Noncommutative geometry on discrete spaces

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The principal motivations

- The following diagram in the category of Banach *-algebras commutes

$$\begin{array}{ccc} A_1 & \xrightarrow{d} & A_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ A_1^{\hbar} & \xrightarrow{d_\hbar} & A_2^{\hbar} \end{array}$$

- We are interested in the question of convergence in norm $\| \cdot \|_\hbar$ when $\hbar \to 0$.
- Discretized operators do not commute in general i.e. $f(d_\hbar g) \neq (d_\hbar g) f$.
- The topology of discrete spaces (lattices, triangulations,...) is ill-behaved.

Definition (Spectral triple)

A *spectral triple* is the data $(\mathcal{A}, \mathcal{H}, D)$ where:

(i) $\mathcal{A}$ is a real or complex $*$-algebra;

(ii) $\mathcal{H}$ is a Hilbert space and a left-representation $(\pi, \mathcal{H})$ of A in $\mathcal{B}(\mathcal{H})$;

(iii) D is a *Dirac operator*, which is a self-adjoint operator on $\mathcal{H}$.

We require in addition that the Dirac operator satisfies the following conditions

a) The resolvent $(D - \lambda)^{-1}$, $\lambda \notin \mathbb{R}$, is a compact operator on H.

b) $[D, a] \in B(\mathcal{H})$, for any $a \in A$.

The 2-points space

Let $a = (a_1, a_2) \in M_2(\mathbb{C})$ and the Dirac operator:

$$D = \frac{i}{\hbar}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad da = \frac{i}{\hbar}\begin{pmatrix} 0 & a_2 - a_1 \\ a_1 - a_2 & 0 \end{pmatrix}.$$

If we define the following distance:

$$d(x,y) = \sup_{a \in A}\{|a(x) - a(y)| : \|[D,a]\| \leq 1\}$$

then one can show that for $X = \{x, y\}$

$$d(x,y) = \hbar.$$

Without prior assumption, we see the emergence of a small parameter $\hbar$ in place of the usual distance Δx.

Preliminary results

The centre of approximately finite C^*-algebras exhaust all possible abelian separable C^*-algebras.

Theorem (Bratteli)

Let $\mathfrak{Z}$ be an abelian separable C^*-algebra with unit. Then there exists an approximately finite-dimensional C^*-algebra $\mathfrak{A}$ having $\mathfrak{Z}$ as center.

One can associate a C^*-algebra A to a triangulation.

Theorem (Behncke and Leptin)

For any (finite) partially ordered set X, there exists a C^*-algebra A such that the primitive spectrum $Prim(A)$ is homeomorphic to X.

Preliminary results

- Associate a separable Hilbert space $H(X)$ to the space X and attach to every point $x \in X$ a subspace $H(x) \subseteq H(X)$:

$$H(x) = H^{-}(x) \otimes H^{+}(x).$$

- Associate to each point $x \in X$ an operator algebra $A(x)$ acting on $H(x)$, extended by zero to the whole space $H(X)$:

$$A(x) = 1_{H^{-}(x)} \otimes \mathcal{K}(H^{+}(x)).$$

- Build the C^*-algebra $A(X)$ associated to X:

$$A(X) = \bigoplus_{x \in X} A(x) \quad \text{acting on} \quad H(X) = \bigoplus_{x \in X} H(x).$$

Sequences of spectral triples

We can draw the following commuting diagram:

$$
\begin{tikzcd}
A_1 \arrow[r, "\phi^*_{12}"] \arrow[d, "id_1"] & A_2 \arrow[r, "\phi^*_{23}"] \arrow[d, "id_2"] & \cdots \arrow[r, "\phi^*_{i-1i}"] & A_i \arrow[r, "\phi^*_{ii+1}"] \arrow[d, "id_i"] & \cdots \arrow[r] & A_\infty \arrow[d] \\
X_1 & X_2 \arrow[l, "\phi_{12}"] & \cdots \arrow[l, "\phi_{23}"] & X_i \arrow[l, "\phi_{i-1i}"] & \cdots \arrow[l, "\phi^*_{ii+1}"] & X_\infty \arrow[l]
\end{tikzcd}
$$

Theorem

The spectrum $Spec(A_\infty)$ equipped with the hull-kernel topology is homeomorphic to the space X_∞ and

$$\varprojlim Spec(A_i) \simeq Spec(\varinjlim A_i).$$

Sequences of spectral triples

The algebra of continuous functions on the manifold M can be obtained as the centre of the limit algebra A_∞.

Theorem (T.)

The limit C^*-algebra A_∞ is isometrically *-isomorphic to C^*-algebra of the complex valued continuous sections $\Gamma(M, A_\infty)$ over the manifold M. The centre $Z(A_\infty)$ is isomorphic to $C(M, \mathbb{C})$.

A similar result is obtained for the representation space $L^2(M)$.

Theorem (T.)

The Hilbert space $L^2(M)$ of square integrable functions over the manifold M is a subspace of H_∞:

$$H_\infty = L^2(M) \oplus H.$$

Definition

Let $D \in M_{2m}(\mathbb{C})$ be an odd and hermitian matrix and let ω_{ij} be the coefficients of the block D^-. We say that D is an admissible Dirac operator associate to X if it satisfies the additional condition:

a) vertices i and j do not share an edge $\Leftrightarrow \omega_{ij} = 0, \ \forall i, j \in \mathfrak{M}$,

b) the eigenvalues μ_n satisfy the asymptotic $\mu_n(D) = O(\hbar^{-1})$.

The prototypical example is given by the *combinatorial Dirac operator*, for which:

$$\omega_{ij} := \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

A first example on the lattice

We define the following algebra A and Dirac operator D:

$$A = M_{2m}(\mathbb{C}), \quad H = \mathbb{C}^{2m}, \quad D = \frac{i}{\hbar}\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with $(D^+)^* = -D^-$ and where D^- is given by

$$D^- = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

A first example on the lattice

We consider a sequence of the block matrix block matrices D_i

$$D_i = \frac{i}{\hbar} \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}$$

Then the limit operator D_∞ acts on A_∞ by the commutator:

$$[D_\infty, a] = ([D_0, a_0], [D_1, a_1], \cdots, [D_i, a_i], \cdots) \in \prod_{i \in I} M_{2m_i}^-(\mathbb{C}).$$

We can compute the spectrum of the commutator $[D_\infty, a]$:

i) $\sigma_{A_\infty}([D_\infty, a]) = \overline{\cup_i \sigma_{A_i}([D_i, a_i])}$

ii) $\| [D_\infty, a] \| = \| d_c a \|_\infty$

A first example on the lattice

Proposition (Spectral convergence)

There exists a finite measure μ and a unitary operator

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}, d\mu) \tag{1}$$

such that,

$$U[D, a]U^{-1}\phi = \frac{da}{dx}\phi, \quad \forall \phi \in L^2(\mathbb{R}), \tag{2}$$

Moreover, the norm of $[D, a]$ is given by $\| \, [D, a] \, \| = \|d_c a\|_\infty$.

This result can be generalized to the d-dimensional lattice Λ. The C^*-algebra $A(\Lambda)$ and the Dirac operator D are obtained through tensor products:

$$A(\Lambda) = A(L) \otimes \cdots \otimes A(L), \quad D_n = \sum_{k=1}^{d} 1 \otimes \cdots \otimes D_n^{(k)} \otimes \cdots \otimes 1.$$

Beyond the lattice case

- It is known that the canonical spectral triple $(C^\infty(M), L^2(S), D)$ on a spin manifold M encodes the metric. The geodesic distance between any two points p and q on M is given by

$$\inf_{\gamma} \int_0^1 \sqrt{g_\gamma(\dot\gamma(t), \dot\gamma(t))}dt = \sup_{f \in \mathcal{A}} \{|f(p) - f(q)| \ : \ \| [D, f] \| \leq 1\}$$

- As it defined the combinatorial Dirac operator does not depend on the metric g of the manifold M.
- Beyond the case of the lattice, the eigenvalues of the commutator $[D, a]$ are not immediately accessible.

Dirac operator as stochastic matrix

If we consider the more general definition of D given by

$$(D)_{ij} := \begin{cases} \omega_{ij} \neq 0 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

where the coefficients ω_{ij} are obtained from a density distribution, a first approach would be to study the convergence in average:

$$S_n^{\hbar_n}(a) := \frac{1}{n} \sum_{k=1}^{n} e_k \left[D_X^k, a_k \right] e_k^*$$

with (e_k) a family of projectors.

F-P Equation and the Von-Mises Fisher distribution

Consider the one-parameter family of meausres $(\mu_{x,t})_t$ satisfy the parabolic equation:

$$\left.\frac{\partial \mu_{x,t}}{\partial t}\right|_{t=0} = L_{A,b}(\mu_{x,t}) \tag{3}$$

in the weak sense, with the operator $L_{A,b}$

$$L_{A,b} f = tr(AD^2 f) + \langle b, \nabla f \rangle , \quad f \in C_c^\infty(M) \tag{4}$$

We consider the von Mises-Fisher distribution on the unit sphere $\mathbb{S}^d$ given by:

$$\rho_d(x; s, \beta) = C_d(\beta) \exp\left(-\beta \langle s, x \rangle\right) \tag{5}$$

where $\beta \geq 0$, $\|s\| = 1$ and $C_d(\beta)$ is a normalization constant.

We show that the von Mises-Fisher distribution satisfies the Fokker-Planck equation:

$$\left.\frac{\partial \rho_{s,t}}{\partial t}\right|_{t=0} = \partial_s(\rho_{s,t}).$$

The distribution can be defined on a normal neighbourhood U_p of the manifold M and satisfies a Fokker-Planck equation.

Proposition

The following limit holds at a point $p \in M$

$$\left.\frac{\partial}{\partial t}\left(C_d(\beta_t)\int_{U_p} e^{\widehat{\Phi}_\beta(s_i,x)} f(x)\mu(x)\right)\right|_{t=0} = \partial_i(f)(p).$$

A first convergence result

We defined the family of projectors e_k such that:

$$e_k D_{X_k} e_k^* = \left(\begin{array}{c|c} 0 & * \quad * \quad \omega_{i_0 j}^k \quad * \quad * \\ \hline \begin{matrix} * \\ * \\ \omega_{i_0 j}^k \\ * \\ * \end{matrix} & 0 \end{array}\right)$$

and the coefficients ω_{ij} are defined $\omega_{ij}^k(\hbar) = C_d(\beta_\hbar) \exp\left(-\frac{\langle x_i^k, s_j \rangle}{\hbar}\right)$.

A first convergence result

Theorem (T.)

Let $\left\{x_{i_0}^k\right\}_{k=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on an open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. Let $\widehat{S}_n^{\hbar_n}$ be the associated operator given by:

$$\widehat{S}_n^{\hbar_n}(a) := \frac{1}{n}\sum_{k=1}^{n} e_k \left[D_X^k, a_k\right] e_k^*.$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then in probability:

$$\lim_{n\to\infty} \sup_{a\in F} \left|\Psi \circ \widehat{S}_n^{\hbar_n}(a)(p) - [\mathcal{D}, a](p)\right| = 0.$$

Theorem (T.)

Let $\{x_i\}_{i=1}^{n}$ be a sequence of i.i.d. sampled points from a uniform distribution on an open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. $\Omega_n^{\hbar_n}$ be the associated operator given by:

$$\Omega_n^{\hbar_n}(a)(p) = \frac{C_d(\beta_\hbar)}{n\hbar^2} \sum_{k=1}^{n} \sum_{j=1}^{d+1} \lambda_j^2 \exp\left(-\frac{\left\langle x_i^k, s_j \right\rangle}{\hbar}\right) \alpha_{ij}(a_k).$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then in probability:

$$\lim_{n\to\infty} \sup_{a\in F} \left| \Omega_n^{\hbar_n}(a)(p) - \Delta_M(a)(p) \right| = 0 \tag{6}$$

Conclusion

Given a compact spin manifold (M, g), we have the following:

- associate to each K_i a C^*-algebra A_i with limit $C(M)$,
- define a differential structure $da = [D_i, a]$ on each A_i,
- for the lattice, (D_i) converges to the usual Dirac operator ∂_M.
- Using the same tools than the continuous case $(C^\infty, L^2(M), \partial_M)$.

Future works:

- convergence results of the (D_i) to the classical Dirac operator,
- provide a unifying framework in the langage of spectral triples.

Thank you !

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