Statistical Fluctuation of Infinitesimal Spaces

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Abstract

This paper is a follow-up on the *noncommutative differential geometry on infinitesimal spaces* [19]. In the present work, we extend the algebraic convergence from [19] to the geometric setting. On the one hand, we reformulate the definition of finite dimensional compatible Dirac operators using Clifford algebras. This definition also leads to a new construction of a Laplace operator. On the other hand, after a brief introduction of the Von Mises-Fisher distribution on manifolds, we show that when the Dirac operators are interpreted as stochastic matrices, the sequence $(D_n)_{n\in\mathbb{N}}$ converges in average to the usual Dirac operator on a spin manifold. The same conclusion can be drawn for the Laplace operator.

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1 Introduction

The approximation theory of partial differential equations (PDE) can take various aspects. Traditionally, numerical analysis proposes different strategies to discretize operators. Depending on the situation, finite differences, finite elements, finite volumes, or spectral methods may be utilized. In this process, the focus is usually analytical. That is, the aim is to control asymptotic convergence of the approximation error in a small parameter $(\Delta t, \Delta x, ...)$. In fact, only a small subset of discretization techniques aim to preserve certain underlying structures (e.g. geometric, algebraic, etc...) of the continuous operator at the discrete level. The present work is a follow-up to [19], in which we provided a general framework to tackle this question.

In our paper, we are interested in the so-called family of compatible discretization, also called geometric discretization. The focus of these approximations is that discrete theory can, and indeed should, possess a geometric description on its own right. Among various types of approaches, one may mention the finite element exterior calculus [1], the discrete exterior calculus [10], methods enforcing group symmetries such as in [4, 5], or conservation laws such as those in [20, 21]. Since a fair amount of the theory of PDEs is developed on (subdomain of) $\mathbb{R}^n$, the later approaches are, at least in their original incarnations, focused on Euclidean spaces. However, examples of partial differential equations arise in a wide variety of applications. As such, extensions of some of the previously mentioned techniques to non-Euclidean domains remains a challenge. Hence, a crucial question in the theory of discretization, is the generalization of classical geometric approaches to smooth manifolds. Also, and still on the topic of convergence analysis of finite elements, [22] studies the cotangent discretization of the Laplace-Beltrami operator; the key result is that mean cruvature vectors converge in the sense of distributions, but fail to converge in L^2. Finally, there are the central research advances on diffusion maps in [15, 7] and the one on random point clouds in [2, 3]. As one can see, approaching the problem of compatible discretizations on manifolds rests heavily on the initial setup chosen to tackle it. The various results are therefore quite different, and perhaps appear disconnected from one another.

In our recent work [19], we derive a general framework to describe finite difference calculus. This framework relies on the tools of C^*-algebras and noncommutative differential geometry (see [8, 9]). More specifically, starting uniquely from a differentiable manifold M, we exhibit a discrete space X and its associate algebra $A(X)$ playing the role of an algebra of function. Then, using the natural setting of C^*-algebra, and its representation theory, we define a differential calculus on the space X. The corner stone of this definition is a so-called Dirac operator. Finally, this construction allows us to study spectral convergence with respect to a positive parameter h in the case where X is a lattice. Doing so may provide a general discrete construction of differential operators on smooth manifolds.
In the present work, we aim at studying the relation between the Dirac operator, D defined in [19] and its continuous counterpart $\mathcal{D}$, thus extending our previous construction beyond the case of a lattice.

To this end, we recall the definition of the derivation d given as a commutator with D:

$$da = [D, a]\,, \tag{1}$$

where a is an element of a C^*-algebra. This operator is analogous to the continuous Dirac operator $\mathcal{D}$. Indeed, in the case of a Riemannian manifold (M, g) with a spinor bundle $\mathscr{S} \to M$, a *Dirac operator* $\mathcal{D}$ on $\mathscr{S}$ is a differential operator whose principal symbol is that of $\mathsf{c} \circ d_{dR}$, where c is the quantization map. In particular, for any $a \in C^\infty(M)$, one has:

$$[\mathcal{D}, a] = i\mathsf{c} \circ d_{dR}(a). \tag{2}$$

The operator $\mathcal{D}$ is defined using the Clifford algebras (see [18, Def. 5.5.12 p.406] for a complete definition). Indeed, in local coordinates on a normal neighbourhood centred at a point p:

$$\mathcal{D}_p = \sum_{j=1}^{d} e_j \left.\frac{\partial}{\partial x_j}\right|_p \tag{3}$$

where $\{e_j \mid j = 1, \ldots, d\}$ is an orthonormal local frame embedded in a Clifford algebra. In [19] , we gave the following definition for the operator D :

$$D = \sum_{i<j} \omega_{ij} e_{ij}, \quad \omega_{ij} \in \mathbb{C} \tag{4}$$

where the e_{ij} are merely matrix elements in $M_{2n}(\mathbb{C})$ associated to a graph. In order to obtain an analogous description of (37) in terms of Clifford elements, in the present work, we redefine finite dimensional Dirac operator as an element in the algebra $M_2(\mathbb{C}) \otimes U(\mathfrak{g})$, where $U(\mathfrak{g})$ is the universal Lie algebra associated to a Lie algebra $\mathfrak{g}$. In addition, this definition leads to a construction of a Laplace operator using the inclusion map $U(\mathfrak{g}) \to Cl(\mathfrak{g})$, where $Cl(\mathfrak{g})$ is the Clifford algebra on $\mathfrak{g}$.

Now, to a fixed triangulation X, one can associate a collection of Dirac operators $(D_t)_{t\in\mathbb{N}}$, where each matrix D_t can be seen as an irreducible matrix associate to the graph G obtained from X. It has n vertices labelled $1, \ldots, n$, and there is an edge from vertex i to a vertex j precisely when $\omega_{ij} \neq 0$. More precisely, in the probabilistic setting, a vertex i is connected to a vertex j with probability ω_{ij}.
Then, if we let $a_t \in \mathrm{Dom}(D_t)$ and define the average operator,

$$S_N = \frac{1}{N} \sum_{t=1}^{N} e_t \left[D_t, a_t\right] e_t^*, \tag{5}$$

where $(e_t)_{t\in\mathbb{N}}$ is some family of projections. The key here is to choose the coefficients ω_{ij}^t associated to D_t in order for the average operator S_N to converge to $[\mathcal{D}, a]$ as $N \to \infty$. The main result of the present work is given by the following theorem.

Theorem 1.1 (Main result)**.** *Let $\left\{x_{i_0}^k\right\}_{k=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on a open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. Let $\widetilde{S}_n^{\hbar_n}$ be the associated operator given by:*

$$\widehat{S}_n^{\hbar_n} : C^\infty(U_p) \to M_2(\mathbb{R}) \otimes U(\mathfrak{gl}_{2m_n}), \qquad \widehat{S}_n^{\hbar_n}(a) := \frac{1}{n} \sum_{k=1}^{n} e_k \left[D_X^k, a_k\right] e_k^*. \tag{6}$$

Put $\hbar_n = n^{-\alpha}$, *where* $\alpha > 0$, *then for* $a \in C^\infty(U_p)$, *in probability:*

$$\lim_{n\to\infty} \Psi \circ \widehat{S}_n^{\hbar_n}(a) = [\mathcal{D}, a]\,(p).$$

Additionally, a similar result is proved for the case of the Laplace operator. These results generalize the previous ones obtained in [19] beyond the case of a lattice. It is worth mentioning at this point, in the realm of noncommutative geometry, the work of [14], where in the same spirit (though in different context and approaches) the Dirac operators are defined as random matrices and form a so-called *Dirac ensembles*. More specifically, the coefficients of the $N \times N$ matrix D are random variables, following a prescribed density function; the spectral properties in the large N limit are then explored. In the present work however, the Dirac operator is associated to a graph and the density functions of the coefficients are specifically chosen to obtain a convergence result with respect to $\mathcal{D}$.

This paper is arranged as follows. In Section 2, we start with a presentation of the main results of [19]. We then introduce Clifford algebras and universal enveloping algebras in order to define and study Dirac operators on finite dimensional spaces. In Section 3, we then introduce the Fokker-Planck equation, the Von Mises-Fisher distribution, and their generalization to manifolds. This is followed by some technical lemmas required to define the coefficients ω_{ij} necessary to prove Theorem 1.1. In Section 4, we prove our main result, Theorem 1.1, and we obtain as a by-product a convergence result for the Laplacian operator.

2 Dirac operators in the algebraic setting

In this section, we introduce two of the main algebraic tools that we are going to use in this study: the Clifford algebras and the universal enveloping Lie algebra. We then define and study Dirac operators on finite spectral triples in terms of root vectors of a Lie algebra $\mathfrak{g}$.

2.1 Noncommutative Geometry on Infinitesimal spaces

In the research paper [19], we show that a discrete topological space X can be identified to the spectrum $Spec(A)$ of a C^*-algebra A. Starting with a Riemannian manifold (M, g), we construct an inverse system of triangulations, (K_n) which become sufficiently fine for large n. Using the Behncke-Leptin construction, we associate to each K_n a C^*-algebra A_n such that the triangulation K_n is identified with the spectrum $Spec(A_n)$. We then form an inductive system (A_n) with limit A_∞.

Theorem 1. *The spectrum* $Spec(A_\infty)$ *equipped with the hull-kernel topology is homeomorphic to the space* X_∞ *and*

$$\lim_{\leftarrow} Spec(A_i) \simeq Spec(\lim_{\rightarrow} A_i). \tag{7}$$

We then show that the centre of A_∞ is isomorphic to the space of continuous function $C(M)$. In this sense, any element $g \in C(M)$ can be uniformly approximated arbitrarily closely by elements a_n in the central subalgebras $\mathfrak{A}_n$.

Theorem 2. *The space of continuous function $C(M)$ is approximated by the system of commutative subalgebras $(\mathfrak{A}_n, \phi^*_{n,\infty})$ in the following sense:*

$$C(M) = \overline{\bigcup_{n\in\mathbb{N}} \phi^*_{n,\infty}(\mathfrak{A}_n)} \cap C(M). \tag{8}$$

In addition, the sequence of representations (H_n) is also considered as a direct system with limit H_∞ containing the space of square integrable functions $L^2(M)$. Finally, we define the spectral triples $(\mathfrak{A}, \mathfrak{h}, D_n)$, where D_n is a so-called Dirac operator. We show that under certain conditions, the sequence (D_n) converges to the multiplication operator by the de Rham differential $d_c a$.

Theorem 3. *(Spectral convergence) There exists a finite measure μ and a unitary operator*

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}, d\mu) \tag{9}$$

such that,

$$U[D,a]U^{-1}\phi = \frac{da}{dx}\phi, \quad \forall \phi \in L^2(\mathbb{R}), \tag{10}$$

Moreover, the norm of the commutator is given by $\| \, [D,a] \, \| = \|d_c a\|_\infty$.

Thus, we have built a correspondence between a given triangulation X and a Dirac operator D: the non-zero coefficients of D are determined by the connectivity between vertices of the graph. We showed that this is however not enough to represent the metric of the manifold. Thus, we ask now the question on how to set the coefficients ω_{ij} of D so that at the limit (in the sense of (44)) the sequence converges.

2.2 Clifford algebras

Let V be a finite dimensional vector space over a commutative field $\mathbb{K}$ of characteristic zero endowed with a quadratic form q. Let $T(V)$ be the tensor algebra over V. Consider the ideal I_q in $T(V)$ generated by all elements of the form $v \otimes v + q(v)$ for $v \in V$. Then the quotient algebra

$$Cl(V,q) = T(V)/I_q. \tag{11}$$

is the Clifford algebra associated to the quadratic space (V,q).
Moreover, we can choose any orthonormal basis Z_i of V with respect to q as a set of generators of $Cl(V)$. We then have the relations,

$$Z_i Z_j = -Z_j Z_i, \quad i \neq j, \quad Z_i^2 = -1. \tag{12}$$

Then the following set

$$Z_{i_1} Z_{i_2} \cdots Z_{i_k} \quad 1 \leq < i_1 < i_2 < \cdots < i_k \leq n = \dim V \tag{13}$$

spans $Cl(V)$. In addition, given a q-orthonormal basis Z_i of V, the mapping

$$1 \mapsto 1, \quad Z_{i_1} \cdots Z_{i_k} \mapsto Z_{i_1} \wedge \cdots \wedge Z_{i_k} \tag{14}$$

yields an isomorphism of vector spaces $Cl(V,q) \simeq \bigwedge V$.

2.3 Dirac operators in the Clifford algebra setting

Let $k = \mathbb{R}, \mathbb{C}$ and let $\mathfrak{g}$ be a Lie algebra over k. We start by recalling the definition of the universal enveloping algebra.

Definition 2.1. The *universal enveloping algebra* of $\mathfrak{g}$ is a map $\varphi : \mathfrak{g} \to U(\mathfrak{g})$, where $U(\mathfrak{g})$ us a unital associative algebra, satisfying the following properties:

1) φ is a Lie algebra homomorphism, i.e. φ is k-linear and

$$\varphi\left([X, Y]\right) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X). \tag{15}$$

2) If A is any associative algebra with a unit and $\alpha : \mathfrak{g} \to A$ is any Lie algebra homomorphism, there is a unique homomorphism of associative algebras $\beta : U(\mathfrak{g}) \to A$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & U(\mathfrak{g}) \\ {\scriptstyle \alpha}\downarrow & \swarrow{\scriptstyle \beta} & \\ A & & \end{array}$$

is commutative, i.e. there is an isomorphism

$$\mathrm{Hom}_{Lie}(\mathfrak{g}, LA) \simeq \mathrm{Hom}_{Ass}(U(\mathfrak{g}), A) \tag{16}$$

We will now give a definition of Dirac operators on finite spectral triples in terms of root vectors of a Lie algebra $\mathfrak{g}$. Then, using the canonical embedding $\mathfrak{g} \hookrightarrow Cl(\mathfrak{g})$ into the Clifford algebra, we define a Laplace-type operator.

Consider the algebra $A = \mathfrak{gl}_{2N}(\mathbb{C})$ of complex matrices with its standard Lie algebra structure. In [19], we have introduced the finite dimensional spectral triple $(\mathfrak{A}, \mathfrak{h}, D)$ given by:

- $\mathfrak{A}$ is a Cartan subalgebra of Lie subalgebra $\mathfrak{g}$ of A,
- $\mathfrak{h} = \mathbb{C}^{2N}$,
- $\gamma = \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix}$.

The chirality element γ induces a decomposition of the representation space $\mathfrak{h}$ into the eigenspaces $\mathfrak{h}^{\pm}$ corresponding to the eigenvalues 1 and -1 such that $\mathfrak{h} = \mathfrak{h}^{+} \oplus \mathfrak{h}^{-}$. Incidentally, one has the decomposition of the algebra A as follows:

$$\mathfrak{gl}_{2N} = \mathfrak{gl}_{2N}^{+} \oplus \mathfrak{gl}_{2N}^{-}. \tag{17}$$

Notice then that the pair $(\mathfrak{gl}_{2N}^{+}, \mathfrak{gl}_{2N}^{-})$ forms a Cartan pair. Any endomorphism $a \in \mathrm{End}(\mathfrak{h})$ defines an endomorphism $\rho_a \in \mathfrak{gl}_{2N}^{+}$ given by

$$\rho_a = \begin{pmatrix} a & 0 \\ 0 & -a^T \end{pmatrix} \in \mathfrak{sp}(2N, \mathbb{C}) \cap \mathfrak{gl}_{2N}^{+}. \tag{18}$$

We consider the compact real case with the embedding

$$\mathfrak{sp}(N) = \mathfrak{sp}(2N, \mathbb{C}) \cap \mathfrak{u}(2N) \hookrightarrow \mathfrak{so}(4N). \tag{19}$$

If a is a diagonal element of $\mathrm{End}(\mathfrak{h}^+)$, the map $a \mapsto \rho_a$ identifies a with an element of the maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{so}(4N)$:

$$\mathfrak{t} = \left\{ \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{pmatrix}, A_j = \begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix} \right\} \tag{20}$$

Consider the Cartan subalgebra $\mathfrak{A} = \mathfrak{t} + i\mathfrak{t}$ of $\mathfrak{so}(4N, \mathbb{C})$. The root vectors are $4N \times 4N$ block matrices having 2×2-matrix C_s, $s \in \{1, \ldots, 4\}$

$$X = \begin{pmatrix} 0 & C_s \\ -C_s^t & 0 \end{pmatrix} \tag{21}$$

in the position (i, j) with $i < j$ and where

$$C_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}.$$

associated to the linear functional in $\mathfrak{H}^*$ given by $i(a_i + a_j)$, $-i(a_i + a_j)$, $i(a_i - a_j)$ and $i(a_j - a_i)$.

We will denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{so}_{4N}$. We then consider the unital associative algebra $M_2(\mathbb{C}) \otimes \mathfrak{gl}_{2N}$ and the homomorphism:

$$\varphi : \mathfrak{g} \to M_2(\mathbb{C}) \otimes \mathfrak{gl}_{2N}, \quad \varphi(X) = \sum_{1 \le i,j \le 2N} X_{ij} \otimes E_{ij}, \tag{22}$$

where E_{ij} is the standard basis in $\mathfrak{gl}_{2N}$ and X_{ij} are the 2×2-submatrix of $X = (x_{rs})$ obtained by keeping $i + 1 \le r \le i + 2$ and $j + 1 \le s \le j + 2$. In addition, φ is a Lie algebra homomorphism with $\varphi([X, Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$.
Then, using the universal property of $U(\mathfrak{g})$, the map φ extends into the homomorphism $\widehat{\varphi} : U(\mathfrak{g}) \to M_2(\mathbb{C}) \otimes \mathfrak{gl}_{2N}$. Furthermore, taking the canonical embedding $h : \mathfrak{gl}_{2N} \to U(\mathfrak{gl}_{2N})$, we get by composing the Lie algebra homomorphism

$$h \circ \widehat{\varphi} : U(\mathfrak{g}) \to M_2(\mathbb{C}) \otimes U(\mathfrak{gl}_{2N}). \tag{23}$$

Let $\{Z_{ij}\}$ be an orthonormal basis of root vectors in $\mathfrak{g}$, associated to the root $-i(a_j + a_k)$, we define the operator W by

$$W = \sum_{i,j} \omega_{ij}^W Z_{ij} \tag{24}$$

as an element of $U(\mathfrak{g})$, where ω_{ij}^W are real coefficients.

Definition 2.2. Given an operator W as in (24), a Dirac operator D_W is an element of $M_2(\mathbb{C}) \otimes U(\mathfrak{gl}_{2N})$ defined by:

$$D_W = \frac{i}{\hbar} \mathfrak{Re}(W), \tag{25}$$

where $\hbar > 0$ is a real parameter.

Remark 2.1. In the previous definition, D_W depends on the choice of element W and in fact, more specifically on the choices of basis elements Z_{ij}. Another definition, independent on the choice of basis elements, of Dirac operators on Lie algebras can be found in [16]

Lemma 2.1. *Let $C_2 = X + iY$ be the root vector associated to the root $-i(a_i + a_j)$. Fix an element W as in (24). Then, for any $a \in \mathfrak{A}$, the exterior derivative can be written as:*

$$[D_W, a] = \frac{i}{\hbar} \sum_{i,j} \omega_{ij}^W \alpha_{ij}(a) Y \otimes E_{ij}, \tag{26}$$

an element of $M_2(\mathbb{C}) \otimes U(\mathfrak{gl}_{2N})$ and with $\alpha_{ij} = a_i - a_j$.

Proof. From the definition of D_W and the definition of root vectors, we get that:

$$[D_W, a] = \frac{i}{2\hbar} \sum_{i,j} \omega_{ij}^W (a_i - a_j) Z_{ij} - \frac{i}{2\hbar} \sum_{i,j} \omega_{ij}^W (a_i - a_j) Z_{ij}^*. \tag{27}$$

Then, using the map $h \circ \widehat{\varphi}$, given by (22) and (23), we can identify a basis element Z_{ij} with an element in $M_2(\mathbb{C}) \otimes U(\mathfrak{g})$ of the form $C_2 \otimes E_{ij}$. Hence, we have that:

$$[D_W, a] = \frac{i}{2\hbar} \sum_{i,j} \omega_{ij}^W (a_i - a_j) C_2 \otimes E_{ij} - \frac{i}{2\hbar} \sum_{i,j} \omega_{ij}^W (a_i - a_j) C_2^* \otimes E_{ij}^t. \tag{28}$$

Simplifying this expression using the fact that $E_{ij}^t = E_{ji}$, we get:

$$[D_W, a] = \frac{i}{\hbar} \sum_{ij} \omega_{ij}^W \alpha_{ij}(a) Y \otimes E_{ij}. \tag{29}$$

with $\alpha_{ij}(a) = a_i - a_j$. □

Furthermore, we recall that there exists a canonical Lie algebra homomorphism $\psi : \mathfrak{gl}_{2N} \to Cl(\mathfrak{gl}_{2N})$ which extends into the map on the universal enveloping Lie algebra:

$$\widehat{\psi} : U(\mathfrak{gl}_{2N}) \to Cl(\mathfrak{gl}_{2N}). \tag{30}$$

We use this map to define a Laplace operator.

Definition 2.3 (Laplacian)**.** Fix an element W. We then define the Laplace operator Δ on $\mathfrak{A}$ using the non-graded commutator. For any $a \in \mathfrak{A}$

$$\Delta(a) := \frac{1}{2} \widehat{\psi}([D_W, [D_W, a]]). \tag{31}$$

Proposition 2.1. *For any $a \in \mathfrak{A}$, the Laplace operator is given by*

$$\Delta(a) = -\Omega_{\mathfrak{g}}(a) \otimes 1 \tag{32}$$

where $\Omega_{\mathfrak{g}} = \frac{1}{\hbar^2} \sum_{i,j} \omega_{ij}^2 J \otimes \alpha_{ij}$ is an element of $End(\mathfrak{A}, M_2(\mathbb{C}))$.

Proof. Let D_W be a Dirac operator, then the bi-commutator of the Laplacian gives::

$$[D_W, [D_W, a]] = D_W [D_W, a] - [D_W, a] D_W. \tag{33}$$

Thus, using Lemma 2.1, we obtain

$$[D_W, [D_W, a]] = \frac{2}{\hbar^2} \sum_{ij} \omega_{ij}^2 \alpha_{ij}(a) J \otimes E_{ij}^2 + \frac{1}{\hbar^2} \sum_{(ij) \neq (kl)} \omega_{ij} \omega_{kl} \alpha_{kl}(a) J \otimes [E_{ij}, E_{kl}]_+ \tag{34}$$

with the bracket $[A, B]_+ = AB + BA$ and where the matrix J is given by:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Finally, applying the map $\widehat{\psi}$, the second term of the left-hand-side in Equation (34) vanishes and we get:

$$\Delta(a) = -\frac{1}{\hbar^2} \sum_{i,j} \omega_{ij}^2 \alpha_{ij}(a) J \otimes 1.$$

□

We have kept the definition of the Dirac operator D_W in (25) very general, however we recall that the operator we are interested in are the *compatible* one with respect to a graph X. In other words, the value ω_{ij} is non-zero if ij is an edge in X.

Now, let us recall how the space X is obtain from a manifold M; more details can be found in [19]. One starts with a triangulation of M and then consider the *dual of the triangulation* that we will call X. In fact, in [19] we used a slightly different terminology and considered the triangulation as a poset, then looked at the opposite poset with reversed order.

Since we are working with a graph X obtained from a dual triangulation, every vertex i has exactly $d+1$ neighbours i.e. only $d+1$ of the ω_{ij} are non-zero for a fixed i. Hence, if we fix an vertex i_0, the definition of the commutator with D_W becomes:

$$([D_W, a])_{i_0} = \frac{i}{\hbar} \sum_{j=1}^{d+1} \omega_{i_0 k_j}^W \alpha_{i_0 k_j}(a) Y \otimes E_{i_0 k_j}, \tag{35}$$

Here, we relabel the index j without lost of generalities and to keep this indexing simple. We will also drop the W index for the same reasons and get:

$$([D, a])_{i_0} = \frac{i}{\hbar} \sum_{j=1}^{d+1} \omega_{i_0 j} \alpha_{i_0 j}(a) Y \otimes E_{i_0 j}, \tag{36}$$

Finally, let us recall the the (true) Dirac operator on a manifold is given in local coordinates on a normal neighbourhood centred at a point p:

$$\mathcal{D}_p = \sum_{j=1}^{d} e_j \left. \frac{\partial}{\partial x_j} \right|_p \tag{37}$$

where $\{e_j \mid j = 1, \ldots, d\}$ is an orthonormal local frame embedded in the Clifford algebra $Cl(\mathbb{R}^d)$ using the natural embedding $\mathbb{R}^d \subset Cl(\mathbb{R}^d)$.

Nevertheless, the Dirac operator as expressed in (36) is not an element of a Clifford algebra. Moreover, the dimensions do not match. Indeed, because of the structure of the triangulation, there are $d+1$ independent vectors in the expression (36), instead of d as the dimension of the manifold M. Since we are trying to approximate the true Dirac operator in (37), we need to re-write Equation (36) in terms of Clifford elements in dimension d. To do so, let us denote by V_{i_0}, the vector space defined by:

$$V_{i_0} := \operatorname{span}\left\{Y \otimes E_{i_0,1}, \cdots, Y \otimes E_{i_0,d+1}\right\}. \tag{38}$$

Then, consider the isomorphism:

$$\tau : V_{i_0} \xrightarrow{\simeq} \mathbb{R}^{d+1} \quad \tau(Y \otimes E_{i_0,j}) = \widehat{e}_j, \ \forall 1 \leq j \leq d+1 \tag{39}$$

where $\{\widehat{e}_j\}_{j=1}^{d+1}$ is the canonical basis on $\mathbb{R}^{d+1}$ with respect to the standard inner product. Moreover, defines the projection p on the subspace spanned by $\{\widehat{e}_j\}_{j=1}^{d}$ and identified with $\mathbb{R}^d$. Finally, if let the embedding $\rho : \mathbb{R}^d \to Cl(\mathbb{R}^d)$, we can compose these maps and define:

$$\Psi := \rho \circ p \circ \tau : V_{i_0} \to Cl(\mathbb{R}^d), \quad \Psi\left([D,a]_{i_0}\right) = \frac{i}{\hbar}\sum_{j=1}^{d} \omega_{i_0 j}\alpha_{i_0 j}(a) e_j \tag{40}$$

which allows us to express the commutator in terms of Clifford elements e_j. We notice, nevertheless, that this construction is not canonical and depends on the choice of isomorphism τ.

2.4 Perron-Frobenius bound on $[D, a]$

To conclude this section, and before being able to show a convergence result to the Dirac operator $\mathcal{D}$, we would like to prove a preliminary result on the commutator $[D, a]$ and its boundedness at the limit when $\hbar \to 0$. This result follows form the correspondence between D and the graph associated, using the Perron-Frobenius theorem. We only need to consider the operator D as a compatible operator in some matrix space, without relying the Clifford algebra setting

We consider an infinite collection $\{\mathfrak{A}_n : n \in \mathbb{N}\}$ of commutative C^*-algebras. In this case, we have identified each of the $\mathfrak{A}_n$ with the Cartan subalgebras h_i inside the finite dimensional algebras $B_n = \mathfrak{so}_{2m_n}(\mathbb{C})$ where $m_n \to \infty$ when $n \to \infty$. We can then construct the product:

$$B_\omega = \prod_{n\in\mathbb{N}} B_n = \{(a_n) : \|a_n\| = \sup \|a_n\| < \infty\}. \tag{41}$$

Let a be an element in $C^\infty(M)$, then there exists a coherent sequence (a_i) such that

$$a = (a_0, a_1, \cdots, a_n, \cdots) \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n. \tag{42}$$

We define a spectral triple on B_ω by introducing the limit Dirac operator D as the sequence

$$D = (D_0, D_1, \cdots, D_n, \cdots) \in \prod_{n\in\mathbb{N}} \mathfrak{gl}^-_{2m_n}(\mathbb{C}), \tag{43}$$

where each D_i is a Dirac operator associated to a poset X_i^{op} in the sense of [19]. This in turns induces a spectral triple on $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$ along with the commutator:

$$d_D a := [D, a] = ([D_0, a_0], [D_1, a_1], \cdots, [D_n, a_n], \cdots) \in \prod_{n\in\mathbb{N}} \mathfrak{gl}^-_{2m_n}(\mathbb{C}). \tag{44}$$

In order to show that $[D, a]$ is a bounded operator, we use Perron-Frobenius theorem, which we start by recalling.

Theorem 2.1 (Perron-Frobenius [17])**.** *Let $A = (a_{ij})$ be an $n \times n$ positive matrix: $a_{ij} > 0$ for $1 \leq i, j \leq n$. Then there exists a positive real number r, called the Perron-Frobenius eigenvalue, such that r is an eigenvalue of A. Moreover, if the spectral radius $\rho(A)$ is equal to r.*
The Perron-Frobenius eigenvalue satisfies the inequalities:

$$\min_i \sum_j a_{ij} \leq r \leq \max_i \sum_j a_{ij}.$$

Proposition 2.2. *For any $a \in \mathfrak{A}$, the spectral radius $\rho(d_D a)$ of $d_D a$ is bounded by*

$$\rho(d_D a) \leq \|d_{dR} a\|_\infty. \tag{45}$$

Proof. We consider the sequence of Dirac operators $(D_\alpha)_{\alpha\in\mathbb{N}}$ associated to D.

Let $\varepsilon > 0$ and $\alpha \in \mathbb{N}$ and define the operator $\widetilde{d_{D_\alpha} a}$ such that

$$(\widetilde{d_D a})_{ij} = \begin{cases} |(d_{D_\alpha} a)_{ij}| & \text{if } (d_{D_\alpha} a)_{ij} \neq 0 \\ \varepsilon & \text{otherwise} \end{cases} \tag{46}$$

The matrix $\widetilde{d_{D_\alpha} a}$ is positive by construction. In addition, we have the upper-bound:

$$\| (d_{D_\alpha} a)^k \|_F^2 \leq \|(\widetilde{d_{D_\alpha} a})^k\|_F^2. \tag{47}$$

Hence, using Theorem 2.1, we deduce that

$$\begin{aligned} \rho(d_{D_\alpha} a)^2 = \lim_{k\infty} \|(d_{D_\alpha} a)^k\|_F^{\frac{2}{k}} &\leq \lim_{k\infty} \|(\widetilde{d_{D_\alpha} a})^k\|_F^{\frac{2}{k}} \\ &= \rho(\widetilde{d_{D_\alpha} a})^2 \\ &\lesssim \max_{1\leq i\leq n} \sum_j |(d_{D_\alpha} a)_{ij}|^2 + N\varepsilon^2. \end{aligned}$$

The value of N is the number of nonzero coefficient in $(d_{D_\alpha} a)_{ij}$ and thus only depends on the number of adjacency vertex in X_α^{op} which by definition equal to $d + 1$, where d is the dimension.

Hence there exists a positive constant C_M, which depends on the maximal length of geodesics (M is compact) but is independent of α, such that

$$\rho(d_{D_\alpha} a)^2 \leq C_M \|d_{dR} a\|_\infty^2 + (d+1)\varepsilon^2. \tag{48}$$

The last inequality holds for an arbitrary $\varepsilon > 0$ and $\alpha \in \mathbb{N}$. The result follows then by taking ε to 0. □

Corollary 2.1. *For each $a \in \mathfrak{A}$, the operator $[D, a]$ is a bounded operator.*

Remark 2.2. It is clear that in the following framework, not only the Dirac operator D define a differential structure but it also plays the role of a transition matrix. This last point will be made clearer in the following section.

3 Von Mises-Fisher integral operators

In this section, we are going to introduce the Fokker-Planck equation (see [6]) in terms of Fourier transform of a measure. Then we will define the Von Mises-Fisher distribution on manifolds and prove technical lemmas.

3.1 The Fourier transform of finite measures

In this section, our main objects are integral operators and semigroup operators. Let M be a locally compact separable subset of $\mathbb{R}^d$ with the induced Lebesgue measure. Consider the Shwartz class

$$\mathcal{S}(\mathbb{R}^d) = \left\{\varphi \in C^\infty(\mathbb{R}^d) : \|\varphi\|_{j,k} < \infty, j,k = 0,1,2,\dots\right\}, \quad \|\varphi\|_{j,k} = \sup|x^j \varphi^{(k)}(x)|.$$

The Fourier transform is a continuous linear isomorphism given by

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), \quad \mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^d} \varphi(x) e^{-i\langle x,\xi\rangle} dx. \tag{49}$$

This map can be restricted to an isomorphism on $\mathcal{S}(M)$.
We then consider the topological dual $\mathcal{S}'$ of $\mathcal{S}$ called the *space of tempered distributions*. The Fourier transform extends by duality to a continuous isomorphism on $\mathcal{S}'$ as follow, for $T \in \mathcal{S}'$:

$$\langle \widehat{T}, \varphi\rangle = \langle T, \widehat{\varphi}\rangle, \qquad \forall \varphi \in \mathcal{S}. \tag{50}$$

If $T \in \mathcal{S}'$ is represented by a function $u \in L^p(\mathbb{R}^d)$, then we denoted by T_u such that

$$T_u(\varphi) = \int_{\mathbb{R}^d} u(x)\varphi(x) dx. \tag{51}$$

Similarly, if we let $M_+(\mathbb{R}^d)$ be the space of finite positive measures on $\mathbb{R}^d$, then for a measure $\mu \in M_+(\mathbb{R}^d)$, using Riesz-Kakutani representation theorem, we can uniquely define $T_\mu \in \mathcal{S}'$

$$T_\mu(\varphi) = \int_{\mathbb{R}^d} \varphi(x) d\mu(x). \tag{52}$$

Then, we can define the Fourier transform of μ by $\widehat{T}_\mu$:

$$\langle \widehat{T}_\mu, \varphi \rangle = \langle T_\mu, \widehat{\varphi} \rangle, \qquad \forall \varphi \in \mathcal{S}. \tag{53}$$

We can also define the Fourier transform of μ as the complex valued function

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle x,\xi \rangle} d\mu(x). \tag{54}$$

Definitions (53) and (54) are equivalent in the following sense

$$\langle T_\mu, \widehat{\varphi} \rangle = \langle \widehat{\mu}, \varphi \rangle, \qquad \forall \varphi \in \mathcal{S}. \tag{55}$$

Again, all these definitions can be restricted to a Fourier transform on M.

Proposition 3.1. *Let μ be a measure in $M_+(\mathbb{R}^d)$ with finite k-th moment, where $k = k_1 + \cdots k_d$. Then the Fourier transform $\mu(\xi)$ has bounded, continuous partial derivatives of order less or equal than k and these are given by the formula:*

$$\partial^k \widehat{\mu}(\xi) = \int_{\mathbb{R}^d} i^k x_1^{k_1} \cdots x_d^{k_d} e^{-i\langle x,\xi \rangle} d\mu(x). \tag{56}$$

Hence, if the function μ is analytic, then we have the expansion on $\mathbb{R}^d$:

$$\widehat{\mu}(\xi) = \sum_{k=k_1+\cdots+k_d} \frac{\xi_1 \cdots \xi_d}{k_1! \cdots k_d!} \partial^k \widehat{\mu}(0). \tag{57}$$

3.2 Family of one-parameter operators

Let $L^2(M)$ considered as a Banach space for the Lebesgue measure ν. Let then U be an open subset of $M \times \mathbb{R}_+$. We denote by $\mathscr{D}(U)$ the set of *test functions* on U and its topological dual $\mathscr{D}'(U)$ the space of *distribution*. A family of one-parameter operators $\{P_t\}_{t \geq 0}$ is a family of linear operators on $L^2(M)$ defined by:

$$P_0 = id, \qquad (P_t f)(x) = \int_M f(y) p_t(x,y) d\nu(y) \tag{58}$$

such that $p_t(x,y)$ is a $\nu \times \nu$-measurable function on $M \times M$. Now let us define the new measure $\mu_{t,x}$, for $t \geq 0$ and $x \in M$, by

$$\mu_{t,x}(A) = \int_A p_t(x,y) d\nu(y), \tag{59}$$

for any ν-measurable subset A. Assume that $\mu_{t,x}$ is a probability measure for every $(t,x) \in \mathbb{R}_+ \times M$. In addition, assume that P_t admits a weak derivative ∂_t. Let $\varphi \in \mathscr{D}(U)$ and define the translation map:

$$R_t : \mathscr{D}(U) \to \mathscr{D}(U), \quad R_t(\varphi)(x,s) = \varphi(x, s+t) \tag{60}$$

extended by duality to a map on $\mathscr{D}'(U)$. If we let $\varphi_0(x) := \varphi(x,t)$, we therefore have on the one hand:

$$\langle P_t, \varphi_0 \rangle = \langle \delta, \varphi_t \rangle. \tag{61}$$

On the other hand, using the Fourier transform (55) of P_t, we have

$$\langle \widehat{P}_t, \varphi \rangle = \langle \widehat{\mu}_{x,t}, \varphi_0 \rangle. \tag{62}$$

Hence, combining Equation (61) and Equation (57) and the fact that $\widehat{\mu}_t(0) = 1$, we obtain:

$$\langle 1, \varphi_t \rangle = \langle 1, \varphi_0 \rangle + \sum_{k=k_1+\cdots+k_d \geq 1} \frac{\partial^k \widehat{\mu}_{x,t}(0)}{k_1! \cdots k_d!} \langle \xi_1 \cdots \xi_d, \varphi_0 \rangle. \tag{63}$$

Finally, dividing by t and taking the limit $t \to 0$, assuming that the following limit:

$$\lim_{t \to 0+} \frac{\partial^k \widehat{\mu}_{x,t}(0)}{t} \quad \text{exists for all } k \tag{64}$$

and the convergence is uniform in t, we have the following formal series

$$\langle 1, \partial_t \varphi_t|_{t=0} \rangle = \sum_{k=k_1+\cdots+k_d \geq 1} \lim_{t \to 0} \frac{1}{t} \frac{\partial^k \widehat{\mu}_{x,t}(0)}{k_1! \cdots k_d!} \langle \xi_1 \cdots \xi_d, \varphi_0 \rangle. \tag{65}$$

In the special case where

$$\lim_{t \to 0+} \frac{\partial^k \widehat{\mu}_{x,t}(0)}{t} = 0, \quad \forall k \geq 3, \tag{66}$$

then $\mu_{x,t}$ satisfy the parabolic equation

$$\left.\frac{\partial \mu_{x,t}}{\partial t}\right|_{t=0} = \mathscr{L}_{A,b}(\mu_{x,t}) \tag{67}$$

in the weak sense, with the operator $\mathscr{L}_{A,b}$

$$\mathscr{L}_{A,b} f = tr(AD^2 f) + \langle b, \nabla f \rangle, \quad f \in C_c^\infty(M) \tag{68}$$

and where $A = (a^{ij})$ is a mapping on M with values in the space of nonnegative symmetric linear operator on $\mathbb{R}^d$ and $b = (b^i)$ is a vector field on M.

3.3 Von Mises-Fisher integral operators

In this section, we are going to identify the coefficients ω_{ij} in (26) using the so-called Von Mises-Fisher distribution in order to obtain the main result in Theorem 1.1.

The probability density function of the von Mises-Fisher distribution on the unit sphere $\mathbb{S}^d$ is given by:

$$\rho_d(x; s, \beta) = C_d(\beta) \exp\left(-\beta \langle s, x \rangle\right) \tag{69}$$

where $\beta \geq 0$, $\|s\| = 1$; the normalization constant $C_d(\beta)$ is given by

$$C_d(\beta) := \frac{\beta^{\frac{d}{2}-1}}{(2\pi)^{\frac{d}{2}} I_{\frac{d}{2}-1}(\beta)} = \int_{\mathbb{S}^d} \rho_d(x; s, \beta) d\mu(x), \tag{70}$$

where I_v denotes the modified *Bessel function* of the first kind at order v. We will refer to the von Mises-Fisher distribution as $VMF(s, \beta)$.
Consider $\mathbb{S}^d$ as a submanifold of $\mathbb{R}^{d+1}$ with the induced Lebesgue measure ν and the euclidean metric. We consider the one-parameter family of integral operators

$$P_t : L^2(\mathbb{S}^d) \to L^2(\mathbb{S}^d), \qquad (P_t f)(x) = C_d(\beta_t) \int_{\mathbb{S}^d} f(x)\rho_d(x; s, \beta_t) d\nu(x) \tag{71}$$

with $\beta = 1/t$.

Proposition 3.2. *The family* $\{P_t\}_{t>0}$ *is a well-defined and satisfies the followings:*

i) $\|P_t\|_{L^2} \leq 1$*,*

ii) $P_0 = id$*.*

Proof. Statement (i) follows immediately from the fact that P_t is induce by a probability measure and therefore, for $f \in \mathscr{D}(\mathbb{S}^d)$ we have for the uniform norm

$$\|P_t(f)\| \leq \|f\|. \tag{72}$$

We now prove (ii). Let $f \in \mathscr{D}(\mathbb{S}^d)$ and consider the difference:

$$|P_t f(x) - f(0)| \leq C_d(\beta_t) \int_{\mathbb{S}^d} |f(x) - f(0)| \rho_d(x; s, \beta_t) d\nu(x). \tag{73}$$

Let $\varepsilon > 0$ and $\delta > 0$ to be chosen. We define the set $S_\delta = \mathbb{S}^d \backslash \mathbb{B}(s, \delta)$, then taking the uniform norm in the last inequality and splitting the right-hand-side, we get

$$\|P_t f - f(0)\| \leq C_d(\beta_t) \int_{S_\delta} |f(x) - f(0)| \rho_d(x; s, \beta_t) d\nu(x) + \|f - f(0)\|_{S_\delta^c}. \tag{74}$$

Now, using the uniform continuity of f, we can chose δ such that

$$\|P_t f - f(0)\| \leq C_d(\beta_t) \int_{S_\delta} |f(x) - f(0)| \rho_d(x; s, \beta_t) d\nu(x) + \frac{\varepsilon}{2}. \tag{75}$$

We recall that the constant $C_d(\beta_t)$ is given by Equation (70) and that for large values of x, $I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}$. Therefore, we have the limit:

$$\lim_{t \to 0+} C_d(\beta_t) \int_{S_\delta} |f(x) - f(0)| \rho_d(x; s, \beta_t) d\nu(x) = 0. \tag{76}$$

Hence, for t small enough, we have that

$$\|P_t f - f(0)\| \leq \varepsilon. \tag{77}$$

□

Lemma 3.1. *Consider the following real valued functions*

$$\begin{aligned}
A(t) &= C_d(1/t)\int_0^1 \frac{r}{tC_d(r/t)}\,\mathrm{d}r,\\
B(t) &= \int_0^1 \frac{rC_d(1/t)}{C_d(r/t)}\,\mathrm{d}r,\\
C(t) &= 2\pi\left[\frac{tC_d(1/t)}{C_{d-2}(1/t)} - \int_0^1 \frac{tC_d(1/\tau)}{C_{d-2}(r/t)}\,\mathrm{d}r\right]s - \frac{(2\pi)^{\frac{d}{2}}C_d(1/t)}{3\Gamma(d/2-1)}s,
\end{aligned}$$

then $B(t)$ and $C(t)$ vanish as $t \to 0^+$ while $A(t)$ converges to 1.

Proof. Recall that for large values of x, $I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}$, hence $C_d(1/t)$ vanishes as $t \to 0$. Therefore, we have:

$$A(t) = \left(C_d(1/t)\int_0^1 \frac{r}{tC_d(r/t)}\,\mathrm{d}r\right) \sim \frac{I_{\frac{d}{2}-2}(1/t)}{I_{\frac{d}{2}-1}(1/t)} \to 1 \tag{78}$$

It follows immediately that $B(t)$ vanishes as $t \to 0$.
Furthermore, we have the following inequalities:

$$0 \le (2\pi)^{\frac{d}{2}}t\int_0^1 r\frac{d}{dr}\left[(r/t)^{2-\frac{d}{2}}I_{d/2-2}(r/t)\right]\mathrm{d}r \le \frac{2\pi t}{C_{d-2}(1/t)} - \frac{(2\pi)^{\frac{d}{2}}t}{\Gamma(d/2-1)}$$

Hence, we have the limit when $t \to 0$,

$$\frac{tC_d(1/t)}{C_{d-2}(1/t)} - \frac{(2\pi)^{\frac{d}{2}}t}{\Gamma(d/2-1)} \to 0. \tag{79}$$

Thus, the coefficient $C(t)$ vanishes as $t \to 0$. □

Theorem 3.1 (Fokker-Planck equation on $\mathbb{S}^d$)**.** *Let $u(x)$ be a smooth function in $C^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R})$. We define the one-parameter family of operators:*

$$P_t(u) = C_d(\beta_t)\int_{\mathbb{S}^d} \exp(-\beta_t\langle s, x\rangle)u(x)\,\mathrm{d}\nu(x) \tag{80}$$

where $\beta_t = 1/t$ and with the initial condition:

$$u(x) = \lim_{t\to 0^+} P_t(u)(x). \tag{81}$$

Then the function $u(x,t) = P_t(u)(x)$ satisfies the infinitesimal equation:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{\partial u}{\partial s}. \tag{82}$$

Proof. According to Equations (66) and (67), to prove the statement, it is enough to show the following:

$$\lim_{t\to 0+}\frac{\nabla\widehat{\mu}_{x,t}(0)}{t} = s, \qquad \lim_{t\to 0+}\frac{\partial^k\widehat{\mu}_{x,t}(0)}{t} = 0, \quad \forall k \ge 2. \tag{83}$$

In fact, we will only show that

$$\lim_{t\to 0+} \frac{\nabla\widehat{\mu}_{x,t}(0)}{t} = s, \qquad \lim_{t\to 0+} \frac{D^2\widehat{\mu}_{x,t}(0)}{t} = 0; \tag{84}$$

the proof for the higher-moments follows mutatis mutandis the case $k = 2$.

We start by computing:

$$\nabla\widehat{\mu}_{x,t}(0) = C_d(\beta_t) \int_{\mathbb{S}^d} \xi e^{-\beta_t\langle \xi, s\rangle}\, \mathrm{d}\xi. \tag{85}$$

Using the divergence theorem on compact manifold, we get

$$\int_{\mathbb{B}_d(0)} \nabla\left(e^{-\beta_t\langle x,s\rangle}\right) \mathrm{d}x = \frac{s}{t}\int_0^1 \frac{r}{C_d(r/t)}\,\mathrm{d}r.$$

Hence, the 1-st moment is given by the following integral,

$$\nabla\widehat{\mu}_{x,t}(0) = \left(C_d(1/t)\int_0^1 \frac{r}{tC_d(r/t)}\,\mathrm{d}r\right) s, \tag{86}$$

where:

$$\int_0^1 \frac{r}{tC_d(r/t)}\,\mathrm{d}r = (2\pi)^{\frac{d}{2}} t \left[\frac{I_{d/2-2}(y)}{y^{\frac{d}{2}-2}}\right]_{\to 0^+}^{\frac{1}{\tau}}.$$

On the other hand, the 2-nd moment is given by:

$$D^2\widehat{\mu}_{x,t}(0) = \int_{\mathbb{S}^d} \exp(-\beta_t\langle s,x\rangle)\xi\otimes\xi\,\mathrm{d}\xi - \nabla\widehat{\mu}_{x,t}(0)\otimes\nabla\widehat{\mu}_{x,t}(0).$$

The first term of the left-hand side can be written as

$$\int_{\mathbb{S}^d} e^{-\beta_t\langle s,\xi\rangle}\xi\otimes\xi\,\mathrm{d}\xi = C_d(\beta_t)\left[\int_{\mathbb{B}_d(0)} e^{-\beta_t\langle s,x\rangle}\frac{s}{t}\otimes x\,\mathrm{d}x + \int_{\mathbb{B}_d(0)} e^{-\beta_t\langle s,x\rangle}\,\mathrm{d}x\right]$$

We can write the first integral as follows

$$\int_{\mathbb{B}_d(0)} e^{-\beta_t\langle s,x\rangle}\frac{s}{t}\otimes x\,\mathrm{d}x = 2\pi\left[\frac{t}{C_{d-2}(1/t)} - \int_0^1 \frac{t\,\mathrm{d}r}{C_{d-2}(r/t)}\right] s - \frac{(2\pi)^{\frac{d}{2}}}{3\Gamma(d/2-1)} s.$$

The second integral can be directly computed as

$$\int_{\mathbb{B}_d(0)} e^{-\beta_t\langle s,x\rangle}\,\mathrm{d}x = \int_0^1\int_{\mathbb{S}^d} e^{-\beta_t r\langle s,x\rangle} r\,\mathrm{d}\xi\,\mathrm{d}r = \int_0^1 \frac{r}{C_d(r/t)}\,\mathrm{d}r$$

The variance can be written in the closed form

$$D^2\widehat{\mu}_{x,t}(0) = B(t)\mathbb{1} + C(t)ss^T. \tag{87}$$

where the coefficients are given by

$$B(t) = \int_0^1 \frac{rC_d(1/t)}{C_d(r/t)}\,\mathrm{d}r,$$

$$C(t) = 2\pi\left[\frac{tC_d(1/t)}{C_{d-2}(1/t)} - \int_0^1 \frac{tC_d(1/\tau)}{C_{d-2}(r/t)}\,\mathrm{d}r\right] s - \frac{(2\pi)^{\frac{d}{2}} C_d(1/t)}{3\Gamma(d/2-1)} s.$$

We then conclude the proof using Lemma (3.1). □

3.4 Wrapped distributions on manifolds

In this section, we extend the previous results to the case where M is a smooth manifold of dimension d. Since we are interested in the Dirac operator over *spin manifolds*, these results are crucial for the rest of the present work.

Given a Lebesgue measure on T_xM and a probability density ν on T_xM whose support is included in U, the density is then push forward at $x \in \phi(U)$ is given by

$$d\mu = d(\phi_* \nu) = \det(d\phi^{-1}) d\nu \tag{88}$$

Among choices of ϕ and interesting candidate is the exponential map, due to its algebraic and geometric properties.

Proposition 3.3. *Let (M, g) be a Riemannian manifold. Fore every point $p \in M$, there is an open subset $W \subseteq M$, with $p \in W$ and a number $\epsilon > 0$, so that:*

$$\exp_q : B(0, \epsilon) \subseteq T_qM \to U_q = \exp(B(0, \epsilon)) \subseteq M \tag{89}$$

is a diffeomorphism for every $q \in W$, with $W \subseteq U_q$.

Definition 3.1 (Normal neighbourhood)**.** Let (M, g) be a Riemannian manifold. For any $q \in M$, an open neighbourhood of q of the form $U_q = \exp_q(B(0, \epsilon))$ where $\exp_q$ is a diffeomorphism from the open ball $B(0, \epsilon)$ onto U_q, is called a *normal neighbourhood.*

Definition 3.2 (Injectivity radius)**.** Let (M, g) be a Riemannian manifold. For every point $p \in M$, the *injectivity radius of M at p*, denoted $\delta(p)$, is the least upper bound of the numbers $r > 0$, such that $\exp_p$ is a diffeomorphism on the open ball $B(0, r) \subseteq T_pM$. The *injectivity radius, $\delta(M)$ of M* is defined as:

$$\delta(M) := \inf_{p \in M} \delta(p). \tag{90}$$

In what will follow, we will simply denote by δ the *injectivity radius* of M. Let $p \in M$, we then consider the exponential map $\exp_p : B(0, \delta) \to U_p = \exp_p(B(0, \delta))$. We can now pushforward the Von Mises-Fisher distribution from the tangent space T_pM at p to the manifold M. Let us define the map Φ_β by:

$$\Phi_\beta : T_pM \times T_pM \to \mathbb{R}, \qquad \Phi_\beta(x, y) \mapsto \beta g(x, y) \tag{91}$$

and its pullback using the inverse of the exponential map at p:

$$\widehat{\Phi}_\beta : U_p \times U_p \to \mathbb{R}, \qquad \widehat{\Phi}_\beta(q, q') := (\exp_p^{-1})^*(\Phi_\beta)(q, q'). \tag{92}$$

We will also denote by $\{e_1, \ldots, e_d\}$ a local orthogonal frame in TM.

Lemma 3.2. *Consider the tangent vector $e_i(p) \in T_pM$ for $i \in \{1, \ldots, d\}$ and let s_i such that $e_i(p) = \exp_p^{-1}(s_i)$. The following pushforward holds:*

$$\int_{U_p} e^{\widehat{\Phi}_\beta(s_i, x)} f(x) \, \mathrm{d}\mu(x) = \int_{B(0,\delta)} e^{\Phi_\beta(e_i(p), y)} \exp_{p*}(f)(y) \det\left(d\exp_p(y)\right) \mathrm{d}\nu(y) \tag{93}$$

where $\exp_p$ is the exponential map on the Riemannian manifold (M, g).

Proof. This follows immediately from the definition of the definition of pullback and the exponential map at p. □

We can then define the map:

$$\Theta : M \to M_+(\mathbb{R}^d) \times \mathbb{R}_+ \qquad p \mapsto \left\{ T_p^t(f) = C_d(\beta_t) \int_{U_p} e^{\widehat{\Phi}_\beta(s_i,x)} f(x)\,\mathrm{d}\mu(x) \right\}_{t>0} \tag{94}$$

where $\beta_t = 1/t$. This associates to any normal neighbourhood a one-parameter family of operators. We are going to show that each family of operators satisfies Equation (3.1).

Definition 3.3 (Jacobi field [11])**.** Let $p \in M$ and $\gamma : [0,a] \to M$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v$. Let $w \in T_v(T_pM)$ with $|w| = 1$. A Jacobi field J along γ given by

$$J(t) = (d \exp_p)_{tv}(tw). \tag{95}$$

Lemma 3.3. *Let J be a Jacobi field. We have the following Taylor expansion about $t = 0$:*

$$\langle w, J(t) \rangle = t + r(t), \tag{96}$$

where $\lim_{t\to 0} \frac{r(t)}{t^2} = 0$.

Proof. From the definition of J and the properties of the exponential map, we have that $J(0) = (d_0 \exp_p)(0) = 0$ and $J'(0) = w$. Hence, the first two coefficients of the Taylor expansion are

$$\langle w, J(0) \rangle = 0,$$
$$\langle w, J'(0) \rangle = 1.$$

As J is a Jacobi field we have $J''(0) = -R(\gamma', J)\gamma'(0) = 0$, where R is the curvature tensor. This yields,

$$\langle w, J''(0) \rangle = 0, \tag{97}$$

which concludes the proof. □

Lemma 3.4. *Define the smooth map:*

$$G : T_pM \to \mathbb{R}, \qquad y \mapsto \det\left(d_y \exp_p\right), \tag{98}$$

then, it satisfies $\nabla(G)(0) = 0$.

Proof. In order to compute $\nabla(G)(0)$, we first use Jacobi's identity

$$\frac{d}{dt} \det\left(d_{ty} \exp_p\right)\bigg|_{t=0} = \det(d_0 \exp_p) \mathrm{tr}\left(d_0 \exp_p^{-1} \frac{d}{dt}\bigg|_{t=0} d_{ty} \exp_p\right) \tag{99}$$

which simplifies into

$$\frac{d}{dt} \det\left(d_{ty} \exp_p\right)\bigg|_{t=0} = \mathrm{tr}\left(\frac{d}{dt}\bigg|_{t=0} d_{ty} \exp_p\right). \tag{100}$$

Using the definition of a Jacobi field and linearity of tr, we have that

$$\begin{aligned}
\operatorname{tr}\left(\left.\frac{d}{dt}\right|_{t=0} d_{ty}\exp_p \right) &= \sum_{i=1}^{d} \left\langle v_i, \left(\left.\frac{d}{dt}\right|_{t=0} d_{ty}\exp_p \right) v_i \right\rangle, \\
&= \left.\frac{d}{dt}\right|_{t=0} \sum_{i=1}^{d} \left\langle v_i, d_{ty}\exp_p(v_i) \right\rangle, \\
&= \left.\frac{d}{dt}\right|_{t=0} \sum_{i=1}^{d} \frac{1}{t} \left\langle v_i, J_i(t) \right\rangle.
\end{aligned}$$

Now using the Taylor expansion obtained in Lemma (3.3), we get:

$$\langle v_i, J_i(t) \rangle = t + r(t) \tag{101}$$

where $r(t) = o(t^2)$, we conclude that:

$$\operatorname{tr}\left(\left.\frac{d}{dt}\right|_{t=0} d_{ty}\exp_p \right) = 0. \tag{102}$$

□

Theorem 3.2. *The following limit holds at $p \in M$*

$$\left.\frac{\partial}{\partial t}\left(C_d(\beta_t) \int_{U_p} e^{\widehat{\Phi}_\beta(s_i,x)} f(x)\, \mathrm{d}\mu(x) \right)\right|_{t=0} = e_i(f)(p). \tag{103}$$

Proof. Using the pushforward map given in Lemma 3.2, the property of Von Mises-Fisher distribution given in Theorem 3.1 and the isomorphism $T_0(T_p(M)) \simeq T_p(M)$, we have:

$$\left.\frac{\partial}{\partial t}\left(\int_{U_p} e^{\widehat{\Phi}_\beta(s_i,x)} f(x)\, \mathrm{d}\mu(x) \right)\right|_{t=0} = e_i\left(\exp_{p*}(f) \det\left(d\exp_p \right) \right)(0). \tag{104}$$

Then using Lemma 3.4, we deduce that:

$$\left.\frac{\partial}{\partial t}\left(C_d(\beta_t) \int_{U_p} e^{\widehat{\Phi}_\beta(s_i,x)} f(x)\, \mathrm{d}\mu(x) \right)\right|_{t=0} = e_i(f)(p). \tag{105}$$

□

3.5 A remark on the semigroup approach

It is interesting to consider semigroup machinery as another approach to the problem of approximation of the Dirac operator. In this section, we consider $L^2(\nu)$ as the Hilbert space $\mathcal{H}$.

Definition 3.4 (Semigroup). A *one-parameter unitary group* is a map $t \to P_t$ from $\mathbb{R}_+$ to $\mathcal{L}(\mathbb{H})$ such that

$$P_0 = 1 \qquad P_{t+s} = P_t P_s, \tag{106}$$

and $t \to P_t$ is continuous in the strong topology, i.e. $U_t \xrightarrow{s} U_{t_0}$ when $t \to t_0$.

Given a semigroup P_t in L^2, define the *generator* $\mathscr{L}$ of the semigroup by

$$\mathscr{L}(f) := \lim_{t \to 0} \frac{f - P_t f}{t}, \tag{107}$$

where the limit is understood in the L^2-norm. The *domain* $\mathrm{dom}(\mathscr{L})$ of the generator $\mathscr{L}$ is the space of functions $f \in \mathcal{H}$ for which the above limit exists. By the Hille-Yosida theorem, $\mathrm{dom}(\mathscr{L})$ is dense in L^2. Moreover, P_t can be recovered from $\mathscr{L}$ as follows:

$$P_t = \exp(-t\mathscr{L}). \tag{108}$$

understood in the sense of spectral theory.
We then consider the operator $L = -i\frac{d}{dx}$ on $\mathcal{H}$ with $\mathrm{dom}(\mathscr{L}) = \{f \in L^2(\mathbb{R}) : \xi \widehat{f} \in L^2(\mathbb{R})\}$. Recall that L is unitary equivalent to the left-multiplication operator M_ξ using the Fourier transform

$$\mathcal{F} L \mathcal{F}^{-1} = \xi \widehat{f}. \tag{109}$$

Then the associated semigroup U_t, so-called *momentum operator*, is given by the left-multiplication operator in Fourier basis: $\mathcal{F} U_t \mathcal{F}^{-1} \widehat{f} = \xi \widehat{f}$. Therefore,

$$U_t f(x) = \mathcal{F}^{-1}(e^{it\xi} \widehat{f})(x) = \int_{\mathbb{R}} e^{i(x+t)\xi} \widehat{f} = f(x+t). \tag{110}$$

4 Statistical fluctuations of differential structures

We are now ready to state and prove Theorem 1.1. We keep the same notations than the previous sections: M is a compact Riemannian manifold of dimension d; we consider a point $p \in M$ and a normal neighbourhood U_p associated to it; we denote by $\{e_1, \ldots, e_d\}$ a local orthogonal frame in TM. If we let $\exp_p : B(0, \delta) \to U_p$ be the exponential map at p. We finally define $\{s_1, \ldots, s_d\}$ such that:

$$e_j(p) = \exp_p^{-1}(s_j), \qquad \forall j \in \{1, \ldots, d\}\,. \tag{111}$$

4.1 Von Mises-Fisher distribution and Dirac operator

We start by recalling that the notation D_X means: a Dirac operator D associated to a graph X in the sense of [19, Def. 4.3]. Now, let n be a positive integers and fix a graph X_n equipped with a Dirac operator D_{X_n} and with set of vertices $\{x_1, \ldots x_n\}$. In addition, we are going to consider n copies of the same graph X_n, each of which is equipped with a Dirac operator D_{X_k} and with a set of vertices denoted by $\{x_1^k, \ldots x_n^k\}$, for $1 \leq k \leq n$. Then, we have a sequence of Dirac operators

$$(D_{X_1}, D_{X_2}, \cdots, D_{X_n}) \in \mathfrak{gl}^{-}_{2m_n}(\mathbb{C})^n, \tag{112}$$

acting on a sequence of diagonal elements $(a_0, a_1, \cdots, a_n)$ with each $a_i \in \mathfrak{A}_n$.
If we denote by $(a_k^i)_{1\leq i\leq n}$ the coefficients of a_k in the block diagonal, then using the projection maps $M \to X_k$ we can identify these values with evaluations of a smooth functions, denoted by a (see [19, Prop. 3.5] for more details):

$$a_k^i = a(x_i^k), \quad \forall i \in \{1, \dots, n\} \tag{113}$$

for some point $x_i^k \in M$. Fix a point $p \in M$ and a neighbourhood U_p of p in M. Then, consider a sequence of points $\{x_1^k, \dots, x_n^k\}$ in U_p, for $1 \leq k \leq n$, such that, for a chosen index i_0 (not depending on k), we have $x_{i_0}^k = p$. We then define the coefficients $(\omega_{ij}^k)_{1\leq i,j\leq n}$ of D_{X_n} as follows:

$$\omega_{ij}^k(\hbar) = C_d(\beta_\hbar) \exp\left(-\frac{\langle x_i^k, s_j\rangle}{\hbar}\right) \quad \text{for } 1 \leq i,j \leq n \text{ and for } 1 \leq k \leq n. \tag{114}$$

Furthermore, for every integer $1 \leq k \leq n$, we define a family of projection elements such that $e_k \in M_{2m_n}(\mathbb{C})$ and we have the following matrix form:

$$e_k D_{X_k} e_k^* = \left(\begin{array}{c|c} 0 & \begin{matrix} & & 0 & & \\ * & * & \omega_{i_0 j}^k & * & * \\ & & 0 & & \end{matrix} \\ \hline \begin{matrix} & * & \\ & * & \\ 0 & \omega_{i_0 j}^k & 0 \\ & * & \\ & * & \end{matrix} & 0 \end{array}\right). \tag{115}$$

Remark 4.1. The non-zero coefficients correspond to the adjacency points of i_0.

Hence, if we recall the expression given by the commutator in Equation (26), we consider the following average of operators over the n copies of X_n:

$$\widehat{S}_n^{\hbar_n}(a) := \frac{1}{n}\sum_{k=1}^n e_k\,[D_{X_k}, a_k]\,e_k^* = \frac{i}{n\hbar}\sum_{k=1}^n\sum_{j=1}^{d+1}\omega_{i_0 j}^k(\hbar_n)\alpha_{i_0 j}(a_k) Y \otimes E_{i_0 j}, \tag{116}$$

where $\alpha_{i_0 j}(a_k) = a(x_j^k) - a(p)$. Moreover, for the purpose of the proof of the main theorem, we define a second operator given by:

$$S_{j,n}^{\hbar_n} : C^\infty(M) \to \mathbb{R}, \qquad S_{j,n}^{\hbar_n}(a) = \frac{1}{n\hbar}\sum_{k=1}^n \omega_{i_0 j}^k(\hbar_n)\alpha_{i_0 j}(a_k). \tag{117}$$

We assume now that the points $\{x_1^k, \dots, x_n^k\}$ are thought as random variables independent and identically distributed (i.i.d.) from a uniform distribution. Let us recall the definition of the map Ψ given in Equation (40):

$$\Psi : V_{i_0} \to Cl(\mathbb{R}^d), \quad \Psi\left([D,a]_{i_0}\right) = \frac{i}{\hbar}\sum_{j=1}^d \omega_{i_0 j}\alpha_{i_0 j}(a)e_j. \tag{118}$$

Then, we can prove the following theorem.

Theorem 4.1. *Let $\{x_{i_0}^k\}_{k=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on a open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. Let $\tilde{S}_n^{\hbar_n}$ be the associated operator given by:*

$$\widehat{S}_n^{\hbar_n} : C^\infty(U_p) \to M_2(\mathbb{R}) \otimes U(\mathfrak{gl}_{2m_n}), \qquad \widehat{S}_n^{\hbar_n}(a) := \frac{1}{n}\sum_{k=1}^n e_k \left[D_X^k, a_k\right] e_k^*. \tag{119}$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then for $a \in C^\infty(U_p)$, in probability:

$$\lim_{n\to\infty} \Psi \circ \widehat{S}_n^{\hbar_n}(a) = [\mathcal{D}, a]\,(p).$$

Proof. We consider the average operator defined by Equation (117). It is then sufficient to prove that for $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, and for $a \in C^\infty(U_p)$, we have:

$$\lim_{n\to\infty} S_{j,n}^{\hbar_n}(a) = e_j(a)(p) \quad \forall 1 \leq j \leq d.$$

in probability and then apply the map $\widehat{\psi}$ defined in (30). Recall that e_j is given in Equation (111).

The expectation value of the random variable $S_n^{\hbar_n}(a)$ is given by:

$$\mathbb{E}S_{j,n}^{\hbar_n}(a)(p) = \frac{C(\beta_{\hbar_n})}{\hbar_n}\int_{U_p} e^{\widehat{\Phi}_{\beta_\hbar}(x,s_j)}(a(x) - a(p))\,\mathrm{d}\mu(x), \tag{120}$$

where we recognize the Von Mises-Fisher distribution. Thus, applying Hoeffding's inequality, we have:

$$\mathbb{P}\Big[\left|S_{j,n}^{\hbar_n}(a)(p) - \mathbb{E}S_{j,n}^{\hbar_n}(a)(p)\right| > \varepsilon\Big] \leq 2\exp\left(-\frac{\varepsilon^2 n}{C_d(n^\alpha)^2}\right). \tag{121}$$

Choosing $\hbar$ as a function of n, such that $\hbar(n) = n^{-\alpha}$, where $\alpha > 0$, we have, for any real number $\varepsilon > 0$:

$$\lim_{n\to\infty}\mathbb{P}\Big[\left|S_{j,n}^{\hbar_n}(a)(p) - \mathbb{E}S_{j,n}^{\hbar_n}(a)(p)\right| > \varepsilon\Big] = 0. \tag{122}$$

Finally, we prove the statement using Theorem 3.2:

$$\lim_{n\to\infty} S_{j,n}^{\hbar_n}(a)(p) = e_j(a)(p), \tag{123}$$

along with the definition of the map Ψ in Equation (40). □

Remark 4.2. Every line in the matrix D_X corresponds then to a point p and a normal neighbourhood U_p obtained from the image of the exponential map of a ball of radius δ. Indeed, since M is compact, we have a finite cover $\{U_{p_i}\}_{i=1}^N$ with center $\{p_i\}_{i=1}^N$ every one of which being associated to a line of D_X.
If we consider a sequence of Dirac operators D_{X_n}, then what we are doing is in fact taking refinements of normal neighbourhoods, increasing with the numbers of vertices in X_n.

4.2 Uniform convergence

It is interesting to mention that the previous result can be extended to have a uniform convergence, following the same steps than [2, Prop. 1]. We then state the result without proof.
Let $\mathscr{F}$ be the space of smooth functions $f \in C^\infty(U_p)$, such that $e_j(f)$ is smooth. Then for each $\hbar > 0$, we have:

$$\lim_{n\to\infty} \mathbb{P}\left[\sup_{a\in\mathscr{F}} \left| S_n^{\hbar_n}(a)(p) - \mathbb{E}S_n^{\hbar_n}(a)(p)\right| > \varepsilon\right] = 0. \tag{124}$$

Theorem 4.2. *Let $\{x_{i_0}^k\}_{k=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on a open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. Let $\widehat{S}_n^{\hbar_n}$ be the associated operator given by:*

$$\widehat{S}_n^{\hbar_n} : C^\infty(U_p) \to M_2(\mathbb{R}) \otimes U(\mathfrak{gl}_{2m_n}), \qquad \widehat{S}_n^{\hbar_n}(a) := \frac{1}{n}\sum_{k=1}^n e_k \left[D_X^k, a_k\right] e_k^*. \tag{125}$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then in probability:

$$\lim_{n\to\infty} \sup_{a\in\mathscr{F}} \left| \Psi \circ \widehat{S}_n^{\hbar_n}(a)(p) - [\mathcal{D}, a]\,(p)\right| = 0. \tag{126}$$

4.3 The Laplacian

In this final section, we want to study the convergence result for the Laplacian defined in Proposition 4.3. The Laplacian Δ_{X_k} obtained from the Dirac operator (115) and acting on an element a_k is given by:

$$\Delta_{X_k}(a_k) = \frac{1}{\hbar^2}\sum_{j=1}^{d+1} (\omega_{ij}^k)^2 \alpha_{ij}(a_k) J. \tag{127}$$

Now assume that the coefficients ω_{ij}^k of the Dirac D_{X_k} are given by:

$$\omega_{ij}^k(\hbar) = C_d(\beta_\hbar)\sqrt{\lambda_j}\exp\left(-\frac{\langle x_i^k, s_j\rangle}{2\hbar}\right), \tag{128}$$

where λ_j are positive numbers to be specified. Then we study the convergence of the averaging operator:

$$\Omega_n^\hbar(a)(p) = \frac{C_d(\beta_\hbar)}{n\hbar^2}\sum_{k=1}^n \sum_{j=1}^{d+1} \lambda_j \exp\left(-\frac{\langle x_i^k, s_j\rangle}{\hbar}\right)\alpha_{ij}(a_k). \tag{129}$$

such that, if $u = \sum_{i=1}^d s_i$, then $s_{d+1} = -u/\|u\|$ and $\lambda_j = 1/(d + \|u\|)$ for $1 \le j \le d$ and $\lambda_{d+1} = \|u\|/(d + \|u\|)$. Notice then that $\sum_{j=1}^{d+1}\lambda_j = 1$.
Moreover, the expectation value of the random variable $\Omega_n^{\hbar_n}(a)$ is given by:

$$\mathbb{E}\Omega_n^\hbar(a)(p) = C_d(\beta_\hbar)\int_{U_p} \rho_d(x; \beta_\hbar)\,(a(x) - a(p))\,\mathrm{d}\mu(x) \tag{130}$$

where the probability density $\rho_d(x;\beta_t)$ is given by:

$$\rho_d(x;\beta_\hbar) = \sum_{j=1}^{d+1} \lambda_j^2 \exp\left(-\frac{\langle x, s_j\rangle}{\hbar}\right). \tag{131}$$

We recognize a convex combination of Von Mises-Fisher distribution. Consequently, following the same steps as in Theorem 3.1, this distribution satisfies the equation:

$$\frac{\partial}{\partial t}\left(C_d(\beta_t)\int_{U_p} \rho_d(x;\beta_t) f(x)\,\mathrm{d}\mu(x)\right)\Bigg|_{t=0} = \varDelta_M(f)(p). \tag{132}$$

Theorem 4.3. *Let $\{x_i\}_{i=1}^n$ be a sequence of i.i.d. sampled points from a uniform distribution on an open normal neighbourhood U_p of a point p in a compact Riemannian manifold M of dimension d. $\varOmega_n^{\hbar_n}$ be the associated operator given by:*

$$\varOmega_n^{\hbar_n} : C^\infty(U_p) \to \mathbb{R}, \qquad \varOmega_n^{\hbar_n}(a)(p) = \frac{C_d(\beta_\hbar)}{n\hbar^2}\sum_{k=1}^{n}\sum_{j=1}^{d+1} \lambda_j^2 \exp\left(-\frac{\langle x_i^k, s_j\rangle}{\hbar}\right)\alpha_{ij}(a_k).$$

Put $\hbar_n = n^{-\alpha}$, where $\alpha > 0$, then in probability:

$$\lim_{n\to\infty} \sup_{a\in\mathscr{F}} \left|\varOmega_n^{\hbar_n}(a)(p) - \varDelta_M(a)(p)\right| = 0 \tag{133}$$

4.4 Discussion

Going back to the definition of the Dirac operator associated to a graph X with non-zero coefficients ω_{ij}, we recall that the goal was to compute the values ω_{ij} in order to obtain a convergence when considering a sequence of refined triangulations. We have exhibited the coefficients

$$\omega_{ij}(\hbar) = C_d(\beta_\hbar)\exp\left(-\frac{\langle x_i, s_j\rangle}{2\hbar}\right) \tag{134}$$

obtained from the Von Mises-Fisher distribution. Hence, we are able to prove a convergence result to the Dirac operator on a normal neighbourhood (Theorem 4.2) as well as a convergence of the Laplace operator (Theorem 4.3). However, as far as the Laplacian is concerned, this choice is not unique, in fact one could take the values of ω_{ij} obtained from a normal distribution and such that:

$$\omega_{ij}^2(\hbar) = \exp\left(\frac{\|x_i - x_j\|^2}{4\hbar}\right) \tag{135}$$

and still get a convergence result. Nevertheless, keeping in mind that we are also interested in the convergence of the square root i.e. to the Dirac operator, it is not clear that such a choice of coefficients would also work.

Moreover, one may also consider classical discretizations of the Laplacian such as the combinatorial one with the choice:

$$\text{vertices } i \text{ and } j \text{ do not share an edge} \Leftrightarrow \omega_{ij} = 0,\ \forall i,j \tag{136}$$

or the *cotangent Laplacian* with the choice

$$\omega_{ij}^2 = \begin{cases} \frac{1}{2}\left(\cot\alpha_{ij} + \cot\beta_{ij}\right) & ij \text{ is an edge,} \\ -\sum_{k\sim i}\omega_{ik}^2 & i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{137}$$

In these two cases the question of convergence to the Laplacian is unclear [22], let alone convergence in the square root.

There is therefore an important direction worth investigating: whether a convergent Laplacian constructed from a specific distribution or obtained from a known discretization implies convergence of its associated Dirac operator.

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