Noncommutative geometry and infinitesimal spaces

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The principal motivations

- The following diagram of Banach *-algebras commutes

$$\begin{array}{ccc} A_1 & \xrightarrow{d} & A_2 \\ \downarrow \pi & & \downarrow \pi' \\ A_1^\hbar & \xrightarrow{d_\hbar} & A_2^\hbar \end{array}$$

- Question of convergence in norm $\|\cdot\|_\hbar$ when $\hbar \to 0$.
- In general, $f(d_\hbar g) \neq (d_\hbar g)f$.
- The topology of discrete spaces is ill-behaved.

Noncommutative geometry $\cap$ Geometric discretization

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Spectral Triples

Definition (Spectral triple)

A *spectral triple* is the data $(\mathcal{A}, \mathcal{H}, D)$ where:

(i) $\mathcal{A}$ is a real or complex $*$-algebra;

(ii) $\mathcal{H}$ is a Hilbert space and a left-representation $(\pi, \mathcal{H})$ of A in $\mathcal{B}(\mathcal{H})$;

(iii) D is a *Dirac operator*, which is a self-adjoint operator on $\mathcal{H}$.

We require in addition that the Dirac operator satisfies the following conditions

a) The resolvent $(D - \lambda)^{-1}$, $\lambda \notin \mathbb{R}$, is a compact operator on H.

b) $[D, a] \in B(\mathcal{H})$, for any $a \in A$.

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Original contributions

To every simplicial complex (poset) X, one can associate a C^*-algebra $A(X)$:

$$\begin{array}{ccc} A(X) & \xrightarrow{\phi^*} & A(X') \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id'} \\ X & \xleftarrow[\phi]{} & X' \end{array}$$

We can draw the following commuting diagram:

$$\begin{array}{ccccccccccc} A_1 & \xrightarrow{\phi^*_{12}} & A_2 & \xrightarrow{\phi^*_{23}} & \cdots & \xrightarrow{\phi^*_{i-1i}} & A_i & \xrightarrow{\phi^*_{ii+1}} & \cdots & \longrightarrow & A_\infty \\ \downarrow{\scriptstyle id_1} & & \downarrow{\scriptstyle id_2} & & & & \downarrow{\scriptstyle id_i} & & & & \downarrow \\ X_1 & \xleftarrow[\phi_{12}]{} & X_2 & \xleftarrow[\phi_{23}]{} & \cdots & \xleftarrow[\phi_{i-1i}]{} & X_i & \xleftarrow[\phi^*_{ii+1}]{} & \cdots & \longleftarrow & X_\infty \end{array}$$

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Original contributions

Theorem

The spectrum $Spec(A_\infty)$ equipped with the hull-kernel topology is homeomorphic to the space X_∞ and

$$\varprojlim Spec(A_i) \simeq Spec(\varinjlim A_i).$$

Proof (sketch).

- From $\phi : X' \to X$, construct $\phi^* : A(X) \to A(X')$.
- The system $\{A_n, \mathbb{N}, \phi^*_{m,n}\}$ forms a direct system.
- Conclude with the GNS construction.

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The algebra of continuous functions on the manifold M can be obtained as the centre of the limit algebra A_∞.

Theorem (T.)

The limit C^*-algebra A_∞ is isometrically *-isomorphic to C^*-algebra of the complex valued continuous sections $\Gamma(M, A_\infty)$ over the manifold M. The centre $Z(A_\infty)$ is isomorphic to $C(M, \mathbb{C})$.

A similar result is obtained for the representation space $L^2(M)$.

Theorem (T.)

The Hilbert space $L^2(M)$ of square integrable functions over the manifold M is a subspace of H_∞:

$$H_\infty = L^2(M) \oplus H.$$

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A first example on the lattice

We define the following algebra A and Dirac operator D:

$$A = M_{2m}(\mathbb{C}), \quad H = \mathbb{C}^{2m}, \quad D = \frac{i}{\hbar}\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with $(D^+)^* = -D^-$ and where D^- is given by

$$D^- = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Original contributions

Proposition (Spectral convergence)

There exists a finite measure μ and a unitary operator

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}, d\mu)$$

such that,

$$U[D, a]U^{-1}\phi = \frac{da}{dx}\phi, \quad \forall \phi \in L^2(\mathbb{R}),$$

Moreover, the norm of $[D, a]$ is given by $\| [D, a] \| = \|d_c a\|_\infty$.

This result can be generalized to the d-dimensional lattice Λ. The C^*-algebra $A(\Lambda)$ and the Dirac operator D are obtained through tensor products:

$$A(\Lambda) = A(L) \otimes \cdots \otimes A(L), \quad D_n = \sum_{k=1}^{d} 1 \otimes \cdots \otimes D_n^{(k)} \otimes \cdots \otimes 1.$$

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Beyond the lattice case

- It is known that the canonical spectral triple $(C^\infty(M), L^2(S), D)$ on a spin manifold M encodes the metric. The geodesic distance between any two points p and q on M is given by

$$\inf_{\gamma} \int_0^1 \sqrt{g_\gamma(\dot\gamma(t), \dot\gamma(t))} dt = \sup_{f \in \mathcal{A}} \{ |f(p) - f(q)| \ : \ \| [D, f] \| \leq 1 \}$$

- As it defined the combinatorial Dirac operator does not depend on the metric g of the manifold M.
- Beyond the case of the lattice, the eigenvalues of the commutator $[D, a]$ are not immediately accessible.

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Dirac operator as stochastic matrix

If we consider the more general definition of D given by

$$(D)_{ij} := \begin{cases} \omega_{ij} \neq 0 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

where the coefficients ω_{ij} are obtained from a density distribution, a first approach would be to study the convergence in average:

$$S_n^{\hbar_n}(a) := \frac{1}{n} \sum_{k=1}^{n} e_k \left[D_X^k, a_k \right] e_k^*$$

with (e_k) a family of projectors.

Original contributions

Theorem (T.)

Let $\left\{x_{i_0}^k\right\}_{k=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on an open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. Let $\widehat{S}_n^{\hbar_n}$ be the associated operator given by:

$$\widehat{S}_n^{\hbar_n}(a) := \frac{1}{n}\sum_{k=1}^{n} e_k \left[D_X^k, a_k\right] e_k^*.$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then in probability:

$$\lim_{n\to\infty} \sup_{a\in\mathscr{F}} \left|\Psi \circ \widehat{S}_n^{\hbar_n}(a)(p) - [\mathcal{D}, a](p)\right| = 0.$$

Original contributions

Proof (sketch).

- The Dirac operator is expressed as $D = \frac{i}{\hbar} \sum_{i,j} \omega_{ij} \alpha_{ij} \otimes E_{ij}$
- The coefficients ω_{ij} are obtained from the Von Mises-Fisher distribution $\omega_{ij}^k(\hbar) = C_d(\beta_\hbar) \exp\left(-\frac{\langle x_i^k, s_j \rangle}{\hbar}\right)$.
- Recall that the VMF distribution is a solution of a Fokker-Planck equation.
- Conclude with the Hoeffding's inequality.

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Original contributions

Theorem (T.)

Let $\{x_i\}_{i=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on an open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. $\Omega_n^{\hbar_n}$ be the associated operator given by:

$$\Omega_n^{\hbar_n}(a)(p) = \frac{C_d(\beta_\hbar)}{n\hbar^2} \sum_{k=1}^{n} \sum_{j=1}^{d+1} \lambda_j^2 \exp\left(- \frac{\left\langle x_i^k, s_j \right\rangle}{\hbar} \right) \alpha_{ij}(a_k).$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then in probability:

$$\lim_{n\to\infty} \sup_{a\in\mathscr{F}} \left| \Omega_n^{\hbar_n}(a)(p) - \Delta_M(a)(p) \right| = 0 \qquad (1)$$

Conclusion and future works

We summarize the list of original contributions:

- Associate to an arbitrary simplicial set K_i a C^*-algebra A_i and show that the limit A_∞ contains $C(M)$,
- Define a differential structure $da = [D_i, a]$ on each A_i,
- In the lattice case, (D_i) converges to the usual derivative $\frac{d}{dx}$.
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Thank you !

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The 2-points space

Let $a = (a_1, a_2) \in M_2(\mathbb{C})$ and the Dirac operator:

$$D = \frac{i}{\hbar} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad da = \frac{i}{\hbar} \begin{pmatrix} 0 & a_2 - a_1 \\ a_1 - a_2 & 0 \end{pmatrix}.$$

If we define the following distance:

$$d(x, y) = \sup_{a \in A} \{ |a(x) - a(y)| : \|[D, a]\| \leq 1 \}$$

then one can show that for $X = \{x, y\}$

$$d(x, y) = \hbar.$$

Without prior assumption, we see the emergence of a small parameter $\hbar$ in place of the usual distance Δx.

Let M be an oriented Riemannian manifold with a $SO(n)$-frame bundle $P \to M$. A spin structure on M is a lift:

$$\tilde{P} \to M, \qquad Spin(n)\text{-frame bundle.}$$

We consider the associate spin bundle $\mathscr{S} = \tilde{P} \times_\gamma \Delta_n$, where $\phi \in \Gamma^\infty(\mathscr{S})$ are called spinors. Let ∇ the lift of the Levi-Civita connection on M to $\tilde{P}$, with ω the associated 1-form.

$$\Gamma^\infty(\mathscr{S}) \xrightarrow{\nabla} T^*X \otimes \mathscr{S} \xrightarrow{g^{-1}} TX \otimes \mathscr{S} \xrightarrow{\mathrm{c}} \Gamma^\infty(\mathscr{S})$$

$$\text{Dirac operator} \qquad D = \mathrm{c} \circ g^{-1} \circ \nabla$$

Spin Geometry

Let $\psi \in \Gamma^\infty(\mathscr{S})$,

$$D\psi = d\psi + \frac{1}{2}\sum_{i<j} \omega_{ij} e_i e_j \psi.$$

We work at the Hilbert space level with $\mathcal{H} = L^2(M, \mathscr{S})$ square integrable spinors

$$\langle \psi, \phi \rangle = \int_M \langle \psi(x), \phi(x) \rangle_x \, dvol_g$$

$\mathcal{C}^\infty(M)$ acting as bounded operators on $\mathcal{H}$.
For $f \in \mathcal{C}^\infty(M)$, we have the commutator $[D, f]\,\psi = -i\mathrm{c}(df)\psi$ as an operator in $B(\mathcal{H})$.

Consider the triple $\mathcal{A} = \mathcal{C}^\infty(M)$, $D = \partial\!\!\!/_M$, $\mathcal{H} = (L^2(M, \mathscr{S}), \pi)$.

$\Omega^1(\mathcal{A}) := \ker(m : A \otimes A \to A), \quad \Omega^n(\mathcal{A}) = \{a_0 da_1 \cdots da_n, a_i \in \mathcal{A}\}$.

$$\text{Connes' differential forms} \quad \Omega^*_D := \Omega^*(\mathcal{A})/J$$

The representation in $B(\mathcal{H})$,
$\pi(a_0 da_1 \cdots da_n) = a_0 [D, a_1] \cdots [D, a_n]$

$$\pi : \Omega^*_D \to \Omega_{dR}(M) \quad a_0 da_1 \cdots da_n \mapsto a_0 d_{dR} a_1 \cdot d_{dR} a_2 \cdots d_{dR} a_n$$

extends to a canonical isomorphism of GDA.

The Behncke-Leptin construction

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2') If $\mathfrak{m}$ is the set of minimal points in X, then $x \in \mathfrak{m}$, one has:

$$H(x) = \mathbb{C} \otimes H^+(x) \simeq H^+(x). \tag{4}$$

3) Associate to $x \in X$ an operator algebra $A(x)$ acting on $H(x)$ (extended by zero to the whole space $H(X)$) such that

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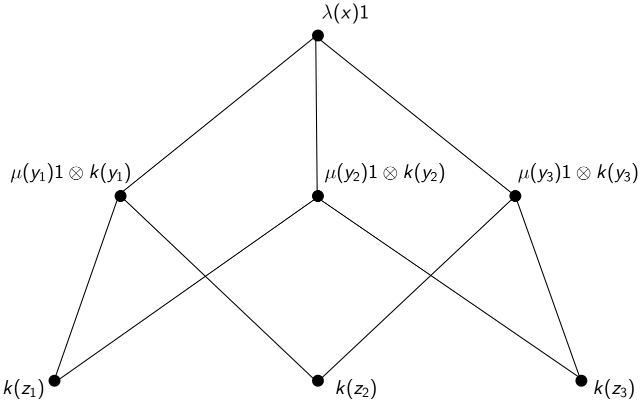
4) Build the C^*-algebra $A(X)$ associated to X as the algebra generated by the subalgebras $A(x)$ when x run over X:

$$A(X) = \bigoplus_{x \in X} A(x) \quad \text{acting on} \quad H(X) = \bigoplus_{x \in X} H(x). \tag{6}$$

The Behncke-Leptin construction: an example



F-P Equation and the Von-Mises Fisher distribution

Consider the one-parameter family of measures $(\mu_{x,t})_t$ satisfying the parabolic equation:

$$\left.\frac{\partial \mu_{x,t}}{\partial t}\right|_{t=0} = \mathscr{L}_{A,b}(\mu_{x,t}) \tag{7}$$

in the weak sense, with the operator $\mathscr{L}_{A,b}$

$$\mathscr{L}_{A,b} f = tr(AD^2 f) + \langle b, \nabla f \rangle , \quad f \in C_c^\infty(M) \tag{8}$$

We consider the von Mises-Fisher distribution on the unit sphere $\mathbb{S}^d$ given by:

$$\rho_d(x; s, \beta) = C_d(\beta) \exp\left(-\beta \langle s, x \rangle\right) \tag{9}$$

where $\beta \geq 0$, $\|s\| = 1$ and $C_d(\beta)$ is a normalization constant.

The Von-Mises Fisher distribution

We show that the von Mises-Fisher distribution satisfies the Fokker-Planck equation:

$$\left.\frac{\partial \rho_{s,t}}{\partial t}\right|_{t=0} = \partial_s(\rho_{s,t}).$$

The distribution can be defined on a normal neighbourhood U_p of the manifold M and satisfies a Fokker-Planck equation.

Proposition

The following limit holds at a point $p \in M$

$$\left.\frac{\partial}{\partial t}\left(C_d(\beta_t)\int_{U_p} e^{\widehat{\Phi}_\beta(s_i,x)} f(x)\mu(x)\right)\right|_{t=0} = \partial_i(f)(p).$$

Theorem (Hoeffding)

Let $X_1, \ldots, X_n$ be independent identically distributed random variables, such that $|X_i| \leq K$. Then

$$P\left[\left|\frac{\sum_i X_i}{n} - \mathbb{E}X_i\right| > \varepsilon\right] < 2\exp\left(-\frac{\varepsilon^2 n}{2K^2}\right).$$