map $\phi_{n,\infty}$, we notice that a_n defines a piecewise-linear function on M:

$$\phi^*_{n,\infty}(a_n) = a_n \circ \phi_{n,\infty} : M \to \mathbb{C}, \quad a_n \circ \phi_{n,\infty}(y) = \lambda_{\phi_{n,\infty}(y)}. \tag{2.44}$$

Therefore, any continuous function $g \in C(M)$ can be uniformly approximated arbitrarily closely by a function of the form $a_n \circ \phi_{n,\infty}$, for some sufficiently large n. $\square$

Finally, using the smooth structure, we can define the subalgebras

$$Z^k(A_\infty) := C^k(M) \tag{2.45}$$

of k-differentiable functions. In the rest of this work, we will focus on the subalgebra $Z^\infty(A_\infty)$ and its approximation given by the equality:

$$Z^\infty(A_\infty) = \overline{\bigcup_{n \in \mathbb{N}} \phi^*_{n,\infty}(\mathfrak{A}_n) \cap Z^\infty(A_\infty)}. \tag{2.46}$$

Direct limit of representations

Similarly, we associate a representation space $H(X_n)$ (defined in Equation (2.16)) to every space X_n. Moreover, a continuous surjection $\phi : X' \to X$ between posets induces a isometry $\psi : H(X) \to H(X')$ between representations. The construction of ψ follows mutatis mutandis the same steps that the one of ϕ^*; therefore, we will keep the same notations and directly state the results. We define $\psi : H(X) \to H(X')$ as follows :

$$H(X) = \oplus_{x \in X} H(x), \quad \psi\left(\oplus_{x \in X} \xi_x\right) = \oplus_{y \in X'} \xi_y \tag{2.47}$$

where,

$$\xi_y = \begin{cases} \xi_\phi(y) & \text{if } \dim(\sigma_y) = \dim(\sigma_{\phi(y)}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.48}$$

Therefore, the inverse system of posets $\{X_n, \mathbb{N}, \phi_{m,n}\}$ induces a direct system of Hilbert spaces $\{H_n, \mathbb{N}, \psi_{m,n}\}$, where H_n denotes the Hilbert space $H(X_n)$.

Proposition 2.2.6. *The system $\{H_n, \mathbb{N}, \psi_{m,n}\}$ forms a direct system.*

Hence, we can construct the direct limit of representations (H_n, ψ_n) as a subspace of the direct sum:

$$\bigoplus_{n\in\mathbb{N}} H_n = \left\{ (h_n)_{n\in\mathbb{N}} : h_n \in H_n, \sum_{n=1}^{\infty} \|h_n\|_{H_n}^2 < \infty \right\} \tag{2.49}$$

equipped with an inner product $\langle ., . \rangle$ given by:

$$\langle g, h \rangle = \sum_{n=1}^{\infty} \langle g_n, h_n \rangle_{H_n}. \tag{2.50}$$

The algebraic direct limit is defined as

$$H' = \left\{ (h_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} H_n : \psi_{n,n+1}(h_n) = h_{n+1} \right\}. \tag{2.51}$$

The resulting Hilbert space is obtained from the closure of H' and will be denoted by

$$H_\infty := \varinjlim (H_n, \psi_n)_{n\in\mathbb{N}}. \tag{2.52}$$

The direct system $\{H_n, \mathbb{N}, \psi_{m,n}\}$ induces a direct system on the irreducible representations $\{\mathcal{H}_n, \mathbb{N}, \psi_{m,n}\}$; we denote the limit $\mathcal{H}_\infty$. We have then the following characterization of this limit space.

Theorem 2.2.3. *The Hilbert space $L^2(M)$ of square integrable functions over the manifold M is a subspace of $\mathcal{H}_\infty$:*

$$\mathcal{H}_\infty = L^2(M) \oplus \mathcal{H}_\omega. \tag{2.53}$$

Proof. Using Axiom 5 in the Behncke-Leptin construction, for any $n \in \mathbb{N}$, we have the following decomposition:

$$\mathcal{H}_n = \bigoplus_{x\in\mathfrak{M}_n} \mathcal{H}_x^n \oplus \bigoplus_{x\in\mathfrak{M}_n^c} \mathcal{H}_x^n \simeq \mathbb{C}^{|\mathfrak{M}_n|} \oplus \bigoplus_{x\in\mathfrak{M}_n^c} \mathcal{H}_x^n. \tag{2.54}$$

Now, let us recall that the commutative subalgebra $\mathfrak{A}_n$ given by

$$\mathfrak{A}_n = \oplus_{x\in\mathfrak{M}_n} 1_{H(x)}, \quad a = \sum_{x\in\mathfrak{M}_n} \lambda(x) 1_{H(x)} \tag{2.55}$$

is completely determined by the representation $(\mathbb{C}^{|\mathfrak{M}_n|}, \oplus_{x\in\mathfrak{M}_n}\pi_x)$:

$$\oplus_{x\in\mathfrak{M}_n} \pi_x : \mathfrak{A}_n \to \mathbb{C}^{|\mathfrak{M}_n|}, \quad a \mapsto (\lambda(x_1), \lambda(x_2), \cdots, \lambda(x_{|\mathfrak{M}_n|})). \tag{2.56}$$

Through this isomorphism of vector spaces, we can identify $\bigoplus_{x\in\mathfrak{M}_n} \mathcal{H}_x^n$ with the image of $\mathfrak{A}_n$ and denote it by $\hat{\mathfrak{A}}_n$:

$$\mathcal{H}_n = \hat{\mathfrak{A}}_n \oplus \bigoplus_{x\in\mathfrak{M}_n^c} \mathcal{H}_x^n. \tag{2.57}$$

Moreover, because $\phi_{n,n+1}(\mathfrak{M}_{n+1}) = \mathfrak{M}_n$ then by definition of $\psi_{n,n+1}$, we have:

$$\psi_{n,n+1}(\mathcal{H}_n) = \phi_{n,n+1}^*(\hat{\mathfrak{A}}_n) \oplus \bigoplus_{x\in\mathfrak{M}_n^c} \psi_{n,n+1}(\mathcal{H}_x^n) \tag{2.58}$$

for every $n \in \mathbb{N}$. Therefore, we have for every $n \in \mathbb{N}$:

$$\psi_{n,\infty}(\mathcal{H}_n) = \phi_{n,\infty}^*(\hat{\mathfrak{A}}_n) \oplus \bigoplus_{x\in\mathfrak{M}_n^c} \psi_{n,n+1}(\mathcal{H}_x^n). \tag{2.59}$$

Hence the direct limit H_∞ decomposes as the direct sum $H \oplus H_\omega$, where H_ω is an infinite dimensional Hilbert space and

$$H = \overline{\oplus_n \phi_{n,\infty}^*(\hat{\mathfrak{A}}_n)} = \overline{\{a \in C(M), \|a\|_{H_\infty} < \infty\}} \equiv L^2(M). \tag{2.60}$$

□

Cubulation: example of a lattice

We now conclude this section with the specific case of a C^*-algebra over a lattice Λ seen as a cubulation of $\mathbb{R}^d$. The lattice Λ can be written as a direct product of a line lattice L. Hence, we can the algebra $A(\Lambda)$ relate them to the tensor product of algebras $A(L)$ over L. First, we need to recall the following result on the structure space of tensor product of C^*-algebras.

Proposition 2.2.7 (Wulfsohn [130]). *Let A and B be separable C^*-algebras and $A \otimes B$ their C^*-tensor product. The mapping*

$$\alpha : Prim(A) \times Prim(B) \to Prim(A \otimes B), \quad \alpha(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} \otimes B + A \otimes \mathfrak{b}$$

is a homeomorphism.

This result immediately gives us that tensor C^*-algebras can be seen as C^*-algebras over Cartesian product of posets.

Corollary 2.2.2. *Let X and Y be topological spaces. If (A, ψ_A) and (B, ψ_B) are separable C^*-algebra over X (respectively over Y), then the pair $(A \otimes B, \psi_A \times \psi_B)$ is a separable C^*-algebra over $X \times Y$ with the product topology.*

Let Λ be the d-dimensional, we can write it as the direct product of d line lattices:

$$\Lambda = L \times \cdots \times L.$$

Let $(A(L), \psi_L)$ be a C^*-algebra over L. Then using 2.2.7 and 2.2.2 we can associate the C^*-algebra over Λ:

$$A(\Lambda) = A(L) \otimes \cdots \otimes A(L), \quad \psi_\Lambda = \Pi \psi_L. \tag{2.61}$$

Similarly to the previous section, we construct a sequence of refined lattice (Λ_n, π_n) and construct the direct limits of C^*-algebras $(A(\Lambda_n), \pi_n^*)$ with their representations (H_n, ψ_n). We can then directly state the following result, which a special case when $M = \mathbb{R}^d$.

Corollary 2.2.3. *The centre of the limit C^*-algebra A_∞, $Z(A_\infty)$ is isometrically *-isomorphic to $\mathcal{C}(\mathbb{R}^n)$ acting on $L^2(\mathbb{R}^n)$ as a subspace of H_∞.*

2.3 Geometry over a triangulation

The last piece remaining to be defined, in order to complete this triptych, is the differential geometry. This will be done using the machinery of noncommutative differential geometry, as explained in the introduction.

2.3.1 Finite spectral triple

Let $(A(X), id)$ be a C^*-algebra over a poset X induced by a triangulation of a compact Riemannian manifold (M, g) of dimension d.

We will denote by $\mathfrak{M}$ the set of maximal points in X and by $\mathfrak{A}$ the commutative subalgebra of A defined by Equation (2.20). We then immediately notice that

$$\mathfrak{h} = \bigoplus_{x \in \mathfrak{M}} \mathbb{C}, \quad \pi = \bigoplus_{x \in \mathfrak{M}} \pi_x. \tag{2.62}$$

defines a faithful representation of $\mathfrak{A}$.

Consider now the pair $(\mathfrak{h}, \mathfrak{h}^*)$ where $\mathfrak{h}$ and $\mathfrak{h}^*$ have both dimension m. Define the even dimensional representation of $\mathfrak{A}$

$$\mathfrak{H}(X) := \mathfrak{h} \oplus \mathfrak{h}^*, \quad \rho = \pi \oplus \pi^* \tag{2.63}$$

where the adjoint representation is given by $\pi^*(a) = -\pi^t(a)$ for any $a \in \mathfrak{A}$. The triple $(\mathfrak{A}, \mathfrak{H}, \rho)$ embeds the commutative algebra $\mathfrak{A}$ into the *Cartan subalgebra* h of the Lie algebra $\mathfrak{gl}(2m, \mathbb{C})$.

The space of bounded operators $B(\mathfrak{H})$ can be identified with $M_{2m}(\mathbb{C})$. We define the parity element $\gamma \in M_{2m}(\mathbb{C})$ such that

$$\gamma = \begin{pmatrix} 1_m & 0 \\ 0 & -1_m \end{pmatrix} \tag{2.64}$$

where the eigenspace decomposition correspond to the splitting (2.63). This in turns defines a $\mathbb{Z}_2$-grading on $M_{2m}(\mathbb{C})$. The space M_{2m} can be accordingly written as a direct sum

$$M_{2m} = M_{2m}^{+} \oplus M_{2m}^{-} \tag{2.65}$$

of even and odd elements, where $a \in M_{2m}$ is even if it commutes with γ and odd if it anticommutes. In fact, even elements will correspond to block diagonal elements and odd elements to off-diagonal with respect to the representation space $\mathfrak{H}$. Under this grading, the algebra $\mathfrak{A}$ is represented as the subspace of diagonal matrices, i.e.

$$\mathfrak{A} \xrightarrow{\rho} \mathsf{h} \hookrightarrow M_{2m}^{+}(\mathbb{C}). \tag{2.66}$$

Remark 2.3.1. The data $(\mathfrak{A}, \mathfrak{H}, \pi)$ can also be localized to an open set $U \subset X$. Consider the restriction functor r_X^U (see [98]) and define the restriction $A(U) := (r_X^U A)$ of A to the open set U. Similarly, $\mathfrak{A}(U)$ defines a restriction of $\mathfrak{A}$ to U. Let $\mathfrak{M}_U$ the subset of $\mathfrak{M}$ of maximal points in U. Again, we have that

$$\mathfrak{H}_U = \bigoplus_{x \in \mathfrak{M}_U} \mathbb{C}, \quad \pi|_U = \bigoplus_{x \in \mathfrak{M}_U} \pi_x,$$

is a representation of $\mathfrak{A}(U)$.

Definition 2.3.1 (Spectral triple)**.** A *spectral triple* is the data $(\mathcal{A}, \mathcal{H}, D)$ where:

(i) $\mathcal{A}$ is a real or complex $*$-algebra;

(ii) $\mathcal{H}$ is a Hilbert space and a left-representation $(\pi, \mathcal{H})$ of A in $\mathcal{B}(\mathcal{H})$;

(iii) D is a *Dirac operator*, which is a self-adjoint operator on $\mathcal{H}$.

If in addition, $\mathcal{H}$ is equipped with a $\mathbb{Z}_2$-grading i.e. there exists a unitary self-adjoint operator $\gamma \in \mathcal{B}(\mathcal{H})$ such that

1) $[\gamma, \pi(a)] = 0$ for all $a \in \mathcal{A}$,

2) γ anticommutes with D,

then the spectral triple is said to be *even*. Otherwise, it is said to be *odd*. In the case where $\mathcal{H}$ is finite dimensional, then the triple $(\mathcal{A}, \mathcal{H}, D)$ is called a discrete spectral triple.

We consider the finite dimensional algebra $(\mathfrak{A}, \mathfrak{H})$ a Dirac operator D chosen as an odd element of $M_{2m}(\mathbb{C})$ of the form

$$D = \frac{i}{\hbar} \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \tag{2.67}$$

where $D^+, D^- \in M_{2m}(\mathbb{R})$ and satisfy $D^- = -(D^+)^*$. We then form the finite spectral triple $(\mathfrak{A}, \mathfrak{H}, D)$; this triple is even with the grading induced by γ.

Using this structure, we can then define a graded derivation da for $a \in M_{2m}(\mathbb{C})$ through a graded commutator,

$$da = -[D, a] := Da - \epsilon_a a D \tag{2.68}$$

where $\epsilon_a = 1$ if a is even and $\epsilon_a = -1$ if a is odd. Using the representation ρ, it also induces a derivation on $\mathfrak{A}$. Furthermore, notice that the derivative d coincides (modulo the grading) with the adjoint operator ad_D. We can then study the differential structure on $\mathfrak{A}$ by identifying $M_{2m}(\mathbb{C})$ as the Lie algebra $\mathfrak{gl}_{2m}(\mathbb{C})$ with Cartan subalgebra h. For convenience, we then equip h with the inner product:

$$\langle h, h' \rangle := Tr(h^* h'). \tag{2.69}$$

We can then identify h with its dual h^* i.e. the set of linear functionals acting on h. Now, recall that a nonzero element $\alpha \in \mathsf{h}$ is a *root* of $\mathfrak{gl}_n(\mathbb{C})$ relative to h if there exists a nonzero $x \in \mathfrak{gl}_n(\mathbb{C})$ such that

$$[x, h] = \alpha(h)x, \tag{2.70}$$

for all $h \in \mathsf{h}$. In particular, the standard matrix basis elements e_{ij} satisfies $he_{ij} = \lambda_i e_{ij}$ and $e_{ij}h = \lambda_j e_{ij}$ for all $h \in \mathsf{h}$. Thus,

$$[h, e_{ij}] = (\lambda_j - \lambda_i)h, \tag{2.71}$$

showing that e_{ij} are simultaneous eigenvectors for ad_h. Now let $a \in \mathfrak{A}$ be described as an element of h through the representation ρ:

$$\rho(a) = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_m & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_m \end{pmatrix}. \tag{2.72}$$

Following the definition, we can write the operator D as a linear combination of elements e_{ij}:

$$D = \sum_{i<j} \omega_{ij} \hat{e}_{ij} \tag{2.73}$$

where $\hat{e}_{ij} = e_{ij} - e_{ji}$. Then the derivation d acts on an element $a \in \mathfrak{A}$ as

$$da = \sum_{i<j} \omega_{ij} \alpha_{ij}(a) \hat{e}_{ij} \tag{2.74}$$

where the roots are given by $\alpha_{ij}(a) = \lambda_j - \lambda_i$.

Graded differential algebras

It is possible to construct over M_{2m}^+ a $\mathbb{N}$-graded differential algebra $\Omega_D^* = \Omega_D^*(M_{2m}^+)$ based on formula (2.68). Define $\Omega_D^0 = M_{2m}^+$ and let

$$\Omega_D^1 = d\Omega_D^0 \subset M_{2m}^- \tag{2.75}$$

be the M_{2m}^+-module generated by the image of Ω_D^0 in M_{2m}^- under d. Then for each p, we let $\operatorname{Im} d^2$ be the submodule of $d\Omega_D^{p-1}$ consisting of those elements which contain a factor which is the image of d^2 and define

$$\Omega_D^p = d\Omega_D^{p-1} / \operatorname{Im} d^2. \tag{2.76}$$

Therefore since $\Omega_D^p \cdot \Omega_D^q \subset \Omega_D^{p+q}$ the complex Ω_D^* define as

$$\Omega_D^* = \bigoplus_{p\geq 0} \Omega_D^p \tag{2.77}$$

is a differential graded algebra. The Ω_D^p need not vanish for large values of p. In addition, it follows by construction that the Ω_D^p are generated by the da as follows

$$\Omega_D^p = \left\{a_0 da_1 \cdots da_n,\ a_i \in M_{2m}^+(\mathbb{C})\ \forall i\right\}. \tag{2.78}$$

However, we would like to restrict to elements in $\mathfrak{A}$ seen as a subset of M_{2m}^+ through the representation ρ. We then define $\Omega_D^*(\mathfrak{A})$ in the exact same. In particular, we have

$$\Omega_D^1(\mathfrak{A}) = \{a_0 da_1,\ a_i \in \mathfrak{A},\ i = 1, 2\}. \tag{2.79}$$

We define an inner product on $B(\mathfrak{H})$ given by

$$(A, B)_{B(\mathfrak{H})} = Tr(B^* A), \tag{2.80}$$

and inducing a Hilbert space structure on $\Omega_D^k(\mathfrak{A})$ for any k.

Laplace operator

Following the definitions, we see that the differential da is not an element of $\mathfrak{A}$ in general, but is in $B(\mathfrak{H})$ nonetheless. Let p be the orthogonal projection operator on $\rho(\mathfrak{A})$ with respect to this inner product:

$$B(\mathfrak{H}) = \rho(\mathfrak{A}) \oplus \rho(\mathfrak{A})^\perp. \tag{2.81}$$

We can now introduce the adjoint operator $\delta : B(\mathfrak{H}) \to \rho(\mathfrak{A})$ of the differential d using Riesz representation theorem

$$(b, da)_{B(\mathfrak{H})} = (a', a)_{\rho(\mathfrak{A})} \tag{2.82}$$

and we set $\delta b := a'$.

Proposition 2.3.1. *The adjoint map δ to the derivation $d : \mathfrak{A} \to B(\mathfrak{H})$ is given by*

$$\delta : B(\mathfrak{H}) \to \mathfrak{A}, \quad \delta(b) = p[D, b].$$

Proof. Using the fact that D is hermitian, we first have

$$(b, da)_{B(\mathfrak{H})} = (b, [D, a])_{B(\mathfrak{H})} = ([D, b], a)_{B(\mathfrak{H})}. \tag{2.83}$$

Then, since $a \in \rho(\mathfrak{A})$ and $p^* = p$, it follows that:

$$([D, b], a)_{B(\mathfrak{H})} = ([D, b], pa)_{B(\mathfrak{H})} = (p[D, b], a)_{\rho(\mathfrak{A})}. \tag{2.84}$$

□

It is then straightforward to define a Laplace operator on $\mathfrak{A}$.

Definition 2.3.2. (Laplacian) The Laplace operator Δ is given by:

$$\Delta : \mathfrak{A} \to \mathfrak{A}, \quad \Delta(a) := -\delta da = -p\left[D, [D, b]\right],$$

where p is the orthogonal projection on $\mathfrak{A}$.

We can now state and prove a Hodge-like decomposition on $\Omega^*_D(\mathfrak{A})$.

Proposition 2.3.2 (Hodge-de Rham decomposition)**.** *The Laplacian Δ on $\Omega_D(\mathfrak{A})$ satisfies the following properties:*

i) $\Delta \geq 0$ *in the Hilbert space* $(\Omega_D(\mathfrak{A}), (\cdot, \cdot))$,

ii) $\Delta\alpha = 0$ *if and only if* $d\alpha = 0$,

iii) $\mathfrak{A} = \delta\Omega_D(\mathfrak{A}) \oplus \ker(\Delta)$ *is an orthogonal decomposition of* $\Omega_D(\mathfrak{A})$ *with respect to* $(\cdot, \cdot)$.

In addition, we will call harmonic these elements $\alpha \in \ker(\Delta)$.

Proof. Let $a \in \mathfrak{A}$, then, $(\Delta(a), a) = \|da\|^2$ which proves $i)$. The inclusion $\ker(\Delta) \subseteq \ker(d)$ follows from $i)$; the inclusion $\ker(d) \subseteq \ker(\Delta)$ is immediate from the definition of Δ. This proves *ii*). Finally, since δ is the adjoint to d, we have the following decomposition in finite dimension

$$\mathfrak{A} = \ker(d) \oplus \delta\Omega_D(\mathfrak{A}); \tag{2.85}$$

thus, *iii*) follows from *ii*). □

2.3.2 Dirac operator associated to a graph

So far, we have worked with a generic Dirac operator D, the only restriction being that D has to be hermitian and odd according to the grading. Nevertheless, one can exhibit a deeper connection between the space X (or equivalently the spectrum of $\mathfrak{A}$) and the Dirac operator. We first need to restrict the space of admissible Dirac operators.

Definition 2.3.3 (Admissible Dirac operators)**.** Let $D \in M_{2m}(\mathbb{C})$ be an odd and hermitian matrix and let ω_{ij} be the coefficients of the block D^-. We say that D is an admissible Dirac operator associate to X if it satisfies the additional condition:

a) vertices i and j do not share an edge $\Leftrightarrow \omega_{ij} = 0, \ \forall i, j \in \mathfrak{M}$,

b) the eigenvalues μ_n satisfy the asymptotic $\mu_n(D) = O(h^{-1})$.

We denote by $\mathcal{D}(X)$ the set of all admissible Dirac operators and by $\mathcal{D}_{\mathbb{R}}(X)$ the set of real admissible Dirac operators.

Example 2.3.1. The prototypical example is given by the *combinatorial Dirac operator,* for which:

$$\omega_{ij} := \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.3.3. *The graded algebra $\Omega_D(M_{2m}^+(\mathbb{C}))$ is invariant under the change $D \mapsto D'$ in $\mathcal{D}(X)$ i.e.*

$$\Omega_D(M_{2m}^+(\mathbb{C})) = \Omega_{D'}(M_{2m}^+(\mathbb{C}))$$

for any $D, D' \in \mathcal{D}(X)$.

Proof. The algebra $\Omega^p(\mathfrak{A})$ is generated by elements of the form $a_0 da_1 \cdots da_n$, with $a_i \in \mathfrak{A}$ for all $0 \leq i \leq n$. Now, we recall that for an element $a \in \mathfrak{A}$,

$$da = \sum_{i<j} \alpha_{ij}(a)\hat{e}_{ij}. \tag{2.86}$$

Therefore, $\Omega^p(\mathfrak{A})$ is generated by basis elements $\{\hat{e}_{ij}\}$ where an element $\hat{e}_{ij}$ is a generator if and only if vertices i a j share an edge. □

2.3.3 A first example: the lattice

We now come back to the example of a C^*-algebra over a line lattice L denoted by $\mathfrak{A}(L)$ before moving to the case of a d-dimensional lattice Λ. For the line lattice, we let the Dirac operator D to be an odd element of $M_{2m}(\mathbb{C})$ of the form:

$$D = \frac{i}{\hbar}\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \tag{2.87}$$

with $(D^+)^* = -D^-$ and where D^- is given by

$$D^- = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \tag{2.88}$$

Proposition 2.3.4. *For every element $da \in \Omega^1_D(\mathfrak{A})$, the spectrum $\sigma(da)$ of the operator da is given by:*

$$\sigma(da) = \left\{\pm\frac{1}{\hbar}(\lambda_{j+1} - \lambda_j) : 1 \leq j \leq m-1\right\} \cup \{0\}$$

Moreover, we have the commutativity relation

$$[da, db] = 0, \quad \forall a, b \in \mathfrak{A}. \tag{2.89}$$

Proof. Using Equation (2.74), we can write the commutator da as

$$da = \sum_{j=1}^{m-1} \alpha_{jj+m+1}(a)\hat{e}_{jj+m+1} \tag{2.90}$$

with the roots:

$$\alpha_{jj+m+1}(a) = \frac{1}{\hbar}(\lambda_{j+1} - \lambda_j), \quad j \in \{1, \cdots, m-1\}. \tag{2.91}$$

Then, we notice that the operator da^*da is a diagonal operator with diagonal entries given by:

$$\beta_{jj} = \frac{1}{\hbar^2}(\lambda_{j+1} - \lambda_j)^2 = \beta_{j+m+1j+m+1}, \tag{2.92}$$

for $j \in \{1, \cdots, m-1\}$ and $\beta_{m,m} = \beta_{m+1,m+1} = 0$. Thus, the eigenvalues of da are obtained as the square roots of the previous diagonal coefficients. Finally, the commutativity follows again from the fact that $dadb$ is a diagonal operator. □

Following the result on the spectrum of da, we can deduce a result on the states. In the finite dimensional case where $A = M_{2m}(\mathbb{C})$, a density matrix ω i.e. an operator with a graded trace $Tr_s(\omega) = 1$ defines a state. over A. Then, we can introduce the *expectation map* $a \mapsto < a >_\omega$ with respect to ω such that

$$< a >_\omega = Tr_s(\omega a). \tag{2.93}$$

Proposition 2.3.5. *There exists a density matrix ω with eigenvalues $\{\mu_k\}_k^{2m}$, such that the expectation value is given by*

$$< da >_\omega = Tr_s(\omega da) = \frac{i}{\hbar}\sum_{k=1}^{2m} \mu_k(\lambda_{k+1} - \lambda_k).$$

for any element $da \in \Omega_D^1(\mathfrak{A})$.

Proof. By Proposition 2.3.4, we know that the algebra generated by da for a running in $\mathfrak{A}$ is commutative. Then, it admits a common spectral decomposition. Therefore, one can choose $\omega = db$, for some element $b \in \mathfrak{A}$ such that the graded trace $Tr_s(db) = 1$, which conclude the proof. □

Remark 2.3.2. The last proposition is of importance in the approximation of differential operators. Indeed, it is well known that any finite difference formula for the first derivative can be written as a convex combination of the two-points approximation. It follows that the perspective is shifted in this context; instead of looking at the pointwise discretization of a derivative, one can study the density matrix ω. It is our hope that this change of paradigm, together with the machinery of C^*-algebra, allows us to produce new results in discretization of differential operators.

Direct limits of spectral triples

We can now complete the construction in the case of the lattice. Recall that we have defined a direct system of C^*-algebras $(A_n, \phi^*_{m,n})$ over an inverse system $(L_n, \phi_{m,n})$ of lattices. We can now associate a Dirac operator D_n to each algebra A_n. We will work on the infinite collection $\{\mathfrak{A}_n : n \in \mathbb{N}\}$ of commutative subalgebras C^*-algebras. In this case, we have identified each of the $\mathfrak{A}_n$ with the Cartan subalgebras h_i inside the finite dimensional algebras $B_n = M_{2m_n}(\mathbb{C})$ where $m_n \to \infty$ when $n \to \infty$. We can then construct the product

$$B_\omega = \prod_{n\in\mathbb{N}} B_n = \{(a_n) : \|a_n\| = \sup \|a_n\| < \infty\}. \tag{2.94}$$

Let a be an element in $C^\infty(\mathbb{R})$, then there exists a sequence (a_i) such that

$$a = (a_0, a_1, \cdots, a_n, \cdots) \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n. \tag{2.95}$$

We define a spectral triple on B_ω by introducing the Dirac operator D as the sequence

$$D = (D_0, D_1, \cdots, D_n, \cdots) \in \prod_{n\in\mathbb{N}} M^-_{2m_n}(\mathbb{C}). \tag{2.96}$$

This in turns induces a spectral triple on $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$ along with the commutator:

$$[D,a] = ([D_0,a_0],[D_1,a_1],\cdots,[D_n,a_n],\cdots) \in \prod_{n\in\mathbb{N}} M^{-}_{2m_n}(\mathbb{C}). \tag{2.97}$$

We can then characterize the operator $[D,a]$ and relate it to the classical differential on $C^\infty(M)$.

Lemma 2.3.1. *The spectrum $\sigma_{B_\omega}(x)$ of an element $x = (\cdots, x_n, \cdots) \in B_\omega$ is given by*

$$\sigma_{B_\omega}(x) = \overline{\cup_n \sigma_{B_n}(x_n)}.$$

Proof. $b = (\cdots, b_n, \cdots) \in B_\omega$ is invertible if and only if each b_n is invertible and $\{\|b_n\|^{-1} : n \in \mathbb{N}\}$ is bounded. Thus, $\sigma_{B_n}(x_n) \subset \sigma_{B_\omega}(x)$ for all $n \in \mathbb{N}$, Therefore, we have the first inclusion $S := \overline{\cup_n \sigma_{B_n}(x_n)} \subseteq \sigma_{B_\omega}(x)$. Reciprocally, if $\lambda \in \mathbb{C}\backslash S$, then $x_n - \lambda 1$ is invertible in B_n for each n and $\|(x_n - \lambda 1)^{-1}\| \leq d(\lambda, S)$, where $d(\lambda, S)$ is the distance from λ to S, therefore

$$(x - \lambda 1)^{-1} = (\cdots, (x_n - \lambda 1)^{-1}, \cdots) \in B_\omega. \tag{2.98}$$

$\square$

Then, for an element $a \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n$, the spectrum $\sigma_{B_\omega}([D,a])$ of the operator $[D,a]$ is given by:

$$\sigma_{B_\omega}([D,a]) = \overline{\cup_n \sigma_{B_n}([D_n,a_n])}. \tag{2.99}$$

We restrict now to an element $a \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n \cap C^\infty(\mathbb{R})$; using Proposition 2.3.4, we have:

$$\sigma_{B_n}([D_n,a_n]) = \left(\cdots, \frac{a_{x^n_{j+1}} - a_{x^n_j}}{h_n}, \cdots\right) = (\cdots, \ell(a)(x^n_j), \cdots) \tag{2.100}$$

where, $a_{x^n_j} = a(y)$, for some y such that $\phi_{n,\infty}(y) = x^n_j$. Then, the map $\phi_{n,n+1} : \mathfrak{M}_{n+1} \to \mathfrak{M}_n$ between maximal sets induces a map, denoted by: $\tilde{\phi}_{n,n+1} : \sigma_{B_{n+1}}([D_{n+1}, a_{n+1}]) \to \sigma_{B_n}([D_n,a_n])$ such that:

$$\tilde{\phi}_{n,n+1}(\ell(a)(x^n_j)) = \ell(a)(\phi_{n,n+1}(x^n_j)). \tag{2.101}$$

Then $\sigma_{B_\omega}([D,a])$ can be identified with the inverse limit given by the inverse system $(\sigma_{B_n}([D_n,a_n]),\mathbb{N},\tilde{\phi}_{n,n+1})$. Recalling that the manifold M is obtained from the maximal points in X_∞, we deduce that:

$$\sigma_{B_\omega}([D,a]) = \left\{ \left.\frac{da}{dx}\right|_x \right\}_{x\in\mathbb{R}}. \tag{2.102}$$

Therefore, if we denote by $d_c a$ the de Rham differential of a on $\mathbb{R}$, then we have the following result.

Proposition 2.3.6. *(Spectral convergence) There exists a finite measure μ and a unitary operator*

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}, d\mu) \tag{2.103}$$

such that,

$$U[D,a]U^{-1}\phi = \frac{da}{dx}\phi, \quad \forall\phi \in L^2(\mathbb{R}), \tag{2.104}$$

Moreover, the norm of $[D,a]$ is given by $\|\,[D,a]\,\| = \|d_c a\|_\infty$.

Proof. This result is an immediate consequence of the spectral theorem on self-adjoint bounded operator (Multiplication operator type) [44, pp.36-37] and the spectrum characterization (2.102). □

The d-dimensional lattice

We start by recalling that the C^*-algebra $A(\Lambda)$ is given by the tensor product,

$$A(\Lambda) = A(L)\otimes\cdots\otimes A(L), \tag{2.105}$$

where $A(L)$ is the algebra associated to the line lattice. Similarly, we consider the sequence of commutative subalgebras

$$\mathfrak{A}_n(\Lambda) = \mathfrak{A}_n(L)\otimes\cdots\otimes\mathfrak{A}_n(L), \tag{2.106}$$

for every $n\in\mathbb{N}$, that we embed it in the tensor product of matrix algebras

$$B_n = M_{2m_n}(\mathbb{C})\otimes\cdots\otimes M_{2m_n}(\mathbb{C}). \tag{2.107}$$

Again, we adjoin a Dirac operator on each algebra B_n given by:

$$D_n = \sum_{k=1}^{d} 1 \otimes \cdots \otimes D_n^{(k)} \otimes \cdots \otimes 1 \tag{2.108}$$

with the commutator on an element b defined as:

$$[D_n, b] = \sum_{k=1}^{d} b_1 \otimes \cdots \otimes [D_n^{(k)}, b_k] \otimes \cdots \otimes b_d. \tag{2.109}$$

This gives a spectral triple structure on B_ω by extending the commutator in the same way as Equation (3.38).

Proposition 2.3.7. *(Spectral convergence) There exists a finite measure μ and a unitary operator*

$$U : \otimes_{i=1}^{d} L^2(\mathbb{R}) \to \otimes_{i=1}^{d} L^2(\mathbb{R}, d\mu),$$

such that

$$U[D, a]U^{-1}\phi = \sum_{k=1}^{d} a_1\phi_1 \otimes \cdots \otimes \frac{\partial a_k}{\partial x_k}\phi_k \otimes \cdots \otimes a_d\phi_d,$$

for all $\phi = \phi_1 \otimes \cdots \phi_k \otimes \cdots \otimes \phi_d$ in $\otimes_{i=1}^{d} L^2(\mathbb{R})$.

Proof. Again, we use the fact that the spectrum $[D_n^{(k)}, a_k]$ is given by

$$\sigma_{B_\omega}([D^{(k)}, a_k]) = \left\{ \left.\frac{\partial a_k}{\partial x_k}\right|_{x_k} \right\}_{x_k \in \mathbb{R}}, \tag{2.110}$$

for very $k \in \{1, \ldots, d\}$; then using the unitary operator given by

$$U = U_1 \otimes \cdots \otimes U_k \otimes \cdots \otimes U_d, \tag{2.111}$$

where for every $k \in \{1, \ldots, d\}$, U_k is the (same) unitary operator given by Proposition 3. □

2.3.4 Beyond the combinatorial Dirac operator: the metric question

Throughout this work the metric of the manifold was assumed but not used explicitly. The construction so far and convergence results were algebraic. We now provide another canonical example to highlight some subtleties and provide a starting point for the second part of this work regarding convergence of the metric.

A second example: the torus $\mathbb{T}^d$

We now turn our attention to the case of the d-dimensional torus $\mathbb{T}^d$; we do not specify the metric yet. This example has two purposes, firstly show how the matrix D depends on the topology of the space X and secondly exhibit certain subtleties with respect to the eigenvalues of the commutator. Indeed on the latter, one has to be able to actually compute them and moreover, these eigenvalues should reflect (in the limit) the metric g on the manifold M.

We start with the case of the circle S^1, since the general case of the torus in an arbitrary dimension d is obtained by direct product (similarly to the approach used above for the lattice in $\mathbb{R}^d$). Hence, we are looking for the Dirac operator D associated to a graph obtained from a triangulation of S^1. The block matrix D^- may be directly read from Figure 2.2 and is given by:

$$D^- = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \tag{2.112}$$

where we notice the non-zero coefficients on the down-left corner. We then compute the eigenvalues of the operator da. Indeed, for every element $da \in \Omega^1_D(\mathfrak{A})$, the spectrum $\sigma(da)$ of the operator da is given by:

$$\sigma(da) = \left\{ \pm\frac{1}{\hbar}(\lambda_{j+1} - \lambda_j) : 1 \leq j \leq m-1 \right\} \cup \left\{ \pm\frac{1}{\hbar}(\lambda_m - \lambda_1) \right\}. \tag{2.113}$$

Taking the limit $h \to 0$, we deduce that there exists a finite measure on S^1 and a unitary operator u acting on $L^2(S^1)$ such that

$$(u[D,a]u^*)\,\phi = \frac{da}{d\theta}\phi, \quad \forall \phi \in L^2(S^1). \tag{2.114}$$

We may now use this result to work out the general case for the d-dimensional torus $\mathbb{T}^d$.

There exists a unitary operator u acting on $L^2(\mathbb{T}^d)$ such that

$$(u[D,a]u^*)\,\phi = \sum_{k=1}^{d} a_1\phi_1 \otimes \cdots \otimes \frac{\partial a_k}{\partial \varphi_k}\phi_k \otimes \cdots \otimes a_d\phi_d,$$

for all $\phi = \phi_1 \otimes \cdots \phi_k \otimes \cdots \otimes \phi_d$ in $\otimes_{i=1}^{d} L^2(S^1) \simeq L^2(\mathbb{T}^d)$.

It is somewhat obvious that up to some topological changes in the Dirac operator, the results obtained for the d-dimensional torus are very similar to those obtained for the lattice in $\mathbb{R}^d$. However, seen as Riemannian manifolds, one may expect the approximation of $\mathbb{R}^d$ with its standard metric on one hand, and of the torus $(\mathbb{T}^d, g)$ with a metric g on the other hand, to reflect the intrinsic geometrical differences between those two manifolds. With this observation in mind, we now highlight some of the required work to be able to have a convergence result that is both algebraic and geometric.

Discussion

As already mentioned, we recall that the above results do not depend on the metric g on the torus. Indeed, one could either look at the flat torus with the metric inherited from a quotient of $\mathbb{R}^{d+1}$ or the metric induced from the ambient space in which it is embedded. However, as it is currently defined, there is no use of the metric in D;

hence one could, for instance on S^1, redefine the matrix D^- using the ansatz

$$D^- = \rho(\theta) \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 0 \end{pmatrix} . \tag{2.115}$$

where ρ is a function depending on the metric. In order to make the discussion more concrete, we consider the 2-dimensional torus $\mathbb{T}^2$ with the usual parameterization,

$$\Psi(\theta, \phi) = \langle (R + r\cos\phi)\cos\theta, (R + r\cos\phi)\sin\theta, r\sin\phi) . \tag{2.116}$$

Then, the inverse of the metric is given (in matrix form) by

$$g^{ij} = (g_{ij})^{-1} = \begin{pmatrix} \frac{1}{R+r\cos\phi} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} . \tag{2.117}$$

Therefore, we could define the Dirac operator in that specific case, following Equation (2.108), by:

$$D = g^{11} (D_1 \otimes 1) + g^{22} (1 \otimes D_2) . \tag{2.118}$$

Nevertheless, this construction is only, in general, a local description i.e. it is valid for a manifold that has an atlas consisting of only one chart (U, φ) . In fact, one could adopt an *extrinsic* point of view of the Dirac operator: given a local chart (U, φ), with a local metric $g|_U$, one could consider a lattice-like approximation of U and define a Dirac operator $D|_U$ following the same construction from Equations (2.115) and (2.118). However, given an atlas containing two coordinate charts (U, φ_U) and (V, φ_V) with associated lattices, say λ_U and λ_V and Dirac operators D_U and D_V, it is unclear how to glue (i.e. transition) them to obtain a lattice $\Lambda_{U\cup V}$ with a Dirac operator $D_{U\cup V}$ that restricts to D_U on Λ_U, respectively D_V on Λ_V.

Now, one may adopt an *intrinsic* point of view instead, where the matrix D is defined globally starting from a graph. However, this time we see that the coefficients, say ω_{ij}, of the Dirac operator must depend on the intrinsic geometry of the manifold i.e. the coefficients ω_{ij} are not simply 0 or 1 but are computed from a priori knowledge of the metric g.

From the perspective of the above observations regarding the metric, and the non-triviality of the manifold (which atlas may comprise more than one chart), is the eigenvalues of a *compatible Dirac operator*. Indeed, in the examples treated above, the correspondence between D and the graph associated to the triangulation gives a commutator for wish the eigenvalues are easy to compute. However, performing the same task in all generality may be difficult. Moreover, there is no evidence that given a sequence of commutators, this sequence will converge to the exterior derivative.

It is now clear that the fundamental question of the choice of coefficients ω_{ij} and their relations with the intrinsic properties of the manifold must be tackled in order to see the desired geometry emerge from a sequence of discrete approximations.

Summary and perspectives So far, we have defined a spectral triple (A, H, D) on a given triangulation X. It is crucial to note at this point that we may rely solely on the spectral triple as it encodes the space X . Indeed, we have shown in Section 2.2 that the algebra A plays the role of functions on X and is enough to recover smooth functions in the limit. Moreover, we have built a correspondence between the given triangulation X and a Dirac operator D: the non-zero coefficients of D are determined by the connectivity between vertices of the graph. Hence, the sparsity pattern of D encodes to some extent the topology of X. The bracket $[D, a]$ can be then represented as a bounded operator acting on the Hilbert space H.

However, the above discussion has shown that this was not enough to represent the metric of the manifold. Thus, we ask now the question of how to set the coefficients ω_{ij} of D so that at the limit (in the sense of (3.38)) the sequence converges. Such a convergence by setting the coefficients ω_{ij} of D while preserving the sparsity pattern of D is precisely the notion of compatible discretization (or structure-preservation) we are pursuing.

Chapter 3

Statistical fluctuations of infinitesimal spaces

3.1 Dirac operators in the algebraic setting

In this section, we introduce two of the main algebraic tools that we are going to use in this study: the Clifford algebras and the universal enveloping Lie algebra. We then define and study Dirac operators on finite spectral triples in terms of root vectors of a Lie algebra $\mathfrak{g}$.

3.1.1 Noncommutative Geometry on Infinitesimal spaces

In the research paper [117], we show that a discrete topological space X can be identified to the spectrum $Spec(A)$ of a C^*-algebra A. Starting with a Riemannian manifold (M, g), we construct an inverse system of triangulations, (K_n) which become sufficiently fine for large n. Using the Behncke-Leptin construction, we associate to each K_n a C^*-algebra A_n such that the triangulation K_n is identified with the spectrum $Spec(A_n)$. We then form an inductive system (A_n) with limit A_∞.

Theorem 1. *The spectrum $Spec(A_\infty)$ equipped with the hull-kernel topology is homeomorphic to the space X_∞ and*

$$\lim_{\leftarrow} Spec(A_i) \simeq Spec(\lim_{\rightarrow} A_i). \tag{3.1}$$

We then show that the centre of A_∞ is isomorphic to the space of continuous function $C(M)$. In this sense, any element $g \in C(M)$ can be uniformly approximated arbitrarily closely by elements a_n in the central subalgebras $\mathfrak{A}_n$.

Theorem 2. *The space of continuous function $C(M)$ is approximated by the system of commutative subalgebras $(\mathfrak{A}_n, \phi^*_{n,\infty})$ in the following sense:*

$$C(M) = \overline{\bigcup_{n\in\mathbb{N}} \phi^*_{n,\infty}(\mathfrak{A}_n)} \cap C(M). \tag{3.2}$$

In addition, the sequence of representations (H_n) is also considered as a direct system with limit H_∞ containing the space of square integrable functions $L^2(M)$. Finally, we define the spectral triples $(\mathfrak{A}, \mathfrak{h}, D_n)$, where D_n is a so-called Dirac operator. We show that under certain conditions, the sequence (D_n) converges to the multiplication operator by the de Rham differential $d_c a$.

Theorem 3. *(Spectral convergence) There exists a finite measure μ and a unitary operator*

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}, d\mu) \tag{3.3}$$

such that,

$$U[D,a]U^{-1}\phi = \frac{da}{dx}\phi, \quad \forall \phi \in L^2(\mathbb{R}), \tag{3.4}$$

Moreover, the norm of the commutator is given by $\| \, [D,a] \, \| = \|d_c a\|_\infty$.

Thus, we have built a correspondence between a given triangulation X and a Dirac operator D: the non-zero coefficients of D are determined by the connectivity between vertices of the graph. We showed that this is however not enough to represent the metric of the manifold. Thus, we ask now the question on how to set the coefficients ω_{ij} of D so that at the limit (in the sense of (3.38)) the sequence converges.

3.1.2 Clifford algebras

Let V be a finite dimensional vector space over a commutative field $\mathbb{K}$ of characteristic zero endowed with a quadratic form q. Let $T(V)$ be the tensor algebra over V.

Consider the ideal I_q in $T(V)$ generated by all elements of the form $v \otimes v + q(v)$ for $v \in V$. Then the quotient algebra

$$Cl(V,q) = T(V)/I_q. \tag{3.5}$$

is the Clifford algebra associated to the quadratic space (V,q).
Moreover, we can choose any orthonormal basis Z_i of V with respect to q as a set of generators of $Cl(V)$. We then have the relations,

$$Z_i Z_j = -Z_j Z_i, \quad i \neq j, \quad Z_i^2 = -1. \tag{3.6}$$

Then the following set

$$Z_{i_1} Z_{i_2} \cdots Z_{i_k} \quad 1 \leq < i_1 < i_2 < \cdots < i_k \leq n = \dim V \tag{3.7}$$

spans $Cl(V)$. In addition, given a q-orthonormal basis Z_i of V, the mapping

$$1 \mapsto 1, \quad Z_{i_1} \cdots Z_{i_k} \mapsto Z_{i_1} \wedge \cdots \wedge Z_{i_k} \tag{3.8}$$

yields an isomorphism of vector spaces $Cl(V,q) \simeq \bigwedge V$.

3.1.3 Dirac operators in the Clifford algebra setting

Let $k = \mathbb{R}, \mathbb{C}$ and let $\mathfrak{g}$ be a Lie algebra over k. We start by recalling the definition of the universal enveloping algebra.

Definition 3.1.1. The *universal enveloping algebra* of $\mathfrak{g}$ is a map $\varphi : \mathfrak{g} \to U(\mathfrak{g})$, where $U(\mathfrak{g})$ us a unital associative algebra, satisfying the following properties:

1) φ is a Lie algebra homomorphism, i.e. φ is k-linear and

$$\varphi([X,Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X). \tag{3.9}$$

2) If A is any associative algebra with a unit and $\alpha : \mathfrak{g} \to A$ is any Lie algebra homomorphism, there is a unique homomorphism of associative algebras β :

$U(\mathfrak{g}) \to A$ such that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & U(\mathfrak{g}) \arrow[dl, "\beta"] \\ A & \end{tikzcd}$$

is commutative, i.e. there is an isomorphism

$$\mathrm{Hom}_{Lie}(\mathfrak{g}, LA) \simeq \mathrm{Hom}_{Ass}(U(\mathfrak{g}), A) \tag{3.10}$$

We will now give a definition of Dirac operators on finite spectral triples in terms of root vectors of a Lie algebra $\mathfrak{g}$. Then, using the canonical embedding $\mathfrak{g} \hookrightarrow Cl(\mathfrak{g})$ into the Clifford algebra, we define a Laplace-type operator.

Consider the algebra $A = \mathfrak{gl}_{2N}(\mathbb{C})$ of complex matrices with its standard Lie algebra structure. In [117], we have introduced the finite dimensional spectral triple $(\mathfrak{A}, \mathfrak{h}, D)$ given by:

- $\mathfrak{A}$ is a Cartan subalgebra of Lie subalgebra $\mathfrak{g}$ of A,
- $\mathfrak{h} = \mathbb{C}^{2N}$,
- $\gamma = \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix}$.

The chirality element γ induces a decomposition of the representation space $\mathfrak{h}$ into the eigenspaces $\mathfrak{h}^{\pm}$ corresponding to the eigenvalues 1 and -1 such that $\mathfrak{h} = \mathfrak{h}^{+} \oplus \mathfrak{h}^{-}$. Incidentally, one has the decomposition of the algebra A as follows:

$$\mathfrak{gl}_{2N} = \mathfrak{gl}_{2N}^{+} \oplus \mathfrak{gl}_{2N}^{-}. \tag{3.11}$$

Notice then that the pair $(\mathfrak{gl}_{2N}^{+}, \mathfrak{gl}_{2N}^{-})$ forms a Cartan pair. Any endomorphism $a \in \mathrm{End}(\mathfrak{h})$ defines an endomorphism $\rho_a \in \mathfrak{gl}_{2N}^{+}$ given by

$$\rho_a = \begin{pmatrix} a & 0 \\ 0 & -a^T \end{pmatrix} \in \mathfrak{sp}(2N, \mathbb{C}) \cap \mathfrak{gl}_{2N}^{+}. \tag{3.12}$$

We consider the compact real case with the embedding

$$\mathfrak{sp}(N) = \mathfrak{sp}(2N, \mathbb{C}) \cap \mathfrak{u}(2N) \hookrightarrow \mathfrak{so}(4N). \tag{3.13}$$

If a is a diagonal element of $\mathrm{End}(\mathfrak{h}^+)$, the map $a \mapsto \rho_a$ identifies a with an element of the maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{so}(4N)$:

$$\mathfrak{t} = \left\{ \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{pmatrix}, A_j = \begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix} \right\} \tag{3.14}$$

Consider the Cartan subalgebra $\mathfrak{A} = \mathfrak{t} + i\mathfrak{t}$ of $\mathfrak{so}(4N, \mathbb{C})$. The root vectors are $4N \times 4N$ block matrices having 2×2-matrix C_s, $s \in \{1, \dots, 4\}$

$$X = \begin{pmatrix} 0 & C_s \\ -C_s^t & 0 \end{pmatrix} \tag{3.15}$$

in the position (i, j) with $i < j$ and where

$$C_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}.$$

associated to the linear functional in $\mathfrak{H}^*$ given by $i(a_i + a_j)$, $-i(a_i + a_j)$, $i(a_i - a_j)$ and $i(a_j - a_i)$.

We will denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{so}_{4N}$. We then consider the unital associative algebra $M_2(\mathbb{C}) \otimes \mathfrak{gl}_{2N}$ and the homomorphism:

$$\varphi : \mathfrak{g} \to M_2(\mathbb{C}) \otimes \mathfrak{gl}_{2N}, \quad \varphi(X) = \sum_{1 \leq i,j \leq 2N} X_{ij} \otimes E_{ij}, \tag{3.16}$$

where E_{ij} is the standard basis in $\mathfrak{gl}_{2N}$ and X_{ij} are the 2×2-submatrix of $X = (x_{rs})$ obtained by keeping $i + 1 \leq r \leq i + 2$ and $j + 1 \leq s \leq j + 2$. In addition, φ is a Lie algebra homomorphism with $\varphi([X, Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$.
Then, using the universal property of $U(\mathfrak{g})$, the map φ extends into the homomorphism $\widehat{\varphi} : U(\mathfrak{g}) \to M_2(\mathbb{C}) \otimes \mathfrak{gl}_{2N}$. Furthermore, taking the canonical embedding

$h : \mathfrak{gl}_{2N} \to U(\mathfrak{gl}_{2N})$, we get by composing the Lie algebra homomorphism

$$h \circ \widehat{\varphi} : U(\mathfrak{g}) \to M_2(\mathbb{C}) \otimes U(\mathfrak{gl}_{2N}). \tag{3.17}$$

Let $\{Z_{ij}\}$ be an orthonormal basis of root vectors in $\mathfrak{g}$, associated to the root $-i(a_j + a_k)$, we define the operator W by

$$W = \sum_{i,j} \omega_{ij}^W Z_{ij} \tag{3.18}$$

as an element of $U(\mathfrak{g})$, where ω_{ij}^W are real coefficients.

Definition 3.1.2. Given an operator W as in (3.18), a Dirac operator D_W is an element of $M_2(\mathbb{C}) \otimes U(\mathfrak{gl}_{2N})$ defined by:

$$D_W = \frac{i}{\hbar}\mathfrak{Re}(W), \tag{3.19}$$

where $\hbar > 0$ is a real parameter.

Remark 3.1.1. In the previous definition, D_W depends on the choice of element W and in fact, more specifically on the choices of basis elements Z_{ij}. Another definition, independent on the choice of basis elements, of Dirac operators on Lie algebras can be found in [97]

Lemma 3.1.1. *Let $C_2 = X + iY$ be the root vector associated to the root $-i(a_i + a_j)$. Fix an element W as in (3.18). Then, for any $a \in \mathfrak{A}$, the exterior derivative can be written as:*

$$[D_W, a] = \frac{i}{\hbar}\sum_{i,j} \omega_{ij}^W \alpha_{ij}(a) Y \otimes E_{ij}, \tag{3.20}$$

an element of $M_2(\mathbb{C}) \otimes U(\mathfrak{gl}_{2N})$ and with $\alpha_{ij} = a_i - a_j$.

Proof. From the definition of D_W and the definition of root vectors, we get that:

$$[D_W, a] = \frac{i}{2\hbar}\sum_{i,j} \omega_{ij}^W (a_i - a_j) Z_{ij} - \frac{i}{2\hbar}\sum_{i,j} \omega_{ij}^W (a_i - a_j) Z_{ij}^*. \tag{3.21}$$

Then, using the map $h \circ \widehat{\varphi}$, given by (3.16) and (3.17), we can identify a basis element

Z_{ij} with an element in $M_2(\mathbb{C}) \otimes U(\mathfrak{g})$ of the form $C_2 \otimes E_{ij}$. Hence, we have that:

$$[D_W, a] = \frac{i}{2\hbar} \sum_{i,j} \omega_{ij}^W (a_i - a_j) C_2 \otimes E_{ij} - \frac{i}{2\hbar} \sum_{i,j} \omega_{ij}^W (a_i - a_j) C_2^* \otimes E_{ij}^t. \tag{3.22}$$

Simplifying this expression using the fact that $E_{ij}^t = E_{ji}$, we get:

$$[D_W, a] = \frac{i}{\hbar} \sum_{ij} \omega_{ij}^W \alpha_{ij}(a) Y \otimes E_{ij}. \tag{3.23}$$

with $\alpha_{ij}(a) = a_i - a_j$. □

Furthermore, we recall that there exists a canonical Lie algebra homomorphism $\psi : \mathfrak{gl}_{2N} \to Cl(\mathfrak{gl}_{2N})$ which extends into the map on the universal enveloping Lie algebra:

$$\widehat{\psi} : U(\mathfrak{gl}_{2N}) \to Cl(\mathfrak{gl}_{2N}). \tag{3.24}$$

We use this map to define a Laplace operator.

Definition 3.1.3 (Laplacian)**.** Fix an element W. We then define the Laplace operator $\varDelta$ on $\mathfrak{A}$ using the non-graded commutator. For any $a \in \mathfrak{A}$

$$\varDelta(a) := \frac{1}{2} \widehat{\psi}([D_W, [D_W, a]]). \tag{3.25}$$

Proposition 3.1.1. *For any $a \in \mathfrak{A}$, the Laplace operator is given by*

$$\varDelta(a) = -\varOmega_{\mathfrak{g}}(a) \otimes 1 \tag{3.26}$$

where $\varOmega_{\mathfrak{g}} = \frac{1}{\hbar^2} \sum_{i,j} \omega_{ij}^2 J \otimes \alpha_{ij}$ is an element of $End(\mathfrak{A}, M_2(\mathbb{C}))$.

Proof. Let D_W be a Dirac operator, then the bi-commutator of the Laplacian gives::

$$[D_W, [D_W, a]] = D_W \, [D_W, a] - [D_W, a] \, D_W. \tag{3.27}$$

Thus, using Lemma 3.1.1, we obtain

$$[D_W,[D_W,a]] = \frac{2}{\hbar^2}\sum_{ij}\omega_{ij}^2\alpha_{ij}(a)J\otimes E_{ij}^2 + \frac{1}{\hbar^2}\sum_{(ij)\neq(kl)}\omega_{ij}\omega_{kl}\alpha_{kl}(a)J\otimes[E_{ij},E_{kl}]_+ \tag{3.28}$$

with the bracket $[A,B]_+ = AB+BA$ and where the matrix J is given by:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Finally, applying the map $\widehat{\psi}$, the second term of the left-hand-side in Equation (3.28) vanishes and we get:

$$\Delta(a) = -\frac{1}{\hbar^2}\sum_{i,j}\omega_{ij}^2\alpha_{ij}(a)J\otimes 1.$$

□

We have kept the definition of the Dirac operator D_W in (3.19) very general, however we recall that the operator we are interested in are the *compatible* ones with respect to a graph X. In other words, the value ω_{ij} is non-zero if ij is an edge in X.

Now, let us recall how the space X is obtained from a manifold M; more details can be found in [117]. One starts with a triangulation of M and then consider the *dual of the triangulation* that we will call X. In fact, in [117] we used a slightly different terminology and considered the triangulation as a poset, then looked at the opposite poset with reversed order.

Since we are working with a graph X obtained from a dual triangulation, every vertex i has exactly $d+1$ neighbours, i.e. only $d+1$ of the ω_{ij} are non-zero for a fixed i. Hence, if we fix a vertex i_0, the definition of the commutator with D_W becomes:

$$([D_W,a])_{i_0} = \frac{i}{\hbar}\sum_{j=1}^{d+1}\omega_{i_0k_j}^W\alpha_{i_0k_j}(a)Y\otimes E_{i_0k_j}, \tag{3.29}$$

Here, we relabel the index j without lost of generalities and to keep this indexing simple. We will also drop the W index for the same reasons and get:

$$([D,a])_{i_0} = \frac{i}{\hbar}\sum_{j=1}^{d+1} \omega_{i_0 j}\alpha_{i_0 j}(a) Y \otimes E_{i_0 j}, \tag{3.30}$$

Finally, let us recall the (true) Dirac operator on a manifold is given in local coordinates on a normal neighbourhood centred at a point p:

$$\mathcal{D}_p = \sum_{j=1}^{d} e_j \left.\frac{\partial}{\partial x_j}\right|_p \tag{3.31}$$

where $\{e_j \mid j = 1,\dots,d\}$ is an orthonormal local frame embedded in the Clifford algebra $Cl(\mathbb{R}^d)$ using the natural embedding $\mathbb{R}^d \subset Cl(\mathbb{R}^d)$.

Nevertheless, the Dirac operator as expressed in (3.30) is not an element of a Clifford algebra. Moreover, the dimensions do not match. Indeed, because of the structure of the triangulation, there are $d+1$ independent vectors in the expression (3.30), instead of d as the dimension of the manifold M. Since we are trying to approximate the true Dirac operator in (3.31), we need to re-write Equation (3.30) in terms of Clifford elements in dimension d. To do so, let us denote by V_{i_0}, the vector space defined by:

$$V_{i_0} := \operatorname{span}\left\{Y \otimes E_{i_0,1}, \cdots, Y \otimes E_{i_0,d+1}\right\}. \tag{3.32}$$

Then, consider the isomorphism:

$$\tau : V_{i_0} \xrightarrow{\simeq} \mathbb{R}^{d+1} \quad \tau(Y \otimes E_{i_0,j}) = \widehat{e}_j, \ \forall 1 \le j \le d+1 \tag{3.33}$$

where $\{\widehat{e}_j\}_{j=1}^{d+1}$ is the canonical basis on $\mathbb{R}^{d+1}$ with respect to the standard inner product. Moreover, defines the projection p on the subspace spanned by $\{\widehat{e}_j\}_{j=1}^{d}$ and identified with $\mathbb{R}^d$. Finally, if let the embedding $\rho : \mathbb{R}^d \to Cl(\mathbb{R}^d)$, we can compose these maps and define:

$$\Psi := \rho \circ p \circ \tau : V_{i_0} \to Cl(\mathbb{R}^d), \quad \Psi\left([D,a]_{i_0}\right) = \frac{i}{\hbar}\sum_{j=1}^{d} \omega_{i_0 j}\alpha_{i_0 j}(a) e_j \tag{3.34}$$

which allows us to express the commutator in terms of Clifford elements e_j. We notice, nevertheless, that this construction is not canonical and depends on the choice of isomorphism τ.

3.1.4 Perron-Frobenius bound on $[D, a]$

To conclude this section, and before being able to show a convergence result to the Dirac operator $\mathcal{D}$, we would like to prove a preliminary result on the commutator $[D, a]$ and its boundedness at the limit when $\hbar \to 0$. This result follows from the correspondence between D and the graph associated, using the Perron-Frobenius theorem. We only need to consider the operator D as a compatible operator in some matrix space, without relying on the Clifford algebra setting

We consider an infinite collection $\{\mathfrak{A}_n : n \in \mathbb{N}\}$ of commutative C^*-algebras. In this case, we have identified each of the $\mathfrak{A}_n$ with the Cartan subalgebras h_i inside the finite dimensional algebras $B_n = \mathfrak{so}_{2m_n}(\mathbb{C})$ where $m_n \to \infty$ when $n \to \infty$. We can then construct the product:

$$B_\omega = \prod_{n\in\mathbb{N}} B_n = \{(a_n) : \|a_n\| = \sup \|a_n\| < \infty\}. \tag{3.35}$$

Let a be an element in $C^\infty(M)$, then there exists a coherent sequence (a_i) such that

$$a = (a_0, a_1, \cdots, a_n, \cdots) \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n. \tag{3.36}$$

We define a spectral triple on B_ω by introducing the limit Dirac operator D as the sequence

$$D = (D_0, D_1, \cdots, D_n, \cdots) \in \prod_{n\in\mathbb{N}} \mathfrak{gl}^-_{2m_n}(\mathbb{C}), \tag{3.37}$$

where each D_i is a Dirac operator associated to a poset X_i^{op} in the sense of [117]. This in turns induces a spectral triple on $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$ along with the commutator:

$$d_D a := [D, a] = ([D_0, a_0], [D_1, a_1], \cdots, [D_n, a_n], \cdots) \in \prod_{n\in\mathbb{N}} \mathfrak{gl}^-_{2m_n}(\mathbb{C}). \tag{3.38}$$

In order to show that $[D, a]$ is a bounded operator, we use Perron-Frobenius theorem, which we start by recalling.

Theorem 3.1.1 (Perron-Frobenius). *Let $A = (a_{ij})$ be an $n \times n$ positive matrix: $a_{ij} > 0$ for $1 \leq i, j \leq n$. Then there exists a positive real number r, called the Perron-Frobenius eigenvalue, such that r is an eigenvalue of A. Moreover, if the spectral radius $\rho(A)$ is equal to r.*
The Perron-Frobenius eigenvalue satisfies the inequalities:

$$\min_i \sum_j a_{ij} \leq r \leq \max_i \sum_j a_{ij}.$$

Proposition 3.1.2. *For any $a \in \mathfrak{A}$, the spectral radius $\rho(d_D a)$ of $d_D a$ is bounded by*

$$\rho(d_D a) \leq \|d_{dR} a\|_\infty. \tag{3.39}$$

Proof. We consider the sequence of Dirac operators $(D_\alpha)_{\alpha \in \mathbb{N}}$ associated to D.

Let $\varepsilon > 0$ and $\alpha \in \mathbb{N}$ and define the operator $\widetilde{d_{D_\alpha} a}$ such that

$$(\widetilde{d_D a})_{ij} = \begin{cases} |(d_{D_\alpha} a)_{ij}| & \text{if } (d_{D_\alpha} a)_{ij} \neq 0 \\ \varepsilon & \text{otherwise} \end{cases} \tag{3.40}$$

The matrix $\widetilde{d_{D_\alpha} a}$ is positive by construction. In addition, we have the upper-bound:

$$\| \, (d_{D_\alpha} a)^k \, \|_F^2 \leq \|(\widetilde{d_{D_\alpha} a})^k\|_F^2. \tag{3.41}$$

Hence, using Theorem 3.1.1, we deduce that

$$\begin{aligned} \rho(d_{D_\alpha} a)^2 = \lim_{k\infty} \|(d_{D_\alpha} a)^k\|_F^{\frac{2}{k}} &\leq \lim_{k\infty} \|(\widetilde{d_{D_\alpha} a})^k\|_F^{\frac{2}{k}} \\ &= \rho(\widetilde{d_{D_\alpha} a})^2 \\ &\lesssim \max_{1 \leq i \leq n} \sum_j |(d_{D_\alpha} a)_{ij}|^2 + N\varepsilon^2. \end{aligned}$$

The value of N is the number of nonzero coefficient in $(d_{D_\alpha} a)_{ij}$ and thus only depends on the number of adjacency vertex in X_α^{op} which by definition equal to $d+1$, where d is the dimension.

Hence there exists a positive constant C_M, which depends on the maximal length of geodesics (M is compact) but is independent of α, such that

$$\rho(d_{D_\alpha} a)^2 \leq C_M \|d_{dR} a\|_\infty^2 + (d+1)\varepsilon^2. \tag{3.42}$$

The last inequality holds for an arbitrary $\varepsilon > 0$ and $\alpha \in \mathbb{N}$. The result follows then by taking ε to 0. □

Corollary 3.1.1. *For each $a \in \mathfrak{A}$, the operator $[D, a]$ is a bounded operator.*

Remark 3.1.2. It is clear that in the following framework, not only the Dirac operator D define a differential structure, but it also plays the role of a transition matrix. This last point will be made clearer in the following section.

3.2 Green's function and integral operators

In this section, we are going to introduce the Green's function of a suitable second-order Cauchy problem. From it, we exhibit a probability distribution that will be used in Section 3.4 in the definition of the Dirac operator. Furthermore, we will extend this probability distribution defined in $\mathbb{R}^d$ to a manifold using the exponential map and prove some technical lemmas.

3.2.1 Hamilton-Jacobi equation with vanishing viscosity

For reasons that will be detailed in this section, we are interested in the following Hamilton-Jacobi equation with vanishing viscosity over the Euclidean space $\mathbb{R}^d$:

$$\begin{aligned} &\partial_t u = s \cdot \nabla u + \varphi(t) \Delta u \\ &u(x, 0) = u_0(x) \end{aligned} \tag{3.43}$$

where s is a unit vector in $\mathbb{R}^d$, u_0 is a smooth initial condition and $\varphi \in C(\mathbb{R}_+)$ satisfying:

$$\varphi(t) = O_{t\to 0^+}(t). \tag{3.44}$$

The fundamental solution of Equation (3.43) is obtained by taking the initial condition to be the Dirac distribution $\delta(y-x)$ for $y \in \mathbb{R}^d$ fixed. The normalized fundamental solution denoted by G is given by:

$$G_t(x,y) = \frac{1}{(4\pi\Phi(t))^{\frac{d}{2}}} \exp\left(-\frac{|y-x+st|^2}{4\Phi(t)}\right) dx, \qquad \text{with } \; \Phi(t) = \int_0^t \varphi(s)ds. \tag{3.45}$$

Hence, the general solution can be obtained by convolution:

$$u(y,t) = (u_0 * G)(y,t) \tag{3.46}$$

then we have the following lemma.

Lemma 3.2.1. *Let u be the solution of Equation(3.43) with initial condition $u_0 \in \mathbb{C}^\infty(\mathbb{R}^d)$, then it satisfies the initial condition:*

$$\partial_t u|_{t=0} = s \cdot \nabla u_0. \tag{3.47}$$

Proof. First, let us notice that the family $\{G_t\}_{t>0}$ is an approximation of the identity:

$$\forall f \in \mathbb{C}^\infty(\mathbb{R}^d), \quad \lim_{t\to 0} \|f - f * G_t\|_\infty. \tag{3.48}$$

In addition, since the differential operator in Equation (3.43) has constant coefficients in the y variable, we have:

$$\partial_t u = (s\cdot\nabla + \varphi\Delta)u_0 * G \tag{3.49}$$

from which we get by taking the limit when t goes to 0:

$$\partial_t u(y,0) = s \cdot \nabla u_0(y). \tag{3.50}$$

□

Therefore, we can define the probability measure $d\nu$ given by

$$d\mu_{y,t}(x) = G_t(x,y)dx \tag{3.51}$$

with respect to the Lebesgue measure on $\mathbb{R}^d$. Hence, we define the distribution:

$$T_{\mu_{y,t}}(\varphi)(y,t) = \int_{\mathbb{R}^d} \varphi(x) d\mu_{y,t}(x), \tag{3.52}$$

that will focus our attention in the next section.

Lemma 3.2.2 (Reduction to a ball). *Consider an open ball $B_\delta \subset \mathbb{R}^n$ of radius delta > 0 such that $p \in B_\delta$. For any function $f \in L^\infty(B_\delta)$ and a smooth extension $\overline{f}$ of f in $L^\infty(\mathbb{R}^d)$, we have, as $t \to 0$:*

$$\left| \int_{B_\delta} G_t(x,y) f(x) dx - \int_{\mathbb{R}^d} G_t(x,y) \overline{f}(x) dx \right| = o(t^d) \tag{3.53}$$

Proof. Without lost of generality, we may take $y = 0$; after a change of variable $u = \frac{x}{2\sqrt{\Phi(t)}}$, we see that:

$$\left| \int_{B_\delta} G_t(x,y) f(x) dx - \int_{\mathbb{R}^d} G_t(x,y) f(x) dx \right| \leq \frac{\|\overline{f}\|_\infty \Phi(t)^{\frac{d}{2}}}{\pi^{\frac{d}{2}}} e^{-\frac{t^2}{4\sqrt{\Phi(t)}}} \int_{B^c_{\delta_t}} e^{-\frac{\delta}{4}|u|} dx$$

with $\delta_t = \frac{\delta}{2\Phi(t)}$ and where $\Phi(t)$ decreases as t tends to zero. □

Thus, we may equivalently consider the distribution T_μ restricted on a ball B_δ.

3.2.2 Some remarks families of one-parameter operators

The previous construction of the Green function of the Cauchy problem (3.43) can be reformulated in the more general setting of families of one-parameter operators and the so-called Fokker-Planck equation [25]. The construction goes as follows.

Let $L^2(M)$ considered as a Banach space for the Lebesgue measure ν. Let then U be an open subset of $M \times \mathbb{R}_+$. We denote by $\mathscr{D}(U)$ the set of *test functions* on U and its topological dual $\mathscr{D}'(U)$ the space of *distribution*. A family of one-parameter operators $\{P_t\}_{t\geq 0}$ is a family of linear operators on $L^2(M)$ defined by:

$$P_0 = id, \qquad (P_t f)(x) = \int_M f(y) p_t(x,y) d\nu(y) \tag{3.54}$$

such that $p_t(x,y)$ is a $\nu \times \nu$-measurable function on $M \times M$. Now let us define the new measure $\mu_{t,x}$, for $t \geq 0$ and $x \in M$, by

$$\mu_{x,t}(A) = \int_A p_t(x,y) d\nu(y), \tag{3.55}$$

for any ν-measurable subset A. Assume that $\mu_{t,x}$ is a probability measure for every $(x,t) \in M \times \mathbb{R}_+$. In addition, we assume that P_t admits a derivative ∂_t. Then, one can associate to any operator P_t a distribution; let $\varphi \in \mathscr{D}(U)$ and define

$$T_{\mu_{x,t}}(f) = \int_M f(y) d\mu_{x,t}(y) \tag{3.56}$$

as a map on $\mathscr{D}'(U)$. In the special case where

$$\lim_{t\to 0+} \frac{\partial^k \widehat{\mu}_{x,t}(0)}{t} = 0, \quad \forall k \geq 3, \tag{3.57}$$

then $\mu_{x,t}$ satisfy the parabolic equation

$$\left.\frac{\partial \mu_{x,t}}{\partial t}\right|_{t=0} = \mathscr{L}_{A,b}(\mu_{x,0}) \tag{3.58}$$

in the weak sense, called the *Fokker-Planck equation*. The operator $\mathscr{L}_{A,b}$ is given by

$$\mathscr{L}_{A,b} f = tr(AD^2 f) + \langle b, \nabla f \rangle\,, \quad f \in C_c^\infty(M) \tag{3.59}$$

and where $A = (a^{ij})$ is a mapping on M with values in the space of nonnegative symmetric linear operator on $\mathbb{R}^d$ and $b = (b^i)$ is a vector field on M. In the special case of Equation (3.43), we have $A = 1$ and $b = s$.

Following this idea, it is interesting to consider semigroup machinery as another approach to the problem of approximation of the Dirac operator. In this section, we consider $L^2(\nu)$ as the Hilbert space $\mathcal{H}$.

Definition 3.2.1 (Semigroup)**.** A *one-parameter unitary group* is a map $t \to P_t$ from $\mathbb{R}_+$ to $\mathcal{L}(\mathbb{H})$ such that

$$P_0 = 1 \qquad P_{t+s} = P_t P_s, \tag{3.60}$$

and $t \to P_t$ is continuous in the strong topology, i.e. $U_t \xrightarrow{s} U_{t_0}$ when $t \to t_0$.

Given a semigroup P_t in L^2, define the *generator* $\mathscr{L}$ of the semigroup by

$$\mathscr{L}(f) := \lim_{t \to 0} \frac{f - P_t f}{t}, \tag{3.61}$$

where the limit is understood in the L^2-norm. The *domain* $\mathrm{dom}(\mathscr{L})$ of the generator $\mathscr{L}$ is the space of functions $f \in \mathcal{H}$ for which the above limit exists. By the Hille-Yosida theorem, $\mathrm{dom}(\mathscr{L})$ is dense in L^2. Moreover, P_t can be recovered from $\mathscr{L}$ as follows:

$$P_t = \exp(-t\mathscr{L}). \tag{3.62}$$

understood in the sense of spectral theory.

We then consider the operator $L = -i\frac{d}{dx}$ on $\mathcal{H}$ with $\mathrm{dom}(\mathscr{L}) = \{f \in L^2(\mathbb{R}) : \xi \widehat{f} \in L^2(\mathbb{R})\}$. Recall that L is unitary equivalent to the left-multiplication operator M_ξ using the Fourier transform

$$\mathcal{F} L \mathcal{F}^{-1} = \xi \widehat{f}. \tag{3.63}$$

Then the associated semigroup U_t, so-called *momentum operator*, is given by the left-multiplication operator in Fourier basis: $\mathcal{F} U_t \mathcal{F}^{-1} \widehat{f} = \xi \widehat{f}$. Therefore,

$$U_t f(x) = \mathcal{F}^{-1}(e^{it\xi} \widehat{f})(x) = \int_{\mathbb{R}} e^{i(x+t)\xi} \widehat{f} = f(x+t). \tag{3.64}$$

We can then use U_t in the definition of the Dirac operator D. This is what we have done to some extend (although not presented in the following framework) in our previous paper [117].

3.3 Integral operators on manifolds

In this section, we extend the previous results to the case where M is a smooth manifold of dimension d. Since we are interested in the Dirac operator over *spin manifolds*, these results are crucial for the rest of the present work.

3.3.1 Wrapped distributions

Let us recall that on a well-suited open neighbourhood of a spin manifold M, the Dirac operator can be written as in Equation (3.31). In the previous section, we have exhibited a probability distribution:

$$T_{\mu_{y,t}}(f)(y,t) = \int_{\mathbb{R}^d} \varphi(x) G_t(x,y) dx \tag{3.65}$$

that we are going to use to approximate the partial derivatives appearing in the expression of the Dirac using the relation:

$$\partial_t|_{t=0} f = \partial_i f \tag{3.66}$$

satisfied by the function f obtained using the Green function G. In order to do so, we will need to extend the distribution T_μ defined on $\mathbb{R}^d$ to a compact manifold M. This will be done using *wrapped distributions*.

Given the Lebesgue measure on $T_x M$ (in this section the tangent space $T_x M$ is identified with $\mathbb{R}^d$), the Riemannian volume form vol_g on M and a diffeomorphism $\phi : M \supset V \to U$, one can define a density h on M (whose support is included in $V \subset M$) to a density on $U \subset T_x M$. This is done by using ϕ as a push-forward map. The construction goes as follows: given a volume form ω written in coordinates as

$$\omega_g = h(x) \mathrm{vol}_g \tag{3.67}$$

Then, the integration on V of this volume form is given by:

$$\int_V \omega_g := \int_U h(\phi(x)) |\det(d_x \phi)| dx_1 \cdots dx_n \tag{3.68}$$

where the integral is written using ϕ as a coordinate chart.

Among choices of ϕ an interesting candidate is the exponential map at point $p \in M$, $\exp_p : T_pM \to M$, due to its algebraic and geometric properties.

Proposition 3.3.1. *Let (M, g) be a Riemannian manifold. Fore every point $p \in M$, there is an open subset $W \subseteq M$, with $p \in W$ and a number $\epsilon > 0$, so that:*

$$\exp_q : B(0, \epsilon) \subseteq T_qM \to U_q = \exp(B(0, \epsilon)) \subseteq M \tag{3.69}$$

is a diffeomorphism for every $q \in W$, with $W \subseteq U_q$.

Definition 3.3.1 (Normal neighbourhood)**.** Let (M, g) be a Riemannian manifold. For any $q \in M$, an open neighbourhood of q of the form $U_q = \exp_q(B(0, \epsilon))$ where $\exp_q$ is a diffeomorphism from the open ball $B(0, \epsilon)$ onto U_q, is called a *normal neighbourhood*.

Definition 3.3.2 (Injectivity radius)**.** Let (M, g) be a Riemannian manifold. For every point $p \in M$, the *injectivity radius of M at p*, denoted $\delta(p)$, is the least upper bound of the numbers $r > 0$, such that $\exp_p$ is a diffeomorphism on the open ball $B(0, r) \subseteq T_pM$. The *injectivity radius, $\delta(M)$ of M* is defined as:

$$\delta(M) := \inf_{p \in M} \delta(p). \tag{3.70}$$

In what will follow, we will simply denote by δ the *injectivity radius* of M; we will also restrict to manifolds with strictly positive injectivity radius. Let $p \in M$, we then consider the exponential map $\exp_p : B(0, \delta) \to \exp_p(B(0, \delta))$. We associate to the density defined in Equation (3.51), the volume form

$$\omega_{0,t} = \frac{1}{(4\pi\Phi(t))^{\frac{d}{2}}} \exp\left(-\frac{|\exp_p^{-1}(x) + st|^2}{4\Phi(t)}\right) \mathrm{vol}_g = G_t(\exp_p^{-1}(x), 0)\mathrm{vol}_g \tag{3.71}$$

We use the inverse of the exponential map to pushforward the Green function from the tangent space T_pM at p (identified with $\mathbb{R}^d$) to the manifold M. Integration on the domain of injectivity given by the injectivity radius, we obtain

$$\int_{\exp_p(B_\delta)} f\omega_{0,t} = \int_{B_\delta} f \circ \exp_p(x) G_{0,t}(x) \det(d_x \exp_p) dx \tag{3.72}$$

If we denote by $\{e_1, \ldots, e_d\}$ a local orthogonal frame in TM, then by taking $s = e_i(p)$, we have proven the following lemma.

Lemma 3.3.1. *Consider the tangent vector $e_i(p) \in T_pM$ for $i \in \{1, \ldots, d\}$ and let $s = e_i(p)$. Then, the following holds:*

$$\int_{\exp_p(B_\delta)} f\omega_{0,t} = \frac{1}{(4\pi\Phi(t))^{\frac{d}{2}}} \int_{B_\delta} \widetilde{f}(x) \exp\left(-\frac{|x + e_i(p)t|^2}{4\Phi(t)}\right) \det(d_x \exp_p) dx \tag{3.73}$$

where $\exp_p$ is the exponential map on the Riemannian manifold (M, g) and the function $\widetilde{f}(x) = f(\exp_p(x))$.

More generally, we can define the following map on the whole manifold M:

$$\Theta : M \to C_0(\mathbb{R}_+), \quad p \mapsto \left\{ t \mapsto T_{\omega_{0,t}}(f) = \int_{\exp_p(B_\delta)} f\omega_{0,t} \right\}_{t>0}.$$

This map associates to any normal neighbourhood a function of the variable t. We are going to show that each family of operators satisfies Equation (3.66).

Definition 3.3.3 (Jacobi field [52]). Let $p \in M$ and $\gamma : [0, a] \to M$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v$. Let $w \in T_v(T_pM)$ with $|w| = 1$. A Jacobi field J along γ given by

$$J(t) = (d \exp_p)_{tv}(tw). \tag{3.74}$$

Lemma 3.3.2. *Let J be a Jacobi field. We have the following Taylor expansion about $t = 0$:*

$$\langle w, J(t) \rangle = t + r(t), \tag{3.75}$$

where $\lim_{t\to 0} \frac{r(t)}{t^2} = 0$.

Proof. From the definition of J and the properties of the exponential map, we have that $J(0) = (d_0 \exp_p)(0) = 0$ and $J'(0) = w$. Hence, the first two coefficients of the

Taylor expansion are

$$\langle w, J(0)\rangle = 0,$$
$$\langle w, J'(0)\rangle = 1.$$

As J is a Jacobi field we have $J''(0) = -R(\gamma', J(0))\gamma'(0) = 0$, where R is the curvature tensor. This yields,

$$\langle w, J''(0)\rangle = 0, \tag{3.76}$$

which concludes the proof. □

Lemma 3.3.3. *Define the smooth map:*

$$G : T_pM \to \mathbb{R}, \qquad y \mapsto \det\left(d_y \exp_p\right), \tag{3.77}$$

then, it satisfies $\nabla(G)(0) = 0$.

Proof. In order to compute $\nabla(G)(0)$, we first use Jacobi's identity

$$\frac{d}{dt}\det\left(d_{ty}\exp_p\right)\bigg|_{t=0} = \det(d_0\exp_p)\mathrm{tr}\left(d_0\exp_p^{-1}\frac{d}{dt}\bigg|_{t=0} d_{ty}\exp_p\right) \tag{3.78}$$

which simplifies into

$$\frac{d}{dt}\det\left(d_{ty}\exp_p\right)\bigg|_{t=0} = \mathrm{tr}\left(\frac{d}{dt}\bigg|_{t=0} d_{ty}\exp_p\right). \tag{3.79}$$

Using the definition of a Jacobi field and linearity of tr, we have that

$$\begin{aligned}
\mathrm{tr}\left(\frac{d}{dt}\bigg|_{t=0} d_{ty}\exp_p\right) &= \sum_{i=1}^{d}\left\langle v_i, \left(\frac{d}{dt}\bigg|_{t=0} d_{ty}\exp_p\right)v_i\right\rangle, \\
&= \frac{d}{dt}\bigg|_{t=0}\sum_{i=1}^{d}\left\langle v_i, d_{ty}\exp_p(v_i)\right\rangle, \\
&= \frac{d}{dt}\bigg|_{t=0}\sum_{i=1}^{d}\frac{1}{t}\left\langle v_i, J_i(t)\right\rangle.
\end{aligned}$$

Now using the Taylor expansion obtained in Lemma (3.3.2), we get:

$$\langle v_i, J_i(t)\rangle = t + r(t) \tag{3.80}$$

where $r(t) = o(t^2)$, we conclude that:

$$\operatorname{tr}\left(\frac{d}{dt}\bigg|_{t=0} d_{ty}\exp_p\right) = 0. \tag{3.81}$$

□

Theorem 3.3.1. *The following limit holds at $p \in M$*

$$\frac{\partial}{\partial t}\left(\int_{\exp_p(B_\delta)} f\omega_{0,t}\right)\bigg|_{t=0} = e_i(f)(p). \tag{3.82}$$

Proof. The result follows from the property of the Green function given in Lemma 3.2.1 and the reduction to an open ball obtained in Lemma . Then, using the integration equality given in Lemma 3.3.1 on the open ball B_δ and the isomorphism $T_0(T_p(M)) \simeq T_p(M)$, we have:

$$\frac{\partial}{\partial t}\left(\int_{\exp_p(B_\delta)} f\omega_{0,t}\right)\bigg|_{t=0} = e_i\left(\exp_{p*}(f)\det\left(d\exp_p\right)\right)(0).$$

Finally, to conclude we use Lemma 3.3.3 and deduce that:

$$\frac{\partial}{\partial t}\left(\int_{\exp_p(B_\delta)} f\omega_{0,t}\right)\bigg|_{t=0} = e_i(f)(p). \tag{3.83}$$

□

3.4 Statistical fluctuations of differential structures

We are now ready to state and prove Theorem 3.4.1. We keep the same notations as the previous sections: M is a compact Riemannian manifold of dimension d; we consider a point $p \in M$ and a normal neighbourhood U_p associated to it; we denote by $\{e_1, \dots, e_d\}$ a local orthogonal frame in TM. We finally define the orthonormal family of vectors $\{s_1, \dots, s_d\}$ such that:

$$s_j = e_j(p), \qquad \forall j \in \{1, \dots, d\}\,. \tag{3.84}$$

3.4.1 The Dirac operator

We start by recalling that the notation D_X means: a Dirac operator D associated to a graph X in the sense of [117, Def. 4.3]. Now, let n be a positive integer and fix a graph X_n equipped with a Dirac operator D_{X_n} and with set of vertices $\{x_1, \dots x_n\}$. In addition, we are going to consider n copies of the same graph X_n, each of which is equipped with a Dirac operator D_{X_k} and with a set of vertices denoted by $\{x_1^k, \dots x_n^k\}$, for $1 \le k \le n$. Then, we have a sequence of Dirac operators

$$(D_{X_1}, D_{X_2}, \cdots, D_{X_n}) \in \mathfrak{gl}^-_{2m_n}(\mathbb{C})^n, \tag{3.85}$$

acting on a sequence of diagonal elements $(a_1, \cdots, a_n)$ with each $a_i \in \mathfrak{A}_n$.
If we denote by $(a_k^i)_{1 \le i \le n}$ the coefficients of a_k in the block diagonal, then using the projection maps $M \to X_k$ we can identify these values with evaluations of a smooth functions, denoted by a (see [117, Prop. 3.5] for more details):

$$a_k^i = a(x_i^k), \quad \forall i \in \{1, \dots, n\} \tag{3.86}$$

for some point $x_i^k \in M$. Fix a point $p \in M$ and a neighbourhood U_p of p in M. Then, consider a sequence of points $\{x_1^k, \dots, x_n^k\}$ in U_p, for $1 \le k \le n$, such that, for a chosen index i_0 (not depending on k), we have $x_{i_0}^k = p$. We then define the coefficients $(\omega_{ij}^k)_{1 \le i,j \le n}$ of D_{X_n} as follows:

$$\omega_{ij}^k(\hbar) = \frac{1}{(4\pi\Phi(\hbar))^{\frac{d}{2}}} \exp\left(-\frac{|y_i^k + s_j\hbar|^2}{4\Phi(\hbar)}\right) \quad \text{for } 1 \le i,j \le n \text{ and for } 1 \le k \le n,$$

with $y_i^k := \exp_p^{-1}(x_i^k)$.

Furthermore, for every integer $1 \leq k \leq n$, we define a family of projection elements such that $e_k \in M_{2m_n}(\mathbb{C})$ and we have the following matrix form:

$$e_k D_{X_k} e_k^* = \left(\begin{array}{c|c} 0 & \begin{array}{c} 0 \\ * \quad * \quad \omega_{i_0 j}^k \quad * \quad * \\ 0 \end{array} \\ \hline \begin{array}{c} * \\ * \\ 0 \quad \omega_{i_0 j}^k \quad 0 \\ * \\ * \end{array} & 0 \end{array}\right). \tag{3.87}$$

Remark 3.4.1. The non-zero coefficients correspond to the adjacency points of i_0.

Hence, if we recall the expression given by the commutator in Equation (3.20), we consider the following average of operators over the n copies of X_n:

$$\widehat{S}_n^{\hbar_n}(a) := \frac{1}{n}\sum_{k=1}^{n} e_k \left[D_{X_k}, a_k\right] e_k^* = \frac{i}{n\hbar_n}\sum_{k=1}^{n}\sum_{j=1}^{d+1} \omega_{i_0 j}^k(\hbar_n)\alpha_{i_0 j}(a_k) Y \otimes E_{i_0 j}, \tag{3.88}$$

where $\alpha_{i_0 j}(a_k) = a(x_j^k) - a(p)$. Moreover, for the purpose of the proof of the main theorem, we define a second operator given by:

$$S_{j,n}^{\hbar_n} : C^\infty(M) \to \mathbb{R}, \qquad S_{j,n}^{\hbar_n}(a) = \frac{1}{n\hbar}\sum_{k=1}^{n} \omega_{i_0 j}^k(\hbar_n)\alpha_{i_0 j}(a_k). \tag{3.89}$$

We assume now that the points $\left\{x_1^k, \ldots, x_n^k\right\}$ are thought as random variables independent and identically distributed (i.i.d.) from a uniform distribution. Let us recall the definition of the map Ψ given in Equation (3.34):

$$\Psi : V_{i_0} \to Cl(\mathbb{R}^d), \quad \Psi\left([D,a]_{i_0}\right) = \frac{i}{\hbar}\sum_{j=1}^{d} \omega_{i_0 j}\alpha_{i_0 j}(a) e_j. \tag{3.90}$$

Then, we can prove the following theorem.

Theorem 3.4.1. *Let $\{x_{i_0}^k\}_{k=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on a open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. Let $\tilde{S}_n^{\hbar_n}$ be the associated operator given by:*

$$\widehat{S}_n^{\hbar_n} : C^\infty(U_p) \longrightarrow M_2(\mathbb{R}) \otimes U(\mathfrak{gl}_{2m_n}), \qquad \widehat{S}_n^{\hbar_n}(a) := \frac{1}{n}\sum_{k=1}^n e_k \left[D_X^k, a_k\right] e_k^*. \tag{3.91}$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then for $a \in C^\infty(U_p)$, in probability:

$$\lim_{n\to\infty} \Psi \circ \widehat{S}_n^{\hbar_n}(a) = [\mathcal{D}, a]\,(p).$$

Proof. We consider the average operator defined by Equation (3.89). It is then sufficient to prove that for $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, and for $a \in C^\infty(U_p)$, we have:

$$\lim_{n\to\infty} S_{j,n}^{\hbar_n}(a) = e_j(a)(p) \quad \forall 1 \le j \le d.$$

in probability, and then apply the map $\widehat{\psi}$ defined in (3.24). Recall that e_j is given in Equation (3.84).

The expectation value of the random variable $S_n^{\hbar_n}(a)$ is given by:

$$\mathbb{E}S_{j,n}^{\hbar_n}(a)(p) = \frac{1}{\hbar_n}\int_{\exp_p(B_\delta)} \omega_{0,\hbar_n}(a - a(p)), \tag{3.92}$$

where we assume, without lost of generality, that the volume of M is equal to one. We recognize then an approximation of the time derivative at 0 in Equation (3.3.1). Thus, applying Hoeffding's inequality, we have:

$$\mathbb{P}\Big[\left|S_{j,n}^{\hbar_n}(a)(p) - \mathbb{E}S_{j,n}^{\hbar_n}(a)(p)\right| > \varepsilon\Big] \le 2\exp\left(-\frac{\varepsilon^2 n}{KC_d(n^\alpha)^2}\right). \tag{3.93}$$

Choosing $\hbar$ as a function of n, such that $\hbar(n) = n^{-\alpha}$, where $\alpha > 0$, we have, for any real number $\varepsilon > 0$:

$$\lim_{n\to\infty} \mathbb{P}\Big[\big|S^{\hbar_n}_{j,n}(a)(p) - \mathbb{E}S^{\hbar_n}_{j,n}(a)(p)\big| > \varepsilon \Big] = 0. \tag{3.94}$$

Finally, we prove the statement using Theorem 3.3.1:

$$\lim_{n\to\infty} S^{\hbar_n}_{j,n}(a)(p) = e_j(a)(p), \tag{3.95}$$

along with the definition of the map Ψ in Equation (3.34). □

Remark 3.4.2. Every line in the matrix D_X corresponds then to a point p and a normal neighbourhood U_p obtained from the image of the exponential map of a ball of radius δ. Indeed, since M is compact, we have a finite cover $\{U_{p_i}\}_{i=1}^N$ with centre $\{p_i\}_{i=1}^N$ every one of which being associated to a line of D_X.
If we consider a sequence of Dirac operators D_{X_n}, then what we are doing is in fact taking refinements of normal neighbourhoods, increasing with the numbers of vertices in X_n.

3.4.2 Uniform convergence

It is interesting to mention that the previous result can be extended to have a uniform convergence, following the same steps as [17, Prop. 6.1]. We then state the result without proof.

Proposition 3.4.1. *Let $\mathscr{F}$ be an equicontinuous family of functions with a uniform bound up to the second derivative. Then for each $\hbar > 0$, we have:*

$$\lim_{n\to\infty} \mathbb{P}\left[\sup_{a\in\mathscr{F}} \left|S^{\hbar_n}_{n}(a)(p) - \mathbb{E}S^{\hbar_n}_{n}(a)(p)\right| > \varepsilon \right] = 0. \tag{3.96}$$

Theorem 3.4.2. *Let $\mathscr{F}$ be a family of smooth functions with uniformly bounded derivatives up to the second order. Let $\{x^k_{i_0}\}_{k=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on an open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. Let $\widehat{S}^{\hbar_n}_n$ be the associated* *operator given by:*

$$\widehat{S}_n^{\hbar_n} : C^\infty(U_p) \to M_2(\mathbb{R}) \otimes U(\mathfrak{gl}_{2m_n}), \qquad \widehat{S}_n^{\hbar_n}(a) := \frac{1}{n} \sum_{k=1}^{n} e_k \left[D_X^k, a_k\right] e_k^*. \tag{3.97}$$

Put $\hbar_n = n^{-\alpha}$*, where* $\alpha > 0$*, then in probability:*

$$\lim_{n\to\infty} \sup_{a\in\mathscr{F}} \left|\Psi \circ \widehat{S}_n^{\hbar_n}(a)(p) - \left[\mathcal{D}, a\right](p)\right| = 0. \tag{3.98}$$

3.4.3 The Laplacian

In this final section, we want to study the convergence result for the Laplacian defined by Equation (3.25). If we take the second derivative in time in the initial value problem (3.43), we see that

$$\partial_t^2 u = s \cdot \nabla \partial_t u + \varphi \Delta \partial_t u + \varphi' \Delta u \tag{3.99}$$

from which we see that if $\varphi(t) = t$ then when we take the limit when t goes to zero and obtain:

$$\partial_t^2 u|_{t=0} = (s \cdot \nabla)^2 u_0 + \Delta u_0. \tag{3.100}$$

Moreover, if we let $\widetilde{G}(x,t)$ defined by

$$\widetilde{G}(x,t) = \sum_{j=1}^{m} \lambda_j G_j(x,t), \qquad \sum_{\lambda_j=1}^{m} \lambda_j = 1 \tag{3.101}$$

where $G_j(x,t)$ is the Green function of Equation (3.43) with unit vector s_j and the λ_j are here to ensure that $\widetilde{G}$ remains a probability distribution. Then, by linearity the function

$$\widetilde{u} = u_0 * \widetilde{G} \tag{3.102}$$

satisfies the equation:

$$\partial_t^2 \widetilde{u}|_{t=0} = (\tilde{s} \cdot \nabla)^2 u_0 + \Delta u_0, \qquad \text{with } \widetilde{s} = \sum_j s_j. \tag{3.103}$$

Hence, if we pick the vectors s_j such that $\widetilde{s}$ is zero, then we are left with the following equations:

$$\partial_t^2 \widetilde{u}|_{t=0} = \Delta u_0, \quad \text{and} \quad \partial_t \widetilde{u}|_{t=0} = 0, \tag{3.104}$$

where, the second equation is obtained from Equation (3.43) after taking t to zero. Therefore, in $\mathbb{R}^d$, we have for any smooth initial condition f, the following limit:

$$\lim_{t \to 0} \frac{1}{t^2} \int_{\mathbb{R}^d} \widetilde{G}(x,t)(f(x) - f(0))dx = \Delta f(0). \tag{3.105}$$

In addition, if the dimension d is greater than two, then the reduction to a ball Lemma 3.3.1 still holds. In the rest of this section, we will assume that $d \geq 2$. Hence, we can extend the distribution $\widetilde{G}$ to a manifold as in Section 3.4. The volume form $\omega_{0,t}$ is now given by:

$$\omega_{0,t} = \frac{1}{(4\pi\Phi(t))^{\frac{d}{2}}} \sum_{j=1}^{d+1} \lambda_j \exp\left(-\frac{|\exp_p^{-1}(x) + s_j t|^2}{4\Phi(t)}\right) \mathrm{vol}_g. \tag{3.106}$$

We recognize a convex combination of the Green function obtained in the previous section. Consequently, following the same steps as in Theorem 3.2.1, we see that this distribution satisfies the equation:

$$\frac{\partial^2}{\partial t^2}\left(\int_{\exp_p(B_\delta)} \omega_{0,t} f\right)\Bigg|_{t=0} = \Delta_M(f)(p). \tag{3.107}$$

Now, the Laplacian Δ_{X_k} obtained from the Dirac operator (3.87) and acting on an element a_k is given by:

$$\Delta_{X_k}(a_k) = \frac{1}{\hbar^2} \sum_{j=1}^{d+1} (\omega_{ij}^k)^2 \alpha_{ij}(a_k) J. \tag{3.108}$$

Then, we assume that the coefficients ω_{ij}^k of the Dirac D_{X_k} are given by:

$$\omega_{ij}^k(\hbar) = \frac{1}{(4\pi\hbar^2)^{\frac{d}{4}}} \sqrt{\lambda_j} \exp\left(-\frac{|y_i^k + s_j \hbar|^2}{8\hbar^2}\right), \tag{3.109}$$

where λ_j are positive numbers to be specified. Therefore, we are lead to study the convergence of the averaging operator:

$$\Omega_n^{\hbar}(a)(p) = \frac{1}{(4\pi\hbar^2)^{\frac{d}{2}} n\hbar^2} \sum_{k=1}^{n} \sum_{j=1}^{d+1} \lambda_j \exp\left(-\frac{|y_i^k + s_j\hbar|^2}{4\hbar^2}\right) \alpha_{ij}(a_k). \tag{3.110}$$

such that, if $u = \sum_{i=1}^{d} s_i$, then $s_{d+1} = -u/\|u\|$ and $\lambda_j = 1/(d + \|u\|)$ for $1 \le j \le d$ and $\lambda_{d+1} = \|u\|/(d + \|u\|)$. Notice then that $\sum_{j=1}^{d+1} \lambda_j = 1$.
Moreover, the expectation value of the random variable $\Omega_n^{\hbar_n}(a)$ is given by:

$$\mathbb{E}\Omega_n^{\hbar}(a)(p) = \int_{\exp_p(B_\delta)} \omega_{0,\hbar_n}\left(a(x) - a(p)\right) \tag{3.111}$$

Theorem 3.4.3. *Let $\mathscr{F}$ be a family of smooth functions with uniformly bounded derivatives up to the third order. Let $\{x_i\}_{i=1}^{n}$ be a sequence of i.i.d. sampled points from a uniform distribution on an open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. $\Omega_n^{\hbar_n} : C^\infty(U_p) \to \mathbb{R}$ be the associated operator given by:*

$$\Omega_n^{\hbar_n}(a)(p) = \frac{1}{(4\pi\Phi(\hbar))^{\frac{d}{2}} n\hbar^2} \sum_{k=1}^{n} \sum_{j=1}^{d+1} \lambda_j exp\left(-\frac{|y_i^k + s_j t|^2}{4\Phi(t)}\right) \alpha_{ij}(a_k).$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then in probability:

$$\lim_{n\to\infty} \sup_{a\in\mathscr{F}} \left|\Omega_n^{\hbar_n}(a)(p) - \Delta_M(a)(p)\right| = 0 \tag{3.112}$$

3.4.4 Discussion

Going back to the definition of the Dirac operator associated to a graph X with non-zero coefficients ω_{ij}, we recall that the goal was to compute the values ω_{ij} in order to obtain a convergence when considering a sequence of refined triangulations. We have exhibited the coefficients

$$\omega_{ij}(\hbar) = \frac{1}{(4\pi\Phi(\hbar))^{\frac{d}{2}}} \exp\left(-\frac{|y_i + s_j\hbar|^2}{4\Phi(\hbar)}\right) \tag{3.113}$$

obtained from the Green function given in Equation (3.45). Hence, we are able to prove a convergence result to the Dirac operator on a normal neighbourhood (Theorem 3.4.2) as well as a convergence of the Laplace operator (Theorem 3.4.3). However, as far as the Laplacian is concerned, this choice is not unique, in fact one could take the values of ω_{ij} obtained from a normal distribution and such that:

$$\omega_{ij}^2(\hbar) = \exp\left(\frac{\|x_i - x_j\|^2}{4\hbar}\right) \tag{3.114}$$

and still get a convergence result. Nevertheless, keeping in mind that we are also interested in the convergence of the square root i.e. to the Dirac operator, it is not clear that such a choice of coefficients would also work.

Moreover, one may also consider classical discretizations of the Laplacian such as the combinatorial one with the choice:

$$\text{vertices } i \text{ and } j \text{ do not share an edge} \Leftrightarrow \omega_{ij} = 0,\ \forall i, j \tag{3.115}$$

or the *cotangent Laplacian* with the choice

$$\omega_{ij}^2 = \begin{cases} \frac{1}{2}\left(\cot\alpha_{ij} + \cot\beta_{ij}\right) & ij \text{ is an edge,} \\ -\sum_{k\sim i}\omega_{ik}^2 & i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.116}$$

In these two cases the question of convergence to the Laplacian is unclear [125], let alone convergence in the square root.

There is therefore an important direction worth investigating: whether a convergent Laplacian constructed from a specific distribution or obtained from a known discretization implies convergence of its associated Dirac operator.

Chapter 4

Future directions

We conclude this thesis by outlining a few projects that will be addresses in the future work of the author.

4.1 A deterministic approach of the Dirac operator

There is an immediate question that arises on the possibility of deriving the convergence results of Chapter 3 from a deterministic approach i.e. where the ω_{ij} are not obtained from random distributions. Indeed, one could characterize the approximation obtained in Theorem 3.4.1 as a Monte-Carlo type of approximation. More specifically, if one is interested by the integral value of the form

$$G = \int g(x) f_X(x) dx \tag{4.1}$$

where f_X is a density distribution, then one consider a sequence of random sample $\{x_1, x_2 \ldots, x_N\}$ and define the approximation through the sum

$$\widehat{g}_N = \frac{1}{N} \sum_{i=1}^{N} g(x_i). \tag{4.2}$$

The convergence to the integral (4.1) is obtained when $N \to \infty$. Hence, one can easily see that Theorem 3.4.1 is the aforementioned type of approximation.

Nevertheless, there is another type of integral approximation through a Riemann sum. Indeed, one could instead defined the value $\widehat{g}_N$ by

$$\widehat{g}_N = \sum_{i=1}^{N} g(x_i) f_X(x_i) \mu(V_i). \tag{4.3}$$

where x_i stands for any arbitrary point contained in the set V_i, and μ is a measure on the underlying set. Then the coefficient ω_{ij} of the Dirac matrix could look like

$$\omega_{ij}(\hbar) = \frac{1}{(4\pi\Phi(\hbar))^{\frac{d}{2}}} \exp\left(-\frac{|y_i + s_j\hbar|^2}{4\Phi(\hbar)}\right) \mu(V_i). \tag{4.4}$$

The dependence of the Dirac operator on the metric then appears in the measure μ.

4.2 A unifying framework ?

In the realm of applied mathematics and approximation theory of partial differential equations, as already mentioned, we conjecture that NDG can serve as a unifying framework to discretization of PDEs. The past decade has seen a change of paradigm in the discretization of PDE community where the general philosophy is that discrete theory can, and indeed, should stand on its own right.

One can mention the pioneer work of Arnold et al. in the finite element exterior calculus [4]. The finite element exterior calculus (FEEC) is the result of this work and aims at studying approximations of PDEs that arise from Hilbert complexes.

Another foundational work is the discrete exterior calculus [45]. The authors Desbrun et al. base their approach on simplicial complexes and its differential calculus on chains and cochains. In that setting, a differential form is an element in the dual of the space of chains.

Thus, we would like to investigate the possibility to describe finite element exterior calculus (FEEC) and discrete exterior calculus (DEC) under a unique framework. We are very excited at the prospect of deriving general discretization results common to what appears to be otherwise distinct approaches.

Bibliography

[1] David H. Adams, *R-torsion and linking numbers from simplicial abelian gauge theories*, arXiv: High Energy Physics - Theory (1996).

[2] P. Ara and M. Mathieu, *Sheaves of C*-algebras: Sheaves of C*-algebras*, Mathematische Nachrichten **283** (2010), no. 1, 21–39.

[3] D. N. Arnold, P. B. Bochev, R. B. Lehoucq, R. A. Nicolaides, and M. Shashkov (eds.), *Compatible Spatial Discretizations*, The IMA Volumes in Mathematics and its Applications, vol. 142, Springer New York, New York, NY, 2006.

[4] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bulletin of the American Mathematical Society **47** (2010), no. 2, 281–354.

[5] D. N. Arnold, Richard S. Falk, and Ragnar Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numerica **15** (2006), 1–155.

[6] W. Asness, *A brief overview of Alexandrov spaces*, Department of Mathematics, University of Chicago (2018), 10.

[7] J. C. Baez and James W. Gilliam, *An algebraic approach to discrete mechanics*, Letters in Mathematical Physics **31** (1994), no. 3, 205–212.

[8] A. P. Balachandran, *Classical topology and quantum states*, J. Phys. **56** (2001), 15.

[9] A. P. Balachandran, G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi, G. Sparano, and P. Teotonio-Sobrinho, *Noncommutative Lattices as Finite Approximations and Their Noncommutative Geometries*, Journal of Geometry and Physics **18** (1996), no. 2, 163–194.

[10] A.P. Balachandran, G. Bimonte, G. Landi, F. Lizzi, and P. Teotonio-Sobrinho, *Lattice gauge fields and noncommutative geometry*, Journal of Geometry and Physics **24** (1998), no. 4, 353–385.

[11] H. Behncke and W. Bos, *A Class of C*-Algebras*, Proceedings of the American Mathematical Society **40** (1973), no. 1, 128.

[12] H. Behncke and G. A. Elliott, *Lexicographic Direct Sums of Elementary C*-Algebras*, Canadian Journal of Mathematics **39** (1987), no. 2, 257–296.

[13] H. Behncke, F. Krauss, and H. Leptin, *C*-algebren mit geordneten ideal folge*, Journal of Functional Analysis **10** (1972), no. 2, 204–211 (de).

[14] H. Behncke and H. Leptin, *C*-algebras with a two-point dual*, Journal of Functional Analysis **10** (1972), no. 3, 330–335.

[15] ———, *C*-algebras with finite duals*, Journal of Functional Analysis **14** (1973), no. 3, 253–268.

[16] ———, *Classification of C*-algebras with a finite dual*, Journal of Functional Analysis **16** (1974), no. 3, 241–257.

[17] M. Belkin and P. Niyogi, *Towards a Theoretical Foundation for Laplacian-Based Manifold Methods*, 15 (en).

[18] M. Belkin, J. Sun, and Y. Wang, *Constructing Laplace Operator from Point Clouds in* $\mathbb{R}^d$, Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, January 2009, pp. 1031–1040 (en).

[19] F. Besnard, *Friendly introduction to the concepts of noncommutative geometry*, (2013), 30.

[20] Alexander Bihlo and Jean-Christophe Nave, *Invariant Discretization Schemes Using Evolution-Projection Techniques*, Symmetry, Integrability and Geometry: Methods and Applications (2013).

[21] ———, *Convecting reference frames and invariant numerical models*, Journal of Computational Physics **272** (2014), 656–663 (en), arXiv:1301.5955 [math-ph, physics:physics].

[22] G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi, G. Sparano, and P. Teotonio-Sobrinho, *Lattices and Their Continuum Limits*, Journal of Geometry and Physics **20** (1996), no. 4, 318–328.

[23] ———, *Noncommutative Lattices and Their Continuum Limits*, Journal of Geometry and Physics **20** (1996), no. 4, 329–348.

[24] B. Blackadar, *Operator algebras: theory of C*-algebras and von Neumann algebras*, Encyclopaedia of mathematical sciences, Operator algebras and non-commutative geometry, no. v. 122. 3, Springer, Berlin ; New York, 2006.

[25] V. Bogachev, N. Krylov, M. Röckner, and S. Shaposhnikov, *Fokker–Planck–Kolmogorov Equations*, Mathematical Surveys and Monographs, vol. 207, American Mathematical Society, Providence, Rhode Island, December 2015 (en).

[26] A. Bossavit, *'Generalized Finite Differences' in Computational Electromagnetics*, Progress In Electromagnetics Research **32** (2001), 45–64.

[27] A. Bossavit and F. Rapetti, *Whitney Forms, from Manifolds to Fields*, Spectral and High Order Methods for Partial Differential Equations - ICOSAHOM 2012 (M. Azaïez, H. El Fekih, and J. S. Hesthaven, eds.), vol. 95, Springer International Publishing, Cham, 2014, Series Title: Lecture Notes in Computational Science and Engineering, pp. 179–189.

[28] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer New York, New York, NY, 1982.

[29] A. Bottcher, *C*-Algebras in Numerical Analysis*, (2000), 77.

[30] O. Bratteli, *Structure spaces of approximately finite-dimensional C*-algebras*, Journal of Functional Analysis **16** (1974), no. 2, 192–204.

[31] Ola Bratteli, *The center of approximately finite-dimensional C^*-algebras*, Journal of Functional Analysis **21** (1976), no. 2, 195–202 (en).

[32] S. H. Christiansen, *A construction of spaces of compatible differential forms on cellular complexes*, Mathematical Models and Methods in Applied Sciences **18** (2008), no. 05, 739–757.

[33] S. H. Christiansen, H. Z. Munthe-Kaas, and B. Owren, *Topics in structure-preserving discretization*, Acta Numerica **20** (2011), 1–119.

[34] E. Clader, *Inverse limits of finite topological spaces*, Homology, Homotopy and Applications **11** (2009), no. 2, 223–227.

[35] Ronald R. Coifman and Stéphane Lafon, *Diffusion maps*, Applied and Computational Harmonic Analysis **21** (2006), no. 1, 5–30 (en).

[36] A. Connes, *Non-commutative differential geometry*, Publications mathématiques de l'IHÉS **62** (1985), no. 1, 41–144.

[37] ———, *Noncommutative geometry*, Academic Press, San Diego, 1994.

[38] ———, *Noncommutative Geometry, the spectral standpoint*, vol. 2, p. 23–84, Cambridge University Press, 2021.

[39] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, Colloquium Publications, vol. 55, American Mathematical Society, Providence, Rhode Island, December 2007.

[40] Alain Connes, *Noncommutative geometry and reality*, Journal of Mathematical Physics **36** (1995), no. 11, 6194–6231.

[41] R Courant, K Friedrichs, and H Lewyt, *On the Partial Difference Equations of Mathematical Physics*, (1967), 20.

[42] R. J. Daverman, *Decompositions of manifolds*, Pure and applied mathematics, no. 124, Academic Press, Orlando, 1986 (en).

[43] K. R. Davidson, *C*-algebras by example*, Fields Institute monographs, no. 6, American Mathematical Society, Providence, R.I, 1996.

[44] E. B. Davies, *Spectral Theory and Differential Operators*, 1 ed., Cambridge University Press, January 1995.

[45] M. Desbrun, A. N. Hirani, M. Leok, and J. E. Marsden, *Discrete Exterior Calculus*, arXiv:math/0508341 (2005).

[46] M. Desbrun, E. Kanso, and Y. Tong, *Discrete Differential Forms for Computational Modeling*, Discrete Differential Geometry (A. I. Bobenko, J. M. Sullivan, P. Schröder, and G. M. Ziegler, eds.), vol. 38, Birkhäuser Basel, Basel, 2008, pp. 287–324.

[47] A. Dimakis and F. Müller-Hoissen, *Discrete differential calculus, graphs, topologies and gauge theory*, Journal of Mathematical Physics **35** (1994), no. 12, 6703–6735.

[48] ———, *Differential Geometry of Group Lattices*, Journal of Mathematical Physics **44** (2003), no. 4, 1781–1821.

[49] A. Dimakis, F. Müller-Hoissen, and T. Striker, *Umbral Calculus, Discretization, and Quantum Mechanics on a Lattice*, Journal of Physics A: Mathematical and General **29** (1996), no. 21, 6861–6876.

[50] A. Dimakis, F. Müller-Hoissen, and F. Vanderseypen, *Discrete Differential Manifolds and Dynamics on Networks*, Journal of Mathematical Physics **36** (1995), no. 7, 3771–3791.

[51] J. Dixmier, *C*-algebras*, rev. ed., North-Holland mathematical library, no. 15, North-Holland Publ. Co, Amsterdam, 1982.

[52] M. P. Do Carmo, *Riemannian geometry*, Mathematics. Theory & applications, Birkhäuser, Boston, 1992 (eng).

[53] R. Doran, *Characterizations of C* Algebras*, CRC Press, 1986.

[54] M. Dubois-Violette, R. Kerner, and J. Madore, *Noncommutative differential geometry of matrix algebras*, Journal of Mathematical Physics **31** (1990), no. 2, 316–322.

[55] N. Dunford, J. T. Schwartz, W. G. Bade, and R. G. Bartle, *Linear operators*, Wiley classics library, J. Wiley, New York Chichester Brisbane [etc.], 1988.

[56] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, December 1952.

[57] G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, Journal of Algebra **38** (1976), no. 1, 29–44.

[58] ———, *On Approximately Finite-Dimensional Von Neuman Algebras, II*, Canadian Mathematical Bulletin **21** (1978), no. 4, 415–418.

[59] G. A. Elliott and D. Olesen, *A Simple Proof of the Dauns-Hofmann Theorem.*, Mathematica Scandinavia **34** (1974), 231.

[60] E. Ercolessi, G. Landi, and P. Teotonio-Sobrinho, *Noncommutative Lattices and the Algebra of their Continuous Functions*, J.Geom.Phys. **20** (1996), 329–348.

[61] M. Forger and D. V. Paulino, *Locally C*-Algebras, C*-Bundles and Noncommutative Spaces*, arXiv:1307.4458 [math-ph] (2013).

[62] E.S. Gawlik, P. Mullen, D. Pavlov, J.E. Marsden, and M. Desbrun, *Geometric, variational discretization of continuum theories*, Physica D: Nonlinear Phenomena **240** (2011), no. 21, 1724–1760.

[63] R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualités scientifiques et industrielles, no. 1252, Hermann, Paris, 1998.

[64] I. Gohberg and S. Goldberg, *Basic Operator Theory*, Birkhäuser Boston, Boston, MA, 1981.

[65] T. J. Grant and P. E. Hydon, *Characteristics of Conservation Laws for Difference Equations*, Foundations of Computational Mathematics **13** (2013), no. 4, 667–692.

[66] A. Grigor'yan, J. Hu, and K-S. Lau, *Heat Kernels on Metric Measure Spaces*, Geometry and Analysis of Fractals, vol. 88, Springer Berlin Heidelberg, Berlin, Heidelberg, 2014, pp. 147–207 (en).

[67] R. Hagen, S. Roch, and B. Silbermann, *C*-algebras and numerical analysis*, Monographs and textbooks in pure and applied mathematics, no. 236, Marcel Dekker, New York, 2001.

[68] E. Hairer, C. Lubich, and G. Wanner, *Geometric numerical integration: structure-preserving algorithms for ordinary differential equations*, 2nd ed., Springer series in computational mathematics, no. 31, Springer Berlin, Heidelberg, 2006.

[69] D. Handelman and J. Lawrence (eds.), *Ring theory*, Lecture notes in mathematics, no. 734, Springer, Berlin Heidelberg, 1978.

[70] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge ; New York, 2002.

[71] C. J. Himmelberg, *On the Product of Quotient Space*, The American Mathematical Monthly **72** (1965), no. 10, 1103.

[72] A. N. Hirani, *Discrete Exterior Calculus*, Ph.D. thesis, Caltech, Pasadena, California, 2003.

[73] P. E. Hydon and E. L. Mansfield, *A Variational Complex for Difference Equations*, Foundations of Computational Mathematics **4** (2004), no. 2, 187–217.

[74] B. Iochum, T. Krajewski, and P. Martinetti, *Distances in Finite Spaces from Noncommutative Geometry*, Journal of Geometry and Physics **37** (2001), no. 1-2, 100–125.

[75] V. Jakšić, *Lecture notes on quantum statistical mechanics*, 2017.

[76] V. Jakšić, Y. Ogata, Y. Pautrat, and C.-A. Pillet, *Entropic fluctuations in quantum statistical mechanics—an introduction*, Quantum Theory from Small to Large Scales (J. Frohlich, M. Salmhofer, V. Mastropietro, W. De Roeck, and L. F. Cugliandolo, eds.), Oxford University Press, May 2012, pp. 213–410.

[77] V. Kadets, *A Course in Functional Analysis and Measure Theory*, Universitext, Springer International Publishing, Cham, 2018.

[78] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras vol. I and II.*, Graduate studies in mathematics, no. v. 15-16, American Mathematical Society, Providence, R.I, 1997.

[79] D. Kastler, J. Madore, and T. Masson, *On finite differential calculi*, Geometry and Nature (H. Nencka and J.-P. Bourguignon, eds.), Contemporary Mathematics, vol. 203, American Mathematical Society, Providence, Rhode Island, 1997.

[80] L. H. Kauffman, *Discrete physics and the derivation of electromagnetism from the formalism of quantum mechanics*, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences **452** (1996), no. 1944, 81–95.

[81] ______, *Noncommutativity and discrete physics*, Physica D: Nonlinear Phenomena **120** (1998), no. 1-2, 125–138.

[82] M. Khalkhali, *Basic noncommutative geometry*, 2nd ed., EMS series of lectures in mathematics, European Mathematical Society, Zürich, Switzerland, 2013.

[83] M. Khalkhali and N. Pagliaroli, *Spectral Statistics of Dirac Ensembles*, arXiv:2109.12741 [hep-th, physics:math-ph] (2021) (en), arXiv: 2109.12741.

[84] G. Kjaergård Pedersen, S. Eilers, and D. Olesen, *C*-algebras and their automorphism groups*, 2 ed ed., Pure and applied mathematics, Academic press, an imprint of Elsevier, London San Diego, 2018.

[85] S. Lafon, *Diffusion Maps and Geometric Harmonics*, Ph.D. thesis, Yale University, May 2004.

[86] G. Landi, *An Introduction to Noncommutative Spaces and their Geometry*, 1st ed., Springer Berlin, January 1997.

[87] J. M. Lee, *Introduction to smooth manifolds*, 2nd ed ed., Graduate texts in mathematics, no. 218, Springer, New York ; London, 2013.

[88] B. Leimkuhler and S. Reich, *Simulating Hamiltonian Dynamics*, 1 ed., Cambridge University Press, February 2005.

[89] J-L. Loday, *Cyclic Homology*, Grundlehren der mathematischen Wissenschaften, vol. 301, Springer Berlin Heidelberg, Berlin, Heidelberg, 1998.

[90] J. Lodder, *Hochschild cohomology of poset algebras and Steenrod operations*, Comptes Rendus Mathematique **354** (2016), no. 4, 339–343.

[91] J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, 2nd ed., Cambridge University Press, June 1999.

[92] S. Majid, *LTCC Lectures on noncommutative differential geometry*, Analysis and Mathematical Physics (S. Bullett, T. Fearn, and F. T. Smith, eds.), LTCC advanced mathematics series, vol. 6, World Scientific, New Jersey, 2017, pp. 139–176.

[93] J. E. Marsden and M. West, *Discrete mechanics and variational integrators*, Acta Numerica **10** (2001), 357–514.

[94] T. Masson, *Géométrie noncommutative et applications à la théroie des champs*, The ESI International Institute for Mathematical Physics (1996), 246 (fr).

[95] ______, *Gauge theories in noncommutative geometry*, AIP Conf. Proc. (Paris, France), 2012, pp. 73–98.

[96] M. C. McCord, *Singular homology groups and homotopy groups of finite topological spaces*, Duke Mathematical Journal **33** (1966), no. 3.

[97] E. Meinrenken, *Clifford Algebras and Lie Theory*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2013 (en).

[98] R. Meyer and R. Nest, *C*-Algebras over Topological Spaces: The Bootstrap Class*, Münster J. of Math. **2** (2009), 215–252.

[99] G. J. Murphy, *C*-algebras and operator theory*, Academic Press, Boston, 1990.

[100] F. Müller-Hoissen, *Introduction to Noncommutative Geometry of Commutative Algebras and Applications in Physics*, Recent developments in gravitation and mathematical physics, Science Network Pub., Konstanz, 1998, p. 37.

[101] Q. Ng Liu, C. An Li, and C. Yi Zhang, *Some inequalities on the Perron eigenvalue and eigenvectors for positive tensors*, no. 2, 2016, pp. 405–414 (en).

[102] H. Olai Milhoj, *AF-algebras and their invariants*, Ph.D. thesis, Faculty of Mathematical Sciences, university of Copenhagen, 2018.

[103] P. J. Olver, *Geometric Foundations of Numerical Algorithms and Symmetry*, Applicable Algebra in Engineering, Communication and Computing **11** (2001), no. 5, 417–436.

[104] G. K. Pedersen, *Pullback and Pushout Constructions in C*-Algebra Theory*, Journal of Functional Analysis **167** (1999), no. 2, 243–344.

[105] T. Pirttimäki, *A survey of Kolmogorov quotients*, arXiv:1905.01157 [math] (2019).

[106] G. Rudolph and M. Schmidt, *Differential Geometry and Mathematical Physics*, Theoretical and Mathematical Physics, Springer Netherlands, Dordrecht, 2013.

[107] ———, *Differential Geometry and Mathematical Physics: Part II. Fibre Bundles, Topology and Gauge Fields*, Theoretical and Mathematical Physics, Springer Netherlands, Dordrecht, 2017.

[108] S. Sakai, *C*-algebras and W*-algebras*, Classics in mathematics, Springer, Berlin ; New York, 1998.

[109] S. Sen, S. Sen, J. C. Sexton, and D. H. Adams, *A geometric discretisation scheme applied to the Abelian Chern-Simons theory*, Physical Review E **61** (2000), no. 3, 3174–3185.

[110] A. G. Sergeev, *Intorduction to noncommutative geometry*, (2016), 138.

[111] A. Sitarz, *Lectures on Noncommutative Geometry*, Acta Polytechnica **48** (2008), no. 2-2008, 22.

[112] ———, *A Friendly Overview of Noncommutative Geometry*, Acta Physica Polonica B **44** (2013), no. 12, 2643.

[113] R. D. Sorkin, *Finitary substitute for continuous topology*, International Journal of Theoretical Physics **30** (1991), no. 7, 923–947.

[114] A. Stern, Y. Tong, M. Desbrun, and J. E. Marsden, *Geometric Computational Electrodynamics with Variational Integrators and Discrete Differential Forms*, **73** (2015), 437–475.

[115] S. Sternberg, *Curvature in mathematics and physics*, Dover Pubications, Inc, Mineola, New York, 2012.

[116] J. M. Swart, *Introduction to Quantum Probability*, (2017), 145.

[117] D. Tageddine and J-C. Nave, *Noncommutative Differential Geometry on Infinitesimal Spaces*, (2022), Publisher: arXiv Version Number: 1.

[118] M. J. Thibault, *Homotopy theory of combinatorial categories*, Ph.D. thesis, University of Chicago, 2013.

[119] L. W. Tu, *Differential Geometry*, Graduate Texts in Mathematics, vol. 275, Springer International Publishing, Cham, 2017.

[120] J. C. Várilly, *An introduction to noncommutative geometry*, EMS Series of lectures in mathematics, European mathematical society, Zürich, 2006.

[121] A. T. S. Wan, *Finding Conservation Laws for Partial Differential Equations*, Ph.D. thesis, The Faculty of Graduate Studies, University of British Columbia, 2010.

[122] A. T. S. Wan, A. Bihlo, and J-C. Nave, *The multiplier method to construct conservative finite difference schemes for ordinary and partial differential equations*, SIAM Journal on Numerical Analysis **54** (2016), no. 1, 86–119.

[123] ———, *Conservative Methods for Dynamical Systems*, SIAM Journal on Numerical Analysis **55** (2017), no. 5, 2255–2285.

[124] A. T. S. Wan and J-C. Nave, *On the arbitrarily long-term stability of conservative methods*, SIAM Journal on Numerical Analysis **56** (2018), no. 5, 2751–2775.

[125] Max Wardetzky, *Convergence of the Cotangent Formula: An Overview*, Discrete Differential Geometry (Alexander I. Bobenko, John M. Sullivan, Peter Schröder, and Günter M. Ziegler, eds.), vol. 38, Birkhäuser Basel, Basel, 2008, Series Title: Oberwolfach Seminars, pp. 275–286.

[126] N. E. Wegge-Olsen, *K-theory and C*-algebras: a friendly approach*, Oxford University Press, Oxford ; New York, 1993.

[127] H. Whitney, *Geometric integration theory*, Princeton University Pres, Place of publication not identified, 1957.

[128] K. G. Wilson, *Confinement of quarks*, Physical Review D **10** (1974), no. 8, 2445–2459.

[129] S. O. Wilson, *Cochain algebra on manifolds and convergence under refinement*, Topology and its Applications **154** (2007), no. 9, 1898–1920.

[130] A. Wulfsohn, *The Primitive Spectrum of a Tensor Product of C*-Algebras*, Proceedings of the American Mathematical Society **19** (1968), no. 5, 1094.