Noncommutative geometry on the Berkovich line

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Table of Contents

General philosophy: before studying a problem over $\mathbb{C}$ or $\mathbb{Q}$, study it on $\mathbb{C}_p$ or $\mathbb{Q}_p$.

$$\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}_p) \to \text{Berkovich projective line}$$

- Put nonarchinedeans places on the footing as archimedean ones
- Study the dynamics of a rational map $\varphi : \mathbb{C} \to \mathbb{C}$ and deduce arithmetic or geometric properties.
- Link with Arakelov geometry: intersection theory for arithmetic surfaces
- Green functions and potential theory over $\mathbb{C}_p$

For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, there is an important function associated to the dirac operator

$$\zeta_D(z) := \mathrm{Tr}(|D|^{-z}) = \sum_{\lambda} \mathrm{Tr}(\Pi(\lambda, |D|))\lambda^{-z}$$

- The Bost-Connes system, $\mathcal{A} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times$

  $$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \zeta(\beta)$$

  with inverse temperature β. The symmetry group is the group of idèles.
- Quantum symmetries of the modular Hecke algebras $A(\Gamma)$ where Γ is a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$.
- Noncommutative geometry and Arakelov theory.

Table of Contents

Non-Archimedean Fields

A **non-archimedean absolute value** in a field K is a function $|\cdot| : K \to [0, \infty)$ such that for all $x, y \in K$,

- $|x| \geq 0, \quad$ with $|x| = 0 \Leftrightarrow x = 0$
- $|xy| = |x| \cdot |y|$,
- $|x + y| \leq \max\{|x|, |y|\}$.

One can define $\mathbb{P}^1(K)$: totally disconnected and not locally compact.

In 1990, Berkovich constructed $\mathbb{P}^1_{\mathrm{Berk}}(K)$ with much nicer properties.

Multiplicative Seminorms on $K[z]$

A **multiplicative seminorm** on $K[z]$ is a function $\|\cdot\|_\zeta : K[z] \to [0, \infty)$ such that

- $\|c\|_\zeta = |c|, \quad$ for all $c \in K$
- $\|fg\|_\zeta = \|f\|_\zeta \cdot \|g\|_\zeta, \quad$ for all $f, g \in K[z]$, and
- $\|f + g\| \leq \|f\|_\zeta + \|g\|_\zeta, \quad$ for all $f, g \in K[z]$.

Definition (Analytic spectrum)

For A a normed ring, its *analytic spectrum* or *Berkovich spectrum* $\mathrm{Spec}_{\mathrm{an}} A$ is the set of all non-zero multiplicative seminorms on A, such that all functions:

$$\mathrm{Spec}_{\mathrm{an}} A \to \mathbb{R}_+, \qquad \zeta \mapsto \|a\|_\zeta$$

for $a \in A$ are continuous.

The Berkovich Projective line: Formal definition

Definition

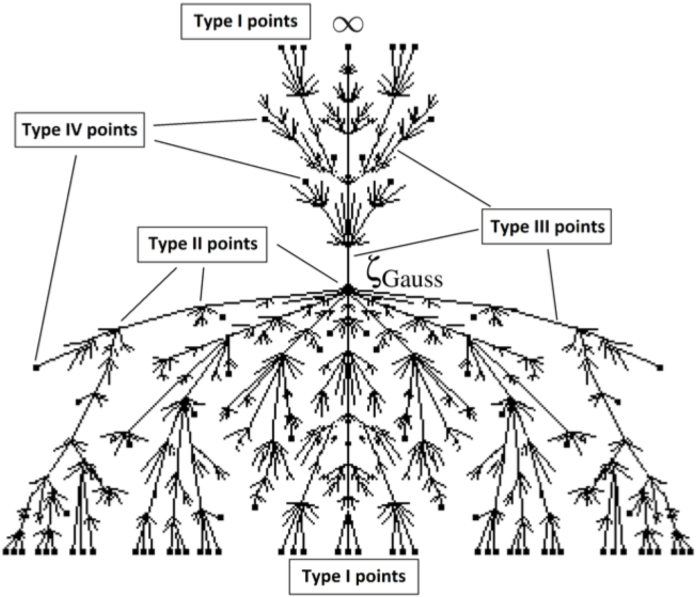
The **Berkovich affine line** $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ is the set of all multiplicative seminorms on $\mathbb{C}_p[z]$.
The **Berkovich projective line** $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ is $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p) \cup \{\infty\}$.
The **Berkovich hyperbolic line** $\mathbb{H}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ is $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \backslash P^1(\mathbb{C}_p)$.

As topological spaces, we equip $\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ and $\mathbb{H}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ with the **Gel'fand topology**.

This is the weakest topology such that for every $f \in \mathbb{C}_p[v]$, the map

$$\mathbb{A}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{R} \qquad \zeta \to \|f\|_\zeta$$

Points classification



Some properties of the Berkovich line

The Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$:

1) is compact, path connected, Hausdorff metric space
   (a) **diameter of a point**: $\mathrm{diam}(\xi) = \inf_{a \in \mathbb{C}_p} \|x - a\|_\xi$
   (b) **metric**: $\rho(\xi, \xi')$
2) is homeomorphic to the inverse limit of finite real-trees (Gromov 0-hyperbolic space).
3) (Favre & Rivera-Letelier; Baker & Rumely; Autissier, Chambert-Loir & Thuiller)
   Let $\phi \in \mathbb{C}_v(z)$ of degree $d \geq 2$:
   i) **Invariant measure**: $\phi^* \mu_\phi = d \cdot \mu_\phi$
   ii) **Green functions**
   iii) **Poisson's formula**

$$f(z) = \int_{\partial D} f d\mu_{z,D}$$

Table of Contents

Spectral triple on the projective B-line

To each finite tree Γ, we associate a spectral triple $(\mathcal{A}_\Gamma, \mathcal{H}_\Gamma, D_\Gamma)$

- $\mathcal{A}_\Gamma$ is $C_{\mathrm{Lip}}(\Gamma)$ Lipschitz continuous functions on Γ
- $\mathcal{H}_\Gamma$ is the representation space $\ell^2(\Gamma) \otimes \mathbb{C}^2$

$$\pi(f)\psi_v = \oplus_{v_+ \sim v} \begin{pmatrix} f(v_+) & 0 \\ 0 & f(v) \end{pmatrix} \psi_v$$

$$\boxed{D\psi_v = \oplus_{v_+ \sim v} \frac{1}{\rho(v, v_+)} \psi_v \otimes \sigma}$$

The pair $(r^*_{\Gamma,\Gamma'}, \iota_{\Gamma\Gamma'})$ induces a morphism of spectral triples

$$(A_\Gamma, \mathcal{H}_\Gamma, D_\Gamma) \xrightarrow{(r^*_{\Gamma,\Gamma'}, \iota_{\Gamma\Gamma'})} (A_{\Gamma'}, \mathcal{H}_{\Gamma'}, D_{\Gamma'})$$

We define the spectral $\left\{(A_j, \mathcal{H}_j, D_j), (r_{jk}^*, \iota_{jk})\right\}_J$ with the following notation:

$$A_j := C_{\text{Lip}}(\Gamma_j), \quad \mathcal{H}_j = \ell^2(\Gamma_j), \quad D_j = D_{\Gamma_j} \tag{1}$$

with the isometric morphism:

$$r_{jk}^* : A_j \to A_k, \quad \iota_{jk} : \mathcal{H} \to \mathcal{H}_k. \tag{2}$$

Theorem

The triple $(C_{\mathrm{Lip}}(\mathbb{P}^1_{\mathrm{Berk}}), \ell^2(\mathbb{P}^1_{\mathrm{Berk}}), D)$ is called the inductive realization of the inductive system $\{(C_{\mathrm{Lip}}(\Gamma_{\mathrm{j}}), \ell^2(\Gamma_{\mathrm{Berk}}), D_j), (\phi_{jk}, I_{jk})\}_J$.

Define the measure μ by

$$\mu(f) = \lim_{s \to s_0} \frac{\mathrm{Tr}(|D|^{-s}\pi(f))}{\mathrm{Tr}(|D|^{-s})}$$

It is possible to define a form Q_s on $L^2(\mathbb{P}^1_{\mathrm{Berk}}, \mu)$

$$Q_s(f,g) := \frac{1}{2}\mathrm{Tr}(|D|^{-s}[D,\pi(f)]^*[D,\pi(g)])$$

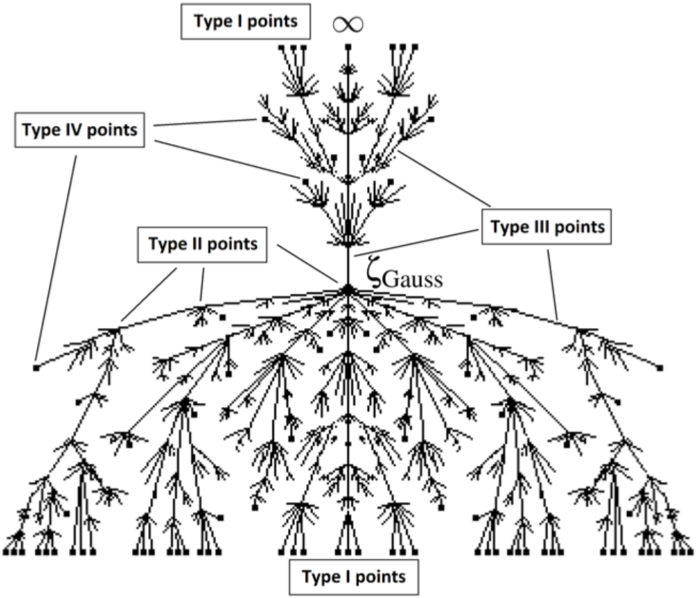
Correspondence: Dirichlet forms and Markovian semigroups.

For $s \in \mathbb{R}$, define a self-adjoint operator Δ_s such that $T_t := \exp(t\Delta_s)$ is a Markovian semigroup

$$\langle \Delta_s f, g \rangle = \frac{1}{2}\mathrm{Tr}(|D|^{-s}[D,\pi(f)]^*[D,\pi(g)]).$$

Table of Contents

Points classification



The Berkovich projective line can be described equivalently as a shift space.

- **Alphabet**: $\mathbb{Q}_1 = \mathbb{Q} \cap (0,1)$
- **Admissible words**: $x = (q_{i_1}^{k_1}, q_{i_2}^{k_2}, q_{i_3}^{k_3}, \ldots, q_{i_{m-1}}^{k_{m-1}}, q_{i_m}^{\infty}), q_{i_\ell} < q_{i_{\ell+1}}$, $q_i \in \mathbb{Q}_1$
- **Shift map**: $\sigma : \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$, $\sigma(x_n) = (x_{n+1})$.

$$Z(q) := \left\{ qx \in \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) : x \in \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \right\} \text{ and}$$
$$F(q) := \left\{ x \in \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) : qx \in \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \right\}$$

- **Right inverse**: $\sigma \circ \sigma_q(x) = x$, where $\sigma_q : F(q) \to Z(q)$

The universal C^*-algebra $\mathcal{O}_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)}$

Define the cylinder sets

$$C(\alpha,\beta) := \{\beta x \in \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) : \alpha x \in \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)\}$$

Let $\mathcal{B}_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)}$ the Boolean algebra generated by the cylinder sets.

The C^*-algebra $\mathcal{O}_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)}$ the universal unital C^*-algebra generated by **projections** $\{p_A : A \in \mathcal{B}_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)}\}$ and **partial isometries** $\{s_q : q \in \mathbb{Q}_1\}$ subject to the relations:

(i) $p_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)} = 1$, $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ and $p_\emptyset = 0$, for every $A, B \in \mathcal{B}_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)}$;

(ii) $s_\beta s_\alpha^* s_\alpha s_\beta^* = p_{C(\alpha,\beta)}$ for all α, β finite words, where $s_\emptyset = 1$ and, for $\alpha = \alpha_1 \dots \alpha_n$, $s_\alpha = s_{\alpha_1} \dots s_{\alpha_n}$.

In particular, $s_\alpha^* s_\alpha = p_{C(\alpha,\emptyset)} = p_{F_\alpha}$ and $s_\beta s_\beta^* = p_{C(\emptyset,\beta)} = p_{Z_\beta}$.

Semibranching system

Consider a measure space (X, μ) and a countable family $\{\sigma_i\}_{i\in\mathbb{N}}$, of measurable maps $\sigma_i : D_i \to X$, defined on measurable subsets $D_i \subset X$. The family $\{\sigma_i\}_{i\in\mathbb{N}}$ is called *a semibranching system* if:

(1) There exists a family $\{R_i\}_{i\in\mathbb{N}}$ of measurable subsets of X s.t.,

$$\mu(X \setminus \cup_i R_i) = 0, \quad \text{and} \quad \mu(R_i \cap R_j) = 0, \quad \text{for } i \neq j \tag{3}$$

where we denote by R_i the range $R_i = \sigma_i(D_i)$.

(2) There is a Radon-Nikodym derivative

$$\Phi_{\sigma_i} = \frac{d(\mu \circ \sigma_i)}{d\mu} > 0, \quad \mu\text{-almost everywhere on } D_i.$$

$\sigma : \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p) \to \mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ and $\{\sigma(q) : F(q) \to Z(q) : q \in \mathbb{Q}_1\}$ define a semibranching system.

Representation of $\mathcal{O}_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)}$

Consider the representation space $\mathcal{H} = L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p))$ and define for $q \in \mathbb{Q}_1$

$$(S_q\psi)(x) = \chi_{Z(q)}(x)\psi \circ \sigma(x)$$

with adjoint

$$(S_q^*\varphi)(x) = \chi_{F(q)}(x)\psi \circ \sigma_q(x)$$

Then, $\pi : s_q \mapsto S_q$ extends to a *-representation of $\mathcal{O}_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)}$

- $S_q S_q^* = P_{Z(q)}$, where $P_{Z(q)}$ is the projection given by multiplication by the characteristic function $\chi_{Z(q)}$
- It satisfies Cuntz-Krieger like relations:

$$\sum_q S_q S_q^* = 1$$

- $S_q^* S_q = P_{F(q)}$, where $P_{F(q)}$ is the projection given by multiplication by $\chi_{F(q)}$.

Projecton-valued Measures

Definition

Let $\mathcal{H}$ be a Hilbert space and

$$\mathcal{P}_{\mathcal{H}} := \left\{ P \in B(\mathcal{H}) : P = P^2 = P^* \right\}$$

An operator-valued map $P : \Sigma(X) \to \mathcal{P}_{\mathcal{H}}$ defined on $\Sigma(X)$ is called a **projection valued measure** if

(1) $P(X) = 1$ and $P(\emptyset) = 0$,

(2) If $B_1, B_2, \ldots$ in $\Sigma(X)$, such that $B_i \cap B_j = \emptyset$ for $i \neq j$, one has

$$P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) \tag{4}$$

in the strong topology sense.

(3) $P(E \cap F) = P(E)P(F)$ for $E, F \in \Sigma(X)$.

Using the Kolmogorov Extention theorem, we get

Proposition (Khalkhali & T.)

The operator-valued map defined on cylinders $Z(q) \mapsto S_q S_q^*$ extends to a spectral measure $P : \Sigma(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)) \to \mathcal{P}_{\mathcal{H}}$.

Let $f \in \mathcal{H}$ be a cyclic vector, then we can define a measure:

$$\mu_f(E) := P^f(E) := \langle P(E)f, f \rangle = \|P(E)f\|^2$$

This measure can be decomposed using the C^*-algebra

$$\boxed{\int_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)} \psi \; d\mu_f = \sum_{q \in \mathbb{Q}_1} \int_{\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)} \psi \circ \sigma_q \; d\mu_{S_q^* f}}$$

Perron-Frobenius operator

The **transfer operator** $T_\sigma : L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)) \to L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p))$ that composes with the coding map denoted by σ,

$$(T_\sigma \psi)(x) = \psi(\sigma(x)).$$

its adjoint P_σ given by

$$\int \psi P_\sigma(\xi) d\mu = \int T_\sigma(\psi)\xi d\mu$$

and called the **Perron-Frobenius operator**. Then, the Perron-Frobenius operator P_σ is of the form

$$\boxed{(P_\sigma \xi)(x) = \sum_{q \in \mathbb{Q}_1} \chi_{Z(q)} \xi(\sigma_q(x)) = \sum_{q \in \mathbb{Q}_1} S_q^*}$$

Graph Laplacian $\quad P_\sigma \xi = \xi \Leftrightarrow \Delta \xi = 0$

Table of Contents

A C^*-dynamical system is a pair (A, σ) with a

- A a C^*-algebra
- $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$

such that $\sigma_0 = \mathrm{id}$, $\sigma_s \circ \sigma_t = \sigma_{s+t}$ and $t \mapsto \sigma_t(a)$ is norm continuous. A *state* of A is a linear functional $\varphi : A \to \mathbb{C}$ such that

$$\varphi(a^* a) \geq 0 \text{ and } \|\varphi\| = \varphi(1) = 1$$

Examples:

- The states of a commutative C^*-algebra $C_0(\Omega_A)$ are in bijection with probability measures μ on Ω_A $\varphi_\mu(f) = \int_{\Omega_A} f d\mu$
- **GNS construction** $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle$

The KMS condition

Definition (KMS states)

A state φ on A satisfies the Kurbo-Martin-Schwinger (KMS) condition with respect to σ at inverse temperature $\beta \neq 0$ (φ is a σ-KMS_β), if

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all σ-analytic elements $a, b \in A$

This a trace-like condition, twisted by σ along imaginary time.

Proposition

If $\beta \neq 0$ and φ is a KMS_β state, then φ is σ-invariant.

Harmonic measures on $\mathbb{D}$: Poisson kernel

$$P(x,\xi) = \frac{1-\|x\|^2}{\|\xi - x\|^2}$$

Poisson transform: $f \mapsto \mathcal{P}f = \int_{\mathbb{S}^1} f(\xi) d\nu_x(\xi)$ defines an isomorphism between $L^\infty(\mathbb{S}^1)$. Equivalently, harmonic measures associated to points of $\mathbb{D}$:

$$\forall \xi \in \mathbb{S}^1 \ \forall x, y \in \mathbb{D}, \ \frac{d\nu_x}{d\nu_y}(\xi) = e^{-b_\xi(y,x)}. \tag{5}$$

where, $b_\xi(x,y)$ is the **Busemann function**

$$\forall \xi \in \mathbb{S}^1 \ \forall x, y \in \mathbb{D}, \ b_\xi(x,y) = \log\left(\frac{P(y,\xi)}{P(x,\xi)}\right)$$

$P(x,\xi)d\sigma(\xi)$ is the **Patterson-Sullivan measure** on $\mathbb{D}$.

$P(x,\xi)d\sigma(\xi)$ is supported on $\mathbb{S}^1$ and is invariant under Möbius transformation $\mathrm{Iso}(\mathbb{D})$.

(Patterson construction) This extends to more general hyperbolic space X (such as Gromov hyperbolic, $\mathbb{R}$-trees). Consider a compactification $\overline{X} = X \cup \partial X$, $\Gamma < \mathrm{Iso}(X)$ a discrete subgroup.

$$\textbf{Busemann funnction}: \quad b_\xi(x,y) = \lim_{z\to\xi}(d(x,z) - d(y,z)).$$

Proposition (Coornaert)

If $\delta(\Gamma) < \infty$, then there exists a (unique) measure Γ-quasiconformal of dimension $\delta(\Gamma)$ with support in ∂X. This measure is called the Patterson-Sullivan measure s.t

$$\forall \xi \in \partial X, \quad \frac{d\mu_{PS,y}}{d\mu_{PS,x}}(\xi) = e^{-\delta b_\xi(y,x)}$$

Hyperbolic space

For $\varphi \in \mathbb{C}_p(T)$ of $\deg = 1$, defines an action on $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$

$$\mathrm{Aut}(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)) \simeq \mathrm{PGL}_2(\mathbb{C}_p)$$

The action is continuous on $\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$ and isometric on $\mathbb{H}^1_{\mathrm{Berk}}(\mathbb{C}_p)$

Let $\Gamma < \mathrm{PGL}_2(\mathbb{C}_p)$ a **Schottky subgroup**;

$$\sigma_g : \mathrm{PGL}_2(\mathbb{C}_p) \times \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p),\ \sigma_g(x) = \begin{pmatrix} \mathrm{diam}(g) & a \\ 0 & 1 \end{pmatrix} \cdot x$$

its limit set is given by

$$\Lambda := \left\{ y \in \partial X \mid y = \lim\{\gamma_m(x)\}\ \{\gamma_m\}\ \in \mathrm{PGL}_2(\mathbb{C}_p) \right\} = \partial\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p)$$

Let $\mathcal{H} = L^2(\mathbb{P}^1_{\mathrm{Berk}}(\mathbb{C}_p))$, (π, U) a covariant representation.

The crossed product algebra:

$$\mathcal{A} = C(\Lambda) \rtimes \Gamma, \quad (\pi_\xi \rtimes U)(f) = \sum_{\gamma' \in \Gamma} \pi_\xi(f_{\gamma'}) U_{\gamma'}$$

One-parameter automorphism on the generators:

$$\sigma_t \left(\sum_\gamma f_\gamma(\xi) U_\gamma \right) = e^{itH} f e^{-itH} = \sum_\gamma p^{itB(x_0, \gamma x_0, \xi)} f_\gamma(\xi) U_\gamma$$

(Busemann function) $B(x_0, \gamma x_0, \xi) = \lim_{x \to \xi} \rho(x_0, x) - \rho(\gamma x_0, x)$.

Then $(\mathcal{A}, \sigma_t)$ defines a C^*-dynamical system with KMS_β states.

KMS & P-S. measure

Unique KMS sates at inverse temperature $\beta = \delta(\Lambda)$:

$$\boxed{\varphi_{\beta,x_0}\left(\sum_{\gamma} f_\gamma(\xi) U_\gamma\right) = \int_\Lambda f_e(\xi) d\mu_{PS,x_0}(\xi)}$$

associated to the Hamiltonian

$$H\left(f_\gamma(\xi) \otimes \gamma\right) = B(x_0, \gamma x_0, \xi) f_\gamma(\xi) \otimes \gamma$$

One can define a 1-cocycle $c : \Gamma \to \mathcal{H}_\pi$ such that

$$B(x_0, \gamma x_0, \xi) = \|c(\gamma)\|^2_{\mathcal{H}_\pi}, \qquad \gamma \in \Gamma$$

In the Hilbert space $\ell^2(\Gamma, \mathcal{H}_\pi)$, define the self-adjoint operator

$$\boxed{Df_\gamma = \|c_\gamma\| f_\gamma} \qquad \Rightarrow \qquad D^2 f_\gamma = B(x_0, \gamma x_0, \xi) f_\gamma$$

Table of Contents

Conclusion

We have constructed three different spectral triples on the B-line, each of which has some geometric and dynamic information:

- Inverse limit of finite spectral triples
- Spectral triple associated to $\Gamma \subset \mathrm{PGL}_2(K)$
- Invariant measures and KMS states on crossed product algebras./

Remaining questions:

- (noncommutative) potential theory
- summability of the spectral triples
- arithmetic/dynamic

Thank you !

- Matthew Baker and Robert Rumely, *Potential Theory on the Berkovich Projective Line* (en).
- Caterina Consani and Matilde Marcolli, *Non-commutative geometry, dynamics, and infinity-adic Arakelov geometry*, October 2003, arXiv:math/0205306.
- Gunther Cornelissen and Matilde Marcolli, *Graph Reconstruction and Quantum Statistical Mechanics*, September 2012.
- Remus Floricel and Asghar Ghorbanpour, *On inductive limit spectral triples*, December 2017.
- M. Greenfield, M. Marcolli, and K. Teh, *Twisted spectral triples and quantum statistical mechanical systems*, P-Adic Numbers, Ultrametric Analysis, and Applications **6** (2014), 81–104.

References II

- Magnus Goffeng, Adam Rennie, and Alexandr Usachev, *Constructing KMS states from infinite-dimensional spectral triples*, Journal of Geometry and Physics **143** (2019), 107–149.
- Masoud Khalkhali and Damien Tageddine, *Noncommutative geometry on the berkovich projective line*, 2024.
- Tsuyoshi Kajiwara and Yasuo Watatani, *C^*-algebras associated with complex dynamical systems*, September 2003.
- John Lott, *Limit Sets as Examples in Noncommutative Geometry*, K-Theory **34** (2005), 283–326.
- Qiuyu Ren, *Linear Fractional Transformations on the Berkovich Projective Line* (en).