Spectral triples on the Berkovich line

D. Tageddine [1]

Joint work with M. Khalkhali (Western Ontario)

[1]Department of Mathematics and Statistics, McGill University

Workshop in Noncommutative Geometry, December 5th, 2023

Motivations: Complex (arithmetic) dynamics

Study the dynamics of a rational map $\varphi : \mathbb{C} \to \mathbb{C}$ and deduce arithmetic or geometric properties.

Arithmetic Geometry		**Dynamical Systems**
rational and integral points on varieties	$\longleftrightarrow$	rational and integral points in orbits
torsion points on abelian varieties	$\longleftrightarrow$	periodic and preperiodic points of rational maps
abelian varieties with complex multiplication	$\longleftrightarrow$	post-critically finite rational maps

General philosophy: before studying a problem over $\mathbb{C}$ or $\mathbb{Q}$, study it on $\mathbb{C}_p$ or $\mathbb{Q}_p$.

Non-Archimedean Fields

A **non-archimedean absolute value** on a field K is a function $|\cdot| : K \to [0, \infty)$ such that for all $x, y \in K$,

- $|x| \geq 0, \quad \text{with } |x| = 0 \Leftrightarrow x = 0$
- $|xy| = |x| \cdot |y|$,
- $|x + y| \leq \max\{|x|, |y|\}$.

One can define $\mathbb{P}^1(K)$: totally disconnected and not locally compact.

In 1990, Berkovich constructed $\mathbb{P}^1_{\mathrm{an}}$ with much nicer properties.

Multiplicative Seminorms on $K[z]$

A **multiplicative seminorm** on $K[z]$ is a function $\|\cdot\|_\zeta : K[z] \to [0, \infty)$ such that

- $\|c\|_\zeta = |c|, \quad$ for all $c \in K$
- $\|fg\|_\zeta = \|f\|_\zeta \cdot \|g\|_\zeta, \quad$ for all $f, g \in K[z]$, and
- $\|f + g\| \leq \|f\|_\zeta + \|g\|_\zeta, \quad$ for all $f, g \in K[z]$.

Definition (Analytic spectrum)

For A a normed ring, its *analytic spectrum* or *Berkovich spectrum* $\mathrm{Spec}_{\mathrm{an}} A$ is the set of all non-zero multiplicative seminorms on A, such that all functions:

$$\mathrm{Spec}_{\mathrm{an}} A \to \mathbb{R}_+, \qquad \zeta \mapsto \|a\|_\zeta$$

for $a \in A$ are continuous.

The Berkovich Projective line: Formal definition

Definition

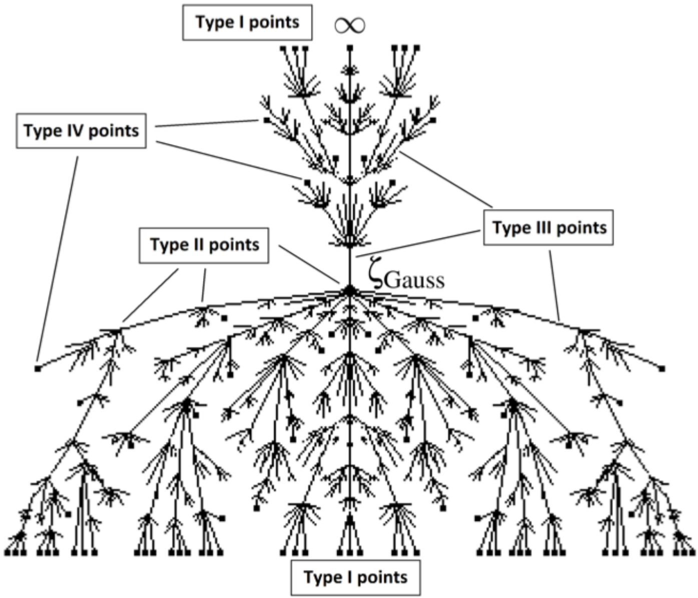
The **Berkovich affine line** $\mathbb{A}^1_{\mathrm{an}}$ is the set of all multiplicative seminorms on $K[z]$.
The **Berkovich projective line** $\mathbb{P}^1_{\mathrm{an}}$ is $\mathbb{A}^1_{\mathrm{an}} \cup \{\infty\}$.
The **Berkovich hyperbolic line** $\mathbb{H}^1_{\mathrm{an}}$ is $\mathbb{P}^1_{\mathrm{an}} \backslash P^1(K)$.

As topological spaces, we equip $\mathbb{A}^1_{\mathrm{an}}$, $\mathbb{P}^1_{\mathrm{an}}$ and $\mathbb{H}^1_{\mathrm{an}}$ with the **Gel'fand topology**.

This is the weakest topology such that for every $f \in K[v]$, the map

$$\mathbb{A}^1_{\mathrm{an}} \to \mathbb{R} \qquad \zeta \to \|f\|_\zeta$$

Points classification



Some properties of the Berkovich line

The Berkovich projective line $\mathbb{P}^1_{\mathrm{an}}$:

1) is compact, path connected, Hausdorff metric space
2) is homeomorphic to the inverse limit of finite real-trees (Gromov 0-hyperbolic space):

(Busemann function) $B(x_0, \gamma x_0, \xi) = \lim_{x \to \xi} d(x_0, x) - d(\gamma x_0, x)$.

3) (Favre & Rivera-Letelier; Baker & Rumely; Autissier, Chambert-Loir & Thuiller)
Let $\phi \in \mathbb{C}_v(z)$ of degree $d \geq 2$. There exists a unique probability measure μ_ϕ on $\mathbb{P}^1_{\mathrm{an}}$ s.t.:
   i) $\phi^* \mu_\phi = d \cdot \mu_\phi$
   ii) $\mu_\phi(E_\phi) = 0$

Spectral triple on the projective B-line

To each finite tree Γ, we associate a spectral triple $(\mathcal{A}_\Gamma, \mathcal{H}_\Gamma, D_\Gamma)$

- $\mathcal{A}_\Gamma$ is $C_{\mathrm{Lip}}(\Gamma)$ Lipschitz continuous functions on Γ
- $\mathcal{H}_\Gamma$ is the repesentation space $\ell^2(\Gamma) \otimes \mathbb{C}^2$

$$\pi(f)\psi_v = \oplus_{v_+ \sim v} \begin{pmatrix} f(v_+) & 0 \\ 0 & f(v) \end{pmatrix} \psi_v$$

$$\boxed{D\psi_v = \oplus_{v_+ \sim v} \frac{1}{d(v, v_+)} \psi_v \otimes \sigma}$$

The pair $(r^*_{\Gamma,\Gamma'}, \iota_{\Gamma\Gamma'})$ induces a morphism of spectral triples

$$(A_\Gamma, \mathcal{H}_\Gamma, D_\Gamma) \xrightarrow{(r^*_{\Gamma,\Gamma'}, \iota_{\Gamma\Gamma'})} (A_{\Gamma'}, \mathcal{H}_{\Gamma'}, D_{\Gamma'})$$

Inverse limit of spectral triples

We define the spectral $\left\{(A_j, \mathcal{H}_j, D_j), (r^*_{jk}, \iota_{jk})\right\}_J$ with the following notation:

$$A_j := C_{\mathrm{Lip}}(\Gamma_j), \quad \mathcal{H}_j = \ell^2(\Gamma_j), \quad D_j = D_{\Gamma_j} \tag{1}$$

with the isometric morphism:

$$r^*_{jk} : A_j \to A_k, \quad \iota_{jk} : \mathcal{H} \to \mathcal{H}_k. \tag{2}$$

Theorem

The triple $(C_{\mathrm{Lip}}(\mathbb{P}^1_{\mathrm{an}}), \ell^2(\mathbb{P}^1_{\mathrm{an}}), D)$ is called the inductive realization of the inductive system $\{(C_{\mathrm{Lip}}(\Gamma_{\mathrm{j}}), \ell^2(\Gamma_{\mathrm{an}}), D_j), (\phi_{jk}, I_{jk})\}_J$.

Dirichlet forms

Define the measure μ by

$$\mu(f) = \lim_{s \to s_0} = \frac{\operatorname{Tr}(|D|^{-s}\pi(f))}{\operatorname{Tr}(|D|^{-s})}$$

It is possible to define a form Q_s on $L^2(\mathbb{P}^1_{\mathrm{an}}, \mu)$

$$Q_s(f,g) := \frac{1}{2}\operatorname{Tr}(|D|^{-s}[D,\pi(f)]^*[D,\pi(g)])$$

Correspondence: Dirichlet forms and Markovian semigroups.

For $s \in \mathbb{R}$, define a self-adjoint operator Δ_s such that $T_t := \exp(t\Delta_s)$ is a Markovian semigroup

$$\langle \Delta_s f, g \rangle = \frac{1}{2}\operatorname{Tr}(|D|^{-s}[D,\pi(f)]^*[D,\pi(g)]).$$

Hyperbolic space

For $\varphi \in K(T)$, we have a continuous action $\varphi : \mathbb{P}^1_{\mathrm{an}} \to \mathbb{P}^1_{\mathrm{an}}$

$$\mathrm{Aut}(\mathbb{P}^1_{\mathrm{an}}) \simeq \mathrm{PGL}_2(K)$$

The action is continuous on $\mathbb{P}^1_{\mathrm{an}}$ and isometrically on $\mathbb{H}^1_{\mathrm{an}}$

Let Γ a discrete subgroup with a limit set Λ;

$$\mathcal{A} = C(\Lambda) \rtimes \Gamma \quad (\pi_\xi \rtimes U)(f) = \sum_{\gamma' \in \Gamma} \pi_\xi(f_{\gamma'}) U_{\gamma'}$$

One-parameter automorphism on the generators:

$$\sigma_t \left(\sum_\gamma f_\gamma(\xi) U_\gamma \right) = \sum_\gamma e^{itd(x_0, \gamma x_0, \xi)} f_\gamma(\xi) U_\gamma$$

The Patterson-Sullivan measure

Unique KMS sates at inverse temperature $\beta = \delta(\Lambda)$:

$$\varphi_{\beta,x_0}\left(\sum_\gamma f_\gamma(\xi)U_\gamma\right) = \int_\Lambda f_e(\xi)d\mu_{PS,x_0}(\xi).$$

associated to the Hamiltonian

$$H\left(f_\gamma(\xi)\otimes\gamma\right) = d(x_0,\gamma x_0,\xi)f_\gamma(\xi)\otimes\gamma$$

One can define a 1-cocycle $c:\Gamma\to\mathcal{H}_\pi$ such that

$$d(x_0,\gamma x_0,\xi) = \|c(\gamma)\|^2_{\mathcal{H}_\pi}, \qquad \gamma\in\Gamma$$

In the Hilbert space $\ell^2(\Gamma,\mathcal{H}_\pi)$, define the self-adjoint operator

$$\boxed{Df_\gamma = f_\gamma c_\gamma} \qquad \Rightarrow \qquad D^2 f_\gamma = d(x_0,\gamma x_0,\xi)f_\gamma$$

Iterated function systems

Consider the iterated function system $(\Lambda_\varphi, \varphi)$. To begin, we define the *graph of* φ to be the set

$$\mathrm{Gr}(\varphi) := \{(\varphi(x), x) \mid x \in \Lambda_\varphi\} \subseteq \Lambda_\varphi \times \Lambda_\varphi,$$

construct a C^*-correspondence for $(\Lambda_\varphi, \varphi)$.

Consider:

- the C^*-algebra $C(\Lambda_\varphi)$
- Right Hilbert A-module $H_\varphi = C(\mathrm{Gr}(\varphi))$

We define the A-valued inner product $(\cdot, \cdot)$ on H_φ

$$(\xi, \eta)(y) = \sum_{x \in \varphi^{-1}(y)} m_\varphi(x) \overline{\xi(x, y)} \eta(x, y)$$

The Watatani-Kajiwara algebra

Define the *Fock space* of H to be:

$$F_A(H_\varphi) = A \oplus H_\varphi \oplus H_\varphi^2 \oplus \cdots$$

From a Toeplitz representation $(\pi, \tau) : (A, H) \to B(F_A)$, define the Cuntz-Pimnser algebra:

$$\mathcal{O}_A(H) = \overline{\mathrm{span}}\,\{\tau(\xi_1)\cdots\tau(\xi_n)\pi(a)\tau(\eta_m)^*\cdots\tau(\eta_1)^* : \xi_i, \eta_i \in H, a \in A\}$$

The Cuntz-Pimnser algebra admits a gauge action

$$\gamma : \mathbb{R} \to \mathrm{Aut}(\mathcal{O}_\varphi) \qquad \gamma_t(S_\xi a S_\eta^*) = e^{it(|\xi|-|\eta|)} S_\xi a S_\eta^*$$

with a unique KMS state at inverse temperature $\beta = \log \deg(R)$

$$\omega_{\varphi,\beta}(a) = \int a d\mu_\varphi$$

Conclusion

We have constructed three different spectral triples on the B-line, each of which has some geometric and dynamic information:

- Inverse limit of finite spectral triples
- Spectral triple associated to $\Gamma \subset \mathrm{PGL}_2(K)$
- D_φ on the Kajiwara-Watatani algebra

Remaining questions:

- (noncommutative) potential theory
- summability of the spectral triples
- arithmetic/dynamic

Thank you !

- Matthew Baker and Robert Rumely, *Potential Theory on the Berkovich Projective Line* (en).
- Caterina Consani and Matilde Marcolli, *Non-commutative geometry, dynamics, and infinity-adic Arakelov geometry*, October 2003, arXiv:math/0205306.
- Gunther Cornelissen and Matilde Marcolli, *Graph Reconstruction and Quantum Statistical Mechanics*, September 2012.
- Remus Floricel and Asghar Ghorbanpour, *On inductive limit spectral triples*, December 2017.
- M. Greenfield, M. Marcolli, and K. Teh, *Twisted spectral triples and quantum statistical mechanical systems*, P-Adic Numbers, Ultrametric Analysis, and Applications **6** (2014), 81–104.

References II

Magnus Goffeng, Adam Rennie, and Alexandr Usachev, *Constructing KMS states from infinite-dimensional spectral triples*, Journal of Geometry and Physics **143** (2019), 107–149.

Tsuyoshi Kajiwara and Yasuo Watatani, *C^*-algebras associated with complex dynamical systems*, September 2003.

John Lott, *Limit Sets as Examples in Noncommutative Geometry*, K-Theory **34** (2005), 283–326.

Qiuyu Ren, *Linear Fractional Transformations on the Berkovich Projective Line* (en).