Noncommutative differential geometry on discrete spaces

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Abstract

In this poster, we will present how the general framework of noncommutative geometry can be used for the discretization of differential operators. We present the exterior derivative as a commutator with a hermitian operator; the so-called Dirac operator. We show that finite difference expressions can be recovered as convex combinations of eigenvalues of this commutator. In addition, we show that under suitable conditions i.e. when the coefficients of the Dirac operator are determined by a suitable distribution, the Laplace operator on a smooth manifold is recovered at the limit.

Introduction

The main objective of this work is to derive *ab initio* finite difference calculus that is compatible with the geometry of the continuous problem. This leads us to use noncommutative differential geometry.

This allows us to extend tools from differential geometry such as differential maps along with their differential complex, affine connections and a Laplace operator. Moreover, we can then study spectral convergence of these objects with respect to a parameter h.

The main objective can be divided into three sub-objectives.

1. Establishing a proper notion of *discrete space* X, obtained from a manifold M
2. Exhibit the algebra of continuous sections over X. Following Gelfand-Naimark's theorem, this should be a C^*-algebra A.
3. Provide an exterior algebra $\Omega(A)$ and a differential calculus.

What is noncommutative geometry ?

Noncommutative geometry is a field of mathematics concerned with noncommutative algebras and their study with a geometric approach. A noncommutative algebra can be thought as an algebra with noncommutative coordinates, i.e. xy does not always equal yx. Noncommutative differential geometry (NDG) is a particular and most notable realization of the program of noncommutative geometry lead by Alain Connes [1].

The theory aims at describing the internal geometry of a certain type of noncommutative "algebra of functions" using a structure called a *spectral triple* (see next section for more details). NDG underlies many applications in several branches of mathematics, among which are operator algebras and number theory. In physics, it has notably produced the noncommutative standard model as a proposed extension of the standard model of particle physics. My own research has been influenced by Connes' elegant description of quantum mechanics using NDG and the beauty of the subject.

Heuristically, non-Hausdorff topological spaces are fuzzy spaces where two points are not always distinguishable. Discrete spaces such as lattices or partially ordered sets (posets) can be modelled as such. This type of space is of paramount importance in discrete theories of differential equations.

Spectral triple

A *spectral triple* is the data $(\mathcal{A}, \mathcal{H}, D)$ where:

i) $\mathcal{A}$ is a real or complex $*$-algebra;
ii) $\mathcal{H}$ is a Hilbert space and a left-representation $(\pi, \mathcal{H})$ of A in $\mathcal{B}(\mathcal{H})$;
iii) D is a *Dirac operator*, which is a self-adjoint operator on $\mathcal{H}$.

We require in addition that the Dirac operator satisfies the following conditions

a) The resolvent $(D-\lambda)^{-1}$, $\lambda \notin \mathbb{R}$, is a compact operator on H.
b) $[D, a] \in B(\mathcal{H})$, for any $a \in A$.

Why noncommutative geometry ?

The discretization of a mathematical structure represented by an arrow $d : A_1 \to A_2$ between two objects A_1 and A_2 in an abelian category is defined by the following commutative diagram:

$$\begin{array}{ccc} A_1 & \xrightarrow{d} & A_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ A_1^\hbar & \xrightarrow{d_\hbar} & A_2^\hbar \end{array}$$

where $A_1^\hbar$ and $A_2^\hbar$ are objects of a subcategories and $d_\hbar : A_1^\hbar \to A_2^\hbar$ depending on a positive parameter $\hbar > 0$. Since we are mainly interested in the convergence in norm $\|\cdot\|_\hbar$ when $\hbar \to 0$ of the preceding diagram, we focus on the category of $*$-Banach algebras. In addition, when d is a differential operator, one can already notice that the discretized differential structure will irremediably differ from its continuous counterpart since functions and forms do not commute anymore: $gd_\hbar f \neq d_\hbar fg$.

Preliminary results

Starting from a manifold M, we construct an inverse system of triangulation, (K_n) which become sufficiently fine for large n. We associate to each space K_n a C^*-algebra A_n such that the triangulation K_n is identified with its spectrum $Spec(A_n)$. The C^*-algebras give a piecewise-linear structure to the triangulations. We then form an inductive system (A_n) with limit a C^*-algebra A_∞ with centre isomorphic to the space of continuous function $C(M)$.

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\phi_{12}^*} & A_2 & \xrightarrow{\phi_{23}^*} & \cdots & \xrightarrow{\phi_{i-1i}^*} & A_i & \xrightarrow{\phi_{ii+1}^*} & \cdots & \longrightarrow & A_\infty \\ \downarrow{\scriptstyle id_1} & & \downarrow{\scriptstyle id_2} & & & & \downarrow{\scriptstyle id_i} & & & & \downarrow \\ X_1 & \xleftarrow[\phi_{12}]{} & X_2 & \xleftarrow[\phi_{23}]{} & \cdots & \xleftarrow[\phi_{i-1i}]{} & X_i & \xleftarrow[\phi_{ii+1}^*]{} & \cdots & \longleftarrow & X_\infty \end{array}$$

Structural results

Theorem [2, Thm 4.1]: The limit C^*-algebra A_∞ is isometrically *-isomorphic to C^*-algebra of the complex valued continuous sections $\Gamma(M, A_\infty)$ over the manifold M. The center $Z(A_\infty)$ is isomorphic to the algebra of continuous functions $C(M, \mathbb{C})$:

$$Z(A_\infty) \simeq C(M, \mathbb{C}).$$

Theorem [2, Thm 4.5]: The Hilbert space $L^2(M)$ of square integrable functions over the manifold M is a subspace of the Hilbert space H_∞:

$$H_\infty = L^2(M) \oplus H.$$

We consider a sequence of the block matrix block matrices D_i

$$D_i = \frac{i}{h} \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}$$

Then the limit operator D_∞ acts on A_∞ by the commutator:

$$[D_\infty, a] = ([D_0, a_0], [D_1, a_1], \cdots, [D_i, a_i], \cdots) \in \prod_{i \in I} M_{2m_i}^-(\mathbb{C}).$$

We can compute the spectrum of the commutator $[D_\infty, a]$:

$$\sigma_{A_\infty}([D_\infty, a]) = \overline{\cup_i \sigma_{A_i}([D_i, a_i])}, \qquad \text{and} \qquad \|\,[D_\infty, a]\,\| = \|d_c a\|_\infty.$$

The limit-operator $[D_\infty, a]$ can be indentified to a multiplication operator using the spectral theorem.

Spectral convergence

There exists a finite measure μ and a unitary operator

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}, d\mu) \tag{1}$$

such that,

$$U[D, a]U^{-1}\phi = \frac{da}{dx}\phi, \quad \forall \phi \in L^2(\mathbb{R}), \tag{2}$$

Moreover, the norm of $[D, a]$ is given by $\|\,[D, a]\,\| = \|d_c a\|_\infty$.

Beyond the lattice example

There are different limitation to the generalization of the example on the lattice. Firstly, let us recall that, on a smooth Riemannian manifold, the Dirac operator encodes the information of the metric g. As it stands, the combinatorial Dirac operator on a triangulation do not contain enough geometric data. Secondly, the computation of the eigenvalues of the matrix D becomes intractable.

A first convergence result

Theorem [3, Thm 4.1]: Let $\{x_{i_0}^k\}$ be a sequence of i.i.d. sampled points from a uniform distribution on a open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. Let $\tilde{S}_n^{\hbar_n}$ be the associated operator given by:

$$\widehat{S}_n^{\hbar_n} : C^\infty(U_p) \to M_2(\mathbb{R}) \otimes U(\mathfrak{gl}_{2m_n}), \qquad \widehat{S}_n^{\hbar_n}(a) := \frac{1}{n}\sum_{k=1}^{n} e_k \left[D_X^k, a_k\right] e_k^*. \tag{3}$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then for $a \in C^\infty(U_p)$, in probability:

$$\lim_{n\to\infty} \Psi \circ \widehat{S}_n^{\hbar_n}(a) = [\mathcal{D}, a]\,(p).$$

The previous result of convergence of the Dirac operator implies a convergence of the Laplace operator. Indeed, one can define the Laplace operator as the bi-commutant with the matrix D:

$$\Delta(a) = [D, [D, a]]\,.$$

Laplace operator

Theorem [3, Thm 4.3]: Let $\{x_i\}_{i=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on an open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. $\Omega_n^{\hbar_n}$ be the associated operator given by:

$$\Omega_n^{\hbar_n}(a)(p) = \frac{C_d(\beta_\hbar)}{n\hbar^2} \sum_{k=1}^{n} \sum_{j=1}^{d+1} \lambda_j^2 \exp\left(-\frac{\langle x_i^k, s_j \rangle}{\hbar}\right) \alpha_{ij}(a_k).$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then in probability:

$$\lim_{n\to\infty} \sup_{a \in F} \left|\Omega_n^{\hbar_n}(a)(p) - \Delta_M(a)(p)\right| = 0 \tag{4}$$

References

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