From Representation Theory to Geometric Discretizations

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A Brief History: Structure and Discretization

- Whitney (1957), *Geometric Integration Theory*

$$C : \Omega^p(M) \to C^p(K, \mathbb{Z}), \quad C(\omega) := \sigma \mapsto \langle \omega, \sigma \rangle$$
$$\mathcal{W} : C^0(K, \mathbb{Z}) \to \Omega^0(M), \quad \mathcal{W}(x_i) = \lambda_i$$

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- Harlow and Welch, MAC grid (1965), K. Yee, Yee's lattice (1966)

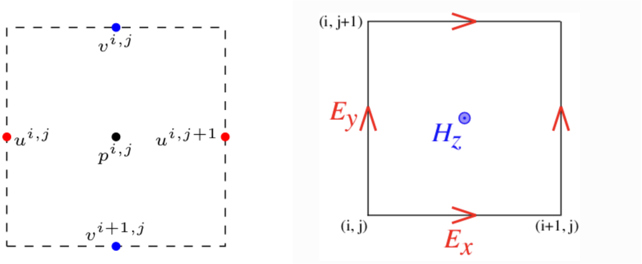

Figure: MAC grid (left) and Yee's lattice (right).

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Question: Identify a unifying picture ?

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- Let K a simplicial complex: $(C^{k}(K), d)$ $\rightarrow$ (DEC model)

Universal Differential Algebra

Let A be an associative k-algebra, then one can define the multiplication map

$$\mu : A \otimes A \to A, \quad \mu(a \otimes b) = ab$$

Then one can define the *universal graded differential algebra* as follow:

$$\Omega^1(A) = \ker(\mu), \quad da := 1 \otimes a - a \otimes 1,$$

$$\Omega^k(A) = \{a_1 da_2 da_3 \cdots da_k, a_i \in A\}, \quad \Omega^*(A) = \bigoplus_k \Omega^k(A).$$

If in addition, we require that A is equipped with an involution $*$

$$(da)^* = d(a^*).$$

C^*-Algebra and Representations

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Theorem (Gelfand-Naimark-Seagal)

Any C^*-algebra A can be identified as a closed *-subalgebra of the algebra of bounded operator $B(H)$ on a separable Hilbert space H.

Spectral Triple

One then needs to introduce a derivation map d on $B(H)$ such that:

$$(da)^* = d(a^*) \quad \text{(self-adjoint)},$$
$$d(ab) = d(a)b + ad(b) \quad \text{(Leibniz rule)}.$$

For that purpose, we define a self-adjoint operator $D : H \to H$ and extend the representation $\pi : A \to B(H)$ to a representation of $\Omega(A)$ in $B(H)$:

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$$\pi(da) = [D, \pi(a)]$$

The data (A, H, D) is called a *spectral triple*, and D a *Dirac operator*.
The definition of a spectral triple has been introduced by A. Connes *Noncommutative Geometry* (1994).

Objectives:

1) Describe a differential calculus using the following ingredients
   - An algebra of "functions" : A
   - A differential map d : Leibniz rule, self-adjoint, exterior algebra ...
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Manifolds and Triangulations

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Let (K_i) be a sequence of triangulations such that

$$K_1 > K_2 > \cdots > K_i > \cdots$$

$$\phi_{ij} : K_j \to K_i, \quad i < j.$$

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Lemma

The topological space M is homeomorphic to the subspace of all the maximal points of the inverse limit of the system (K_i, ϕ_{ij}).

C^*-Algebra and Space

We recall the interplay between C^*-algebra and topological spaces:

Theorem (Gelfand)

Let A be a commutative unital C^*-algebra, then there exists a compact Hausdorff topological space X such that:

$$A \simeq C(X).$$

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Claim

One can associate to every C^*-algebra A a topological space X, namely its spectrum $Spec(A)$.

We have the following identification:

$$Spec(A) \xrightarrow{\simeq} K$$

Take It to the Limit

We can draw the following commuting diagram:

$$\begin{array}{ccccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_i & \longrightarrow & \cdots & \longrightarrow & A_\infty \\ \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\ K_1 & \longleftarrow & K_2 & \longleftarrow & \cdots & \longleftarrow & K_i & \longleftarrow & \cdots & \longleftarrow & K_\infty \end{array}$$

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\end{array}
$$

Theorem A

The limit C^*-algebra A_∞ is isometrically *-isomorphic to C^*-algebra of the complex valued continuous sections $\Gamma(M, A_\infty)$ over the manifold M. The center $Z(A_\infty)$ is isomorphic to $C(M, \mathbb{C})$.

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We extend this construction to the representation H_i of the algebra A_i.
We have the following sequence:

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Theorem B

The Hilbert space $L^2(M)$ of square integrable functions over the manifold M is a subspace of H_∞:

$$H_\infty = L^2(M) \oplus H.$$

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We have introduced the following:

- A sequence of C^*-algebras A_i with limit $C(M)$
- A sequence of representations H_i with limit $L^2(M)$
- A so-called Dirac operator D and an exterior algebra $\Omega_D(A)$

If (M, g) is a compact spin manifold then data $(C^\infty(M), L^2(S), \partial_M)$ is enough to recover the geometric structure.

A First Example: the Lattice

We define the following algebra A and Dirac operator D:

$$A = M_{2m}(\mathbb{C}), \quad H = \mathbb{C}^{2m}, \quad D = \frac{i}{h}\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with $(D^+)^* = -D^-$ and where D^- is given by

$$D^- = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

A First Example: the Lattice

Proposition 1

For every element $da \in \Omega^1_D(A)$, we have

$$\sigma(da) = \left\{ \pm \frac{i}{h}(\lambda_{i+1} - \lambda_i) : 0 \leq i \leq d-1 \right\}$$

Moreover, we have the commutativity relation

$$[da, db] = 0, \quad \forall a, b \in A.$$

Case $n = 2$

Let $a = (a_1, a_2) \in M_2(\mathbb{C})$ and the Dirac operator:

$$D = \frac{i}{h}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad da = \frac{i}{h}\begin{pmatrix} 0 & a_2 - a_1 \\ a_1 - a_2 & 0 \end{pmatrix}.$$

If we define the following distance:

$$d(x, y) = \sup_{a \in A} \{|a(x) - a(y)| : \|[D, a]\| \leq 1\}$$

then one can show that for $X = \{x, y\}$

$$d(x, y) = h.$$

Without prior assumption, we see the emergence of a small parameter h in place of the usual distance Δx.

A First Example: the Lattice

Let ρ be a positive matrix with positive eigenvalues (μ_k) such that

$$Tr(\rho) = \sum_k \mu_k = 1$$

Then ρ is called a density matrix.

Proposition 2

Let ρ be a density matrix, then the expectation value is given by

$$< da >_\rho = Tr(\rho da) = \frac{i}{h} \sum_k \mu_k (a_{k+1} - a_k)$$

for any element $da \in \Omega^1_D(A)$.

General Case and Direct Limits

In the general case of a triangulation K_i, we define D_i as the block matrix

$$D_i = \frac{i}{h} \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}$$

where D_i^- is the adjacency matrix associated to K_i.

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Then the limit operator D_∞ acts on A_∞ by the commutator:

$$[D_\infty, a] = ([D_0, a_0], [D_1, a_1], \cdots, [D_i, a_i], \cdots) \in \prod_{i \in I} M^-_{2m_i}(\mathbb{C}).$$

Proposition 3 (lattice)

i) $\sigma_{A_\infty}([D_\infty, a]) = \overline{\cup_i \sigma_{A_i}([D_i, a_i])}$

ii) $\| [D_\infty, a] \| = \|d_c a\|_\infty$

Conclusion

We have the following results: given a compact spin manifold (M, g),

- associate to each K_i a C^*-algebra A_i with limit $C(M)$,
- define a differential structure $da = [D_i, a]$ on each A_i,
- for the lattice, (D_i) converges to the usual Dirac operator ∂_M.
- Using the same tools than the continuous case $(C^\infty, L^2(M), \partial_M)$!

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Future works:

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- derive non-trivial discretizations (MAC, Yee..),
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Stay tuned !

References

D. Tageddine, J-C. Nave
Noncommutative geometry on Infinitesimal Spaces
in preparation, https://www.math.mcgill.ca/dtagedine/.

Arnold, D. N. and Falk, Richard S. and Winther, Ragnar
Finite element exterior calculus, homological techniques, and applications
Acta Numerica (2006).

Desbrun, M. and Hirani, A. N. and Leok, M. and Marsden, J. E.
Discrete Exterior Calculus
http://arxiv.org/abs/math/0508341 (2005).

Whitney, H.
Geometric integration theory
Princeton University Pres.

References

A. Connes
Noncommutative geometry
Academic Press (1994).

Balachandran, A. P. and Bimonte, G. and Ercolessi, E. and Landi, G. and Lizzi, F. and Sparano, G. and Teotonio-Sobrinho, P.
Noncommutative Lattices as Finite Approximations and Their Noncommutative Geometries
Journal of Geometry and Physics (1996), pp. 163-194.

Blackadar, B.
Operator algebras: theory of C*-algebras and von Neumann algebras
Encyclopaedia of mathematical sciences, Operator algebras and non-commutative geometry (2006), Berlin ; New York.