Morse Theory:

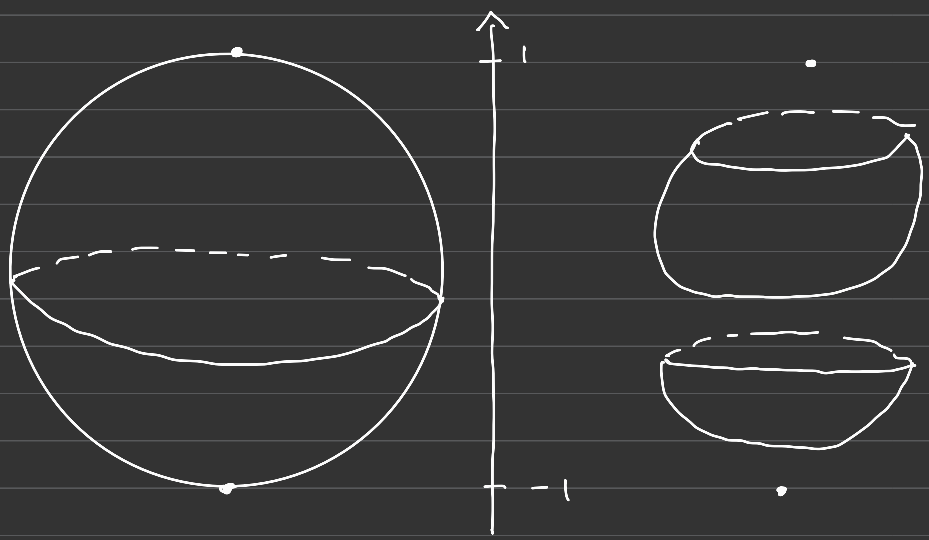
main objective:

top. invariant features of space using critical points of a Morse function f and the trajectories of a gradient field. $-\nabla f$

Ex: "height function"

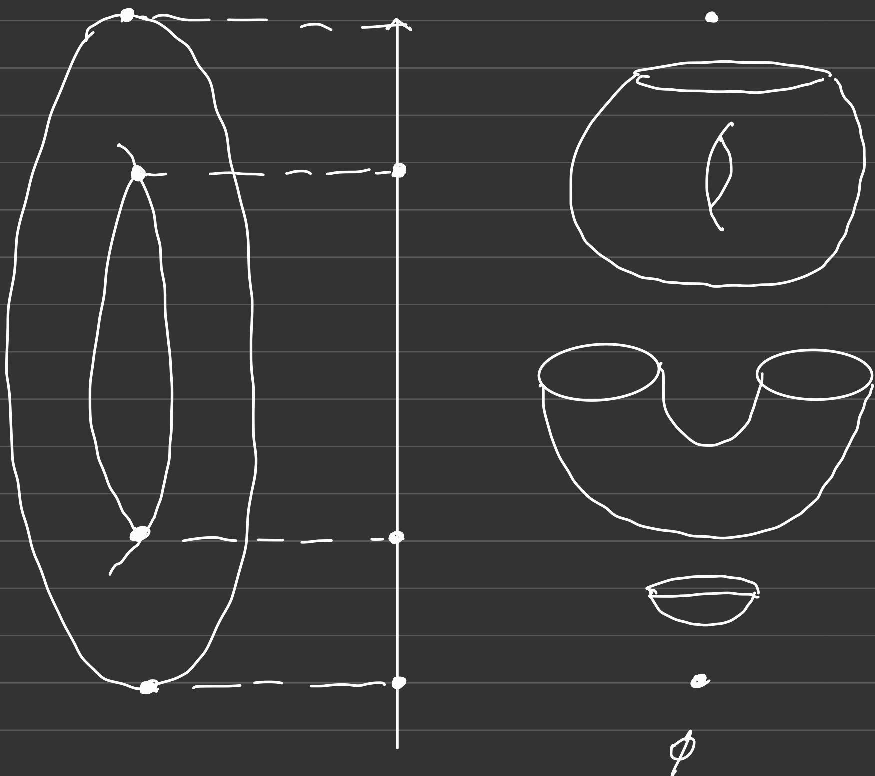
$$\mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$(x,y,z) \longmapsto z$$



$\emptyset$

{critical points} $\longleftrightarrow$ {Change in topology}



1) Morse function
2) Existence and Density
3) Morse lemma and Index of a Crit Point.
4) Homotopy type in terms of crit values

5) Morse inequality

7) Morse Homology

8) Applications :

→ top. data analysis

→ PDE ⇝ Conley index.

Def: (Critical Points)

$(d_x f) = 0 \qquad f : M \longrightarrow \mathbb{R}$

$(J_x F)$ not full rank $\qquad f : M \to \mathbb{R}^n$.

Def: (Hessian at a Critical Point)

$$d_x^2 f(X, Y) \qquad X, Y \in T_x M$$

is a well defined, symmetric, bilinear form.

Def: $B(x, y)$ non degenerate if

$$\begin{cases} B(x,y) = 0 \quad \forall y \implies x = 0 \\ B(x,y) = 0 \quad \forall x \implies y = 0 \end{cases}$$

(

Def: (Morse Function)

A Morse function is a function that has only non degenerate critical points.

Ex:

- $f_p(x) = \|x - p\|^2$
- $F : (x, y, z) \longmapsto z$ (height)

  (S^2, T^2)
- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

  $(x, y) \longmapsto \cos(2\pi x) + \cos(2\pi y)$

  on the Torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$

Theorem:

Let V be a compact manifold

$\{$ Morse functions $\}$ is dense

open (C^2 Top) in $C^\infty(V, \mathbb{R})$.

Index of a Critical Point:

Given h and ∇.

Define $W^u(a) = \{x \in V \mid \lim_{s \to -\infty} \varphi^s(x) = a\}$

φ^s is the flow of $-\nabla h$.

$$W^s(a) = \{x \in V \mid \lim_{s \to \infty} \varphi^s(x) = a\}$$

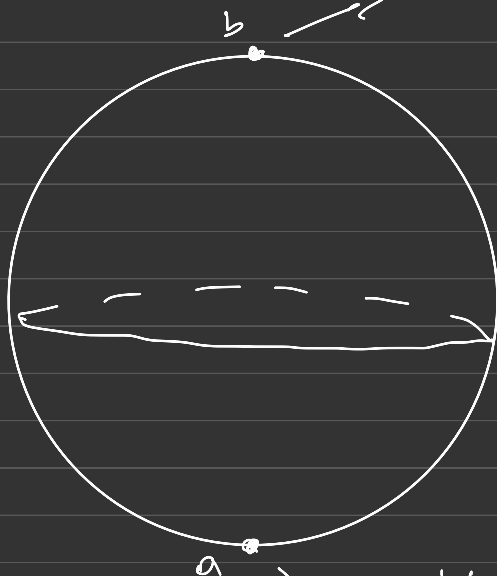
$\operatorname{Ind}(a) = \dim W^u(a)$

$= \{\#$ negative eigenvalues of $d_x^2 f\}$

Ex :

S^2

$\mathrm{Ind}(b) = 2$

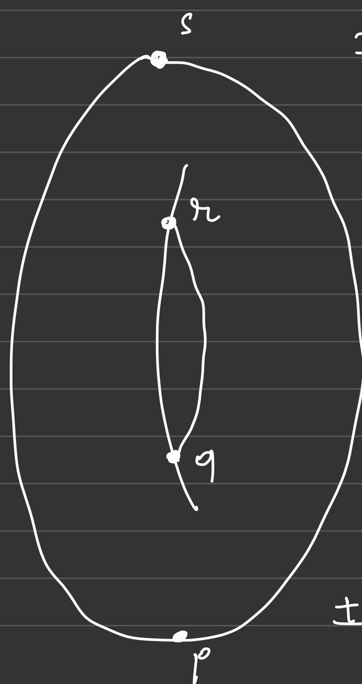


$\mathrm{Ind}(a) = 0$

T^2

$\mathrm{Ind}(s) = 2$

$\mathrm{Ind}(r) = 1$



$\mathrm{Ind}(r) = 1$

$\mathrm{Ind}(p) = 0$

Morse lemma:

b a non degenerate crit. of $f: M \to \mathbb{R}$

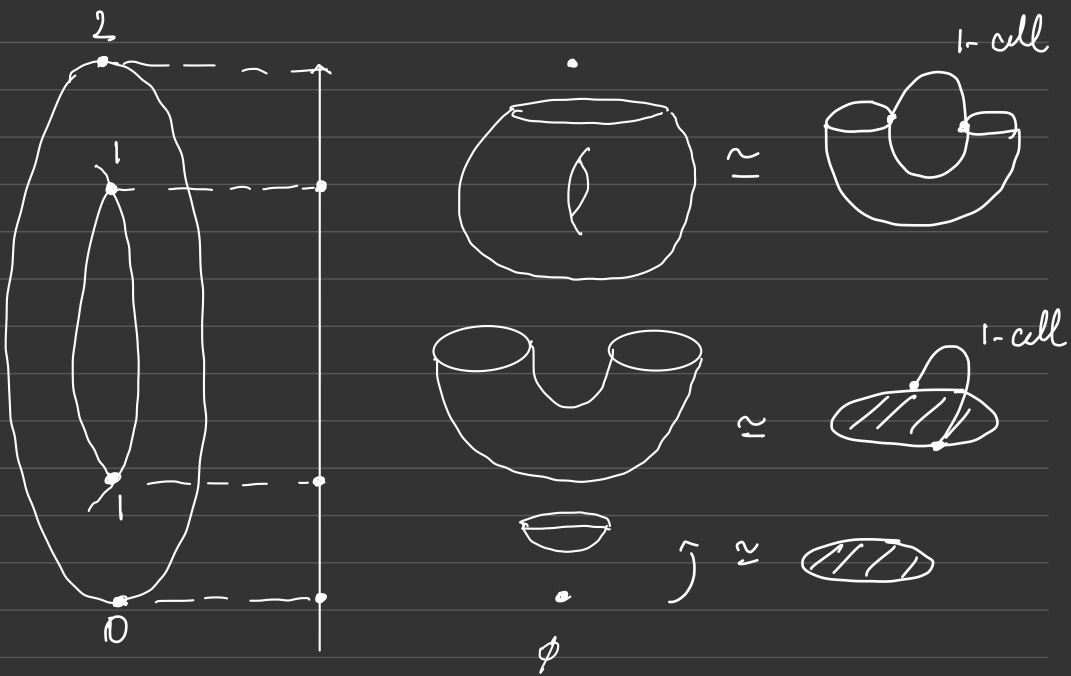
$\exists$ a chart $(x_1, --, x_n)$ in a nbhd

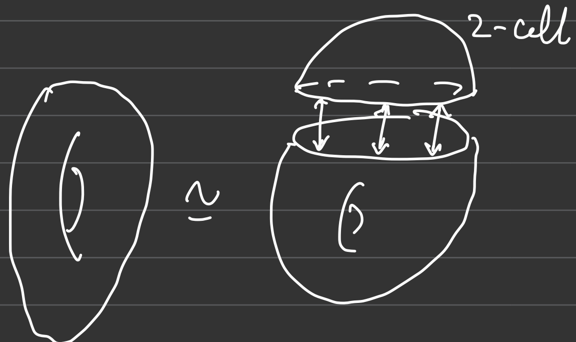
U of b s.t $x_i(b) = 0$ $\forall i$ and

$$f(x) = f(b) - x_1^2 - \cdot -- - x_\alpha^2 + x_{\alpha+1}^2 + \cdots + x_n^2$$

$$\boxed{\mathrm{Ind}(b) = \alpha}$$

Homotopy Type in terms of Critical Points





Define $M^a = f^{-1}((-\infty, a])$

M^α doesn't change except when α passes the height of a crit point

when α pass this height with ind γ
a γ - cell is attached to M^{α}.

<u>Theorem</u>: Suppose $f \in C^{\infty}(M, \mathbb{R})$
p is a nondegenerate crit of f
of index γ $f(p) = q$.
Suppose $f^{-1}([q-\varepsilon, q+\varepsilon])$ is compact
and contains only p as a crit
point. Then $M^{q+\varepsilon} \simeq M^{q-\varepsilon} + \gamma$ cell attached.

If no crit value $M^{q+\varepsilon} \simeq M^{q-\varepsilon}$.

Morse Homology:

$Cr(f)$ with a grading (μ)

$$C_k(f) := \mathrm{Span}_{\mathbb{Z}/2}(Cr(f))$$

Let $X = -\nabla f$

$$\partial_X : C_k(f) \longrightarrow C_{k-1}$$

$$\partial_X(a) = \sum_{b \in C_{k-1}} n_X(a,b)\, b$$

with $n_X(a,b) \in \mathbb{Z}/2$

$$n_X(a,b) = (\#\text{ Trajectory } a \to b)/2$$

Define $H_k(f, \mathbb{Z}/2) := \mathrm{Ker}\, \partial^k / \mathrm{Im}\, \partial^{k+1}$

Theorem : M compact h Morse

$$MH_\bullet(h, \mathbb{Z}/2) \simeq H_\bullet(M, \mathbb{Z}/2).$$

Morse Inequality :

Def : (Morse Polynomial)

$$M_h(t) = \sum_{p \in Cr(h)} t^{\mu(p)}$$

Poincaré Polynomial :

$$P(t) = \sum_i \beta_i t^i$$

Theorem : (Strong Morse Inequality)

$$M_h(t) = P(t) + (1+t) Q(t)$$

with $Q \in \mathbb{N}[t]$

Corollary : (Euler Characteristic)

$t = -1$

$$M_h(-1) = P(-1) = \chi(M)$$

$$\Rightarrow \quad \chi(M) = \sum_{p \in C(f)} (-1)^{\mu(p)} .$$

Morse Theory:

→ topological data analysis (persistent homology)

→ rigorous computing (dynamical systems)

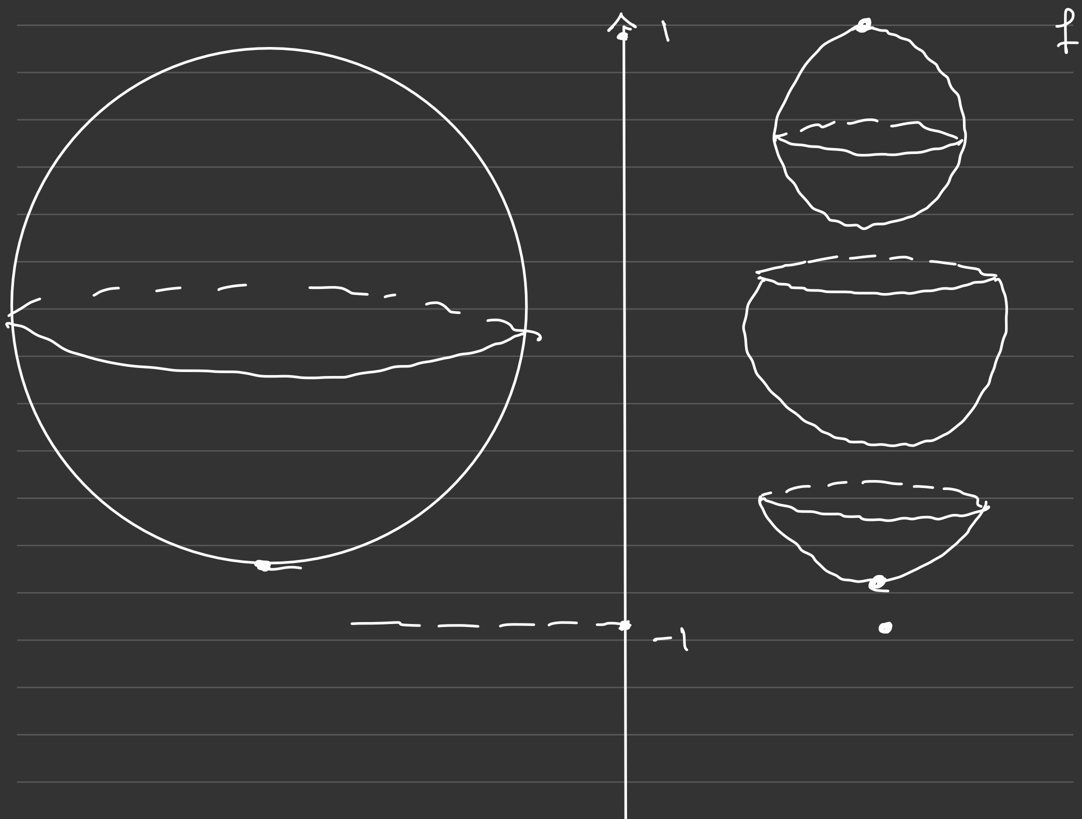
→ PDE / ODE : $u_t = u_{xx} + f(x, u, u_x)$

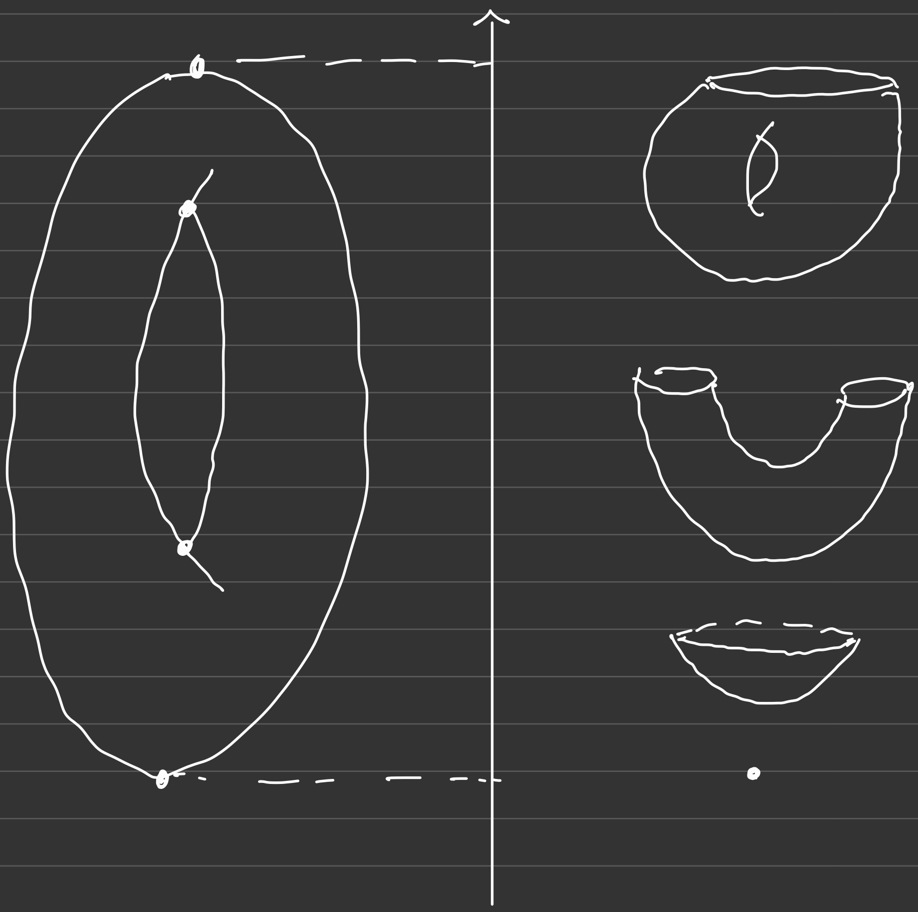
"travelling wave" solution.

$$\{\text{Shapes, topology}\} \longleftrightarrow \left\{\begin{array}{l}\text{Morse functions } f \\ -\nabla f\end{array}\right\}$$

Ex: f : "height function"

$$(x, y, z) \longmapsto z$$





$\{\text{topology}\} \longleftrightarrow \left\{\begin{array}{c}\text{critical points}\\ \text{of } f\end{array}\right\}$

Def: (Critical Points) $f : M \to \mathbb{R}$

$(d_x f) = 0$

$(J_x F)$ not full rank $(F : M \to \mathbb{R}^n)$

Def: (Hessian at critical points)

$d_x^2 f(X, Y)$ $X, Y \in T_x M$

is well defined, symmetric, bilinear form.

Def: A critical point is non-degenerate if $d_x^2 f$ is non degenerate.

Def: (Morse functions)

A Morse function is a function that only nondegenerate critical points.

Ex:

- $f_p(x) = \| x - p \|^2$
- $F(x, y, z) \longmapsto z$ (S^2, T^2)
- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (on T^2!)

$$(x, y) \longmapsto \cos(2\pi x) + \cos(2\pi y)$$

Theorem:

Let V a compact manifold

$\{$Morse functions on $V\}$ is open dense in $C^\infty(M, \mathbb{R})$

Index of critical points:

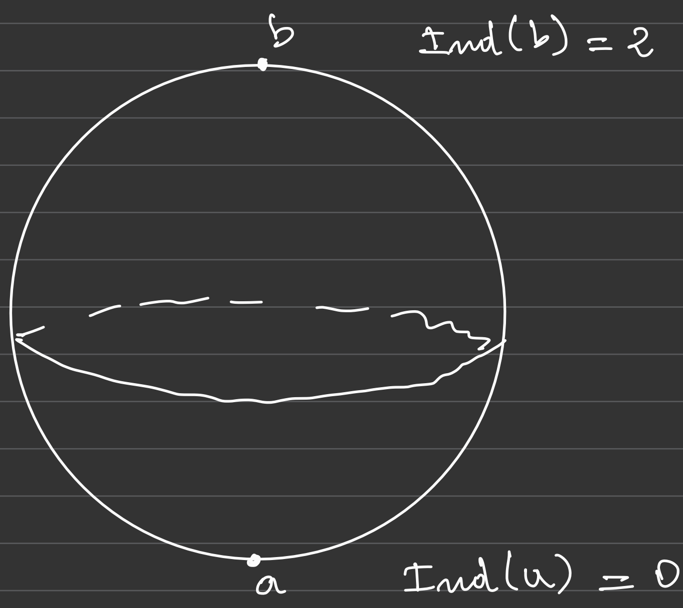
Given h smooth Morse, φ^s the flow of $-\nabla h$, $a \in Cr(h)$

Define $W^u(a) = \{ x \in V \mid \lim_{s \to -\infty} \varphi^s(x) = a \}$

$\mathrm{Ind}(a) = \dim W^u(a)$

$= \#$ negative eigenvalues of $d^2_a h$.

Ex:



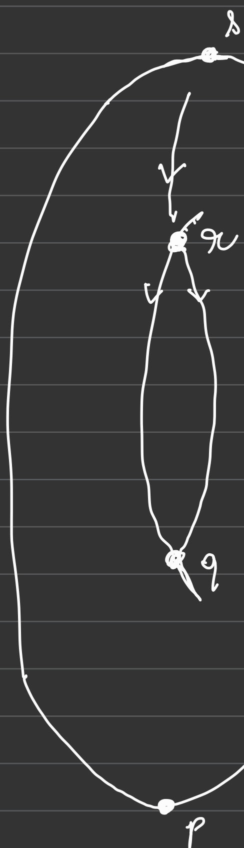
Ex:

Ind(s) = 2

Ind(r) = 1

Ind(q) = 1

Ind(p) = 0



Morse Lemma:

b a nondegenerate crit pt of $f: M \to \mathbb{R}$

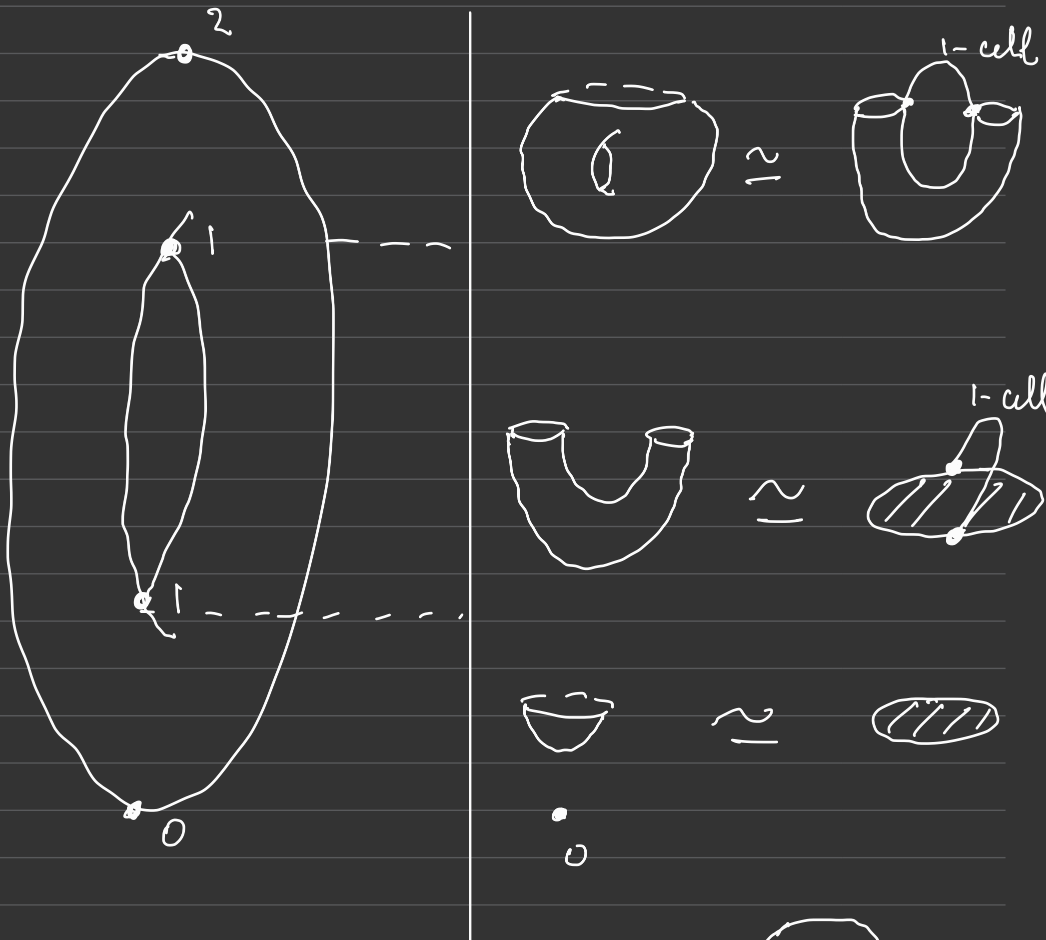
$\exists$ chart $(x_1, \dots, x_n)$ in nbhd of b

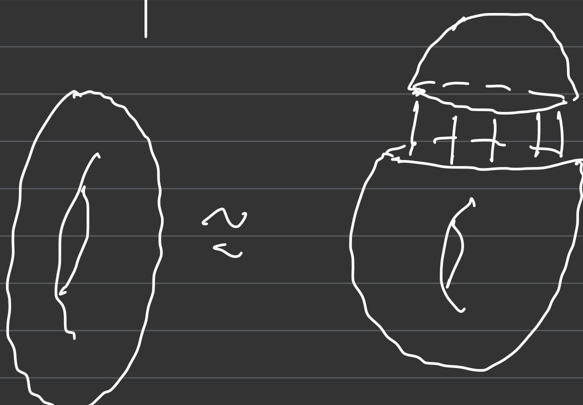
$x_i(b) = 0 \quad \forall i$ and

$$f(x) = f(b) - x_1^2 - \dots - x_\alpha^2 + x_{\alpha+1}^2 + \dots + x_n^2$$

$$\operatorname{Ind}(b) = \alpha$$

Homotopy Type in terms of Critical Points:


2
1
1
0
1-cell
1-cell
0
≃
≃
≃


≃

Let f be a Morse function

Define $M^a := f^{-1}((-\infty, a])$

M^a doesn't change except when a passes the height of crit. point.

When a passes this height with ind γ, a γ-cell is attached to M^a.

$$\left\{ \begin{matrix} \text{critical points} \\ \text{of } f \\ \text{with ind } \gamma \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{homotopy} \\ \text{of the} \\ \text{manifold} \end{matrix} \right\}$$

Morse Homology:

$Cr(f)$: $\{$ critical points of f $\}$

Natural grading on $Cr(f)$ given by $\mu(a) = \mathrm{Ind}(a)$ with $a \in Cr(f)$.

Build $C_k(f) := \mathrm{Span}_{\mathbb{Z}/2}\left(Cr(f)_{\mu(p) \leq k} \right)$

$$C_k \xrightarrow{\partial_X} C_{k-1}$$

$$\partial_X(a) = \sum_{b \in C_{k-1}} \underbrace{n_X(a,b)}_{\in \mathbb{Z}/2} \cdot b$$

$$n_X(a,b) = \{\#\text{ trajectories } a \to b\}/2$$

its a finite number.

Define : $H_k(f, \mathbb{Z}/2) := \operatorname{Ker} \partial^k / \operatorname{Im} \partial^{k+1}$

Theorem : M compact , h Morse

$$MH_\bullet(h, \mathbb{Z}/2) \cong H_\bullet(M, \mathbb{Z}/2)$$

Morse inequality :

Def : (Morse Polynomial)

$$M_h(t) = \sum_{p \in Cr(h)} t^{\mu(p)}$$

(Poincaré Polynomial)

$$P(t) = \sum_i \beta_i t^i$$

$$\beta_i = \dim H_i(M, \mathbb{Z}/2)$$

Theorem:

$$M_h(t) = P(t) + (1+t)\,Q(t)$$

$Q \in \mathbb{N}[t]$

Corollary: (At $t = -1$)

$$M_h(-1) = \sum_{p \in Cr(h)} (-1)^{\mu(p)}$$

$$P(-1) = \sum_i \beta_i (-1)^i = \chi(M)$$

$\chi(M) :=$ Euler Characteristic of M.

$$\chi(M) = \sum_{p \in Cr(h)} (-1)^{\mu(p)}$$

Go further :

→ Conley Index (Dynamical Systems)

→ Floer Homology (Hamiltonian Systems ; Symplectic Geometry)

→ top structure of data (Discrete Morse Theory)