

4. Non-Stationary and Seasonal Models

Models with structured non-stationary components can be constructed as straightforward extensions to ARMA models

- Integrated Models: ARIMA
- Seasonal Models: SARIMA

It is also possible to introduce time-varying or stochastic mean or variance models.

4.1 ARIMA Models

If d is a non-negative integer, then $\{X_t\}$ is an $\text{ARIMA}(p, d, q)$ process if $\{Y_t\}$ defined by

$$Y_t = (1 - B)^d X_t$$

is a (causal) $\text{ARMA}(p, q)$ process, that is

$$\Phi(B)(1 - B)^d X_t = \Theta(B)Z_t$$

where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$; we have stationarity only if $d = 1$.

Such a process can be made stationary by d th order differencing.

4.1 ARIMA Models

Example (ARIMA(1, 1, 0) vs ARIMA(2, 0, 0))

If $\{Z_t\} \sim \text{WN}(0, \sigma^2)$:

- **Stationary** ARIMA(2, 0, 0) $\equiv$ ARMA(2, 0) $\equiv$ AR(2)

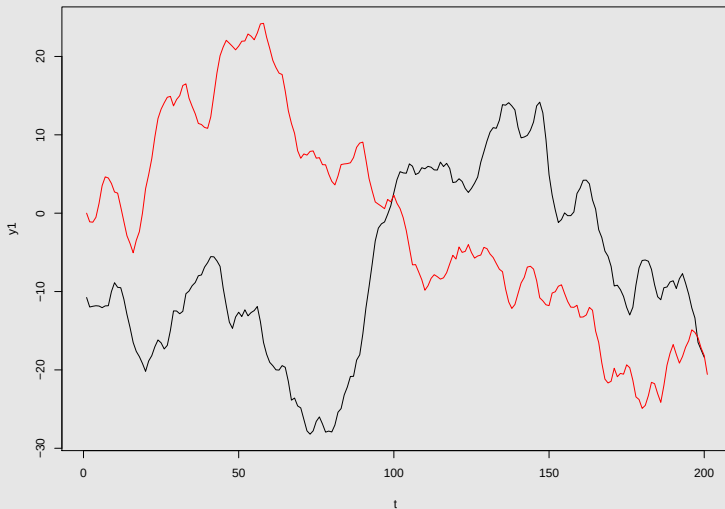
$$(1 - 0.6B)(1 - 0.99B)X_t = Z_t$$

- **Non-stationary** ARIMA(1, 1, 0)

$$(1 - 0.6B)(1 - B)X_t = Z_t$$

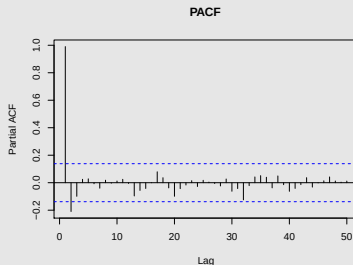
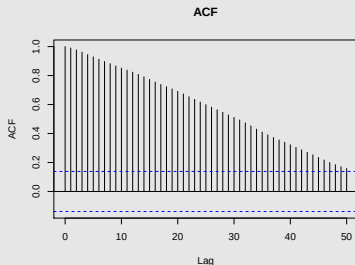
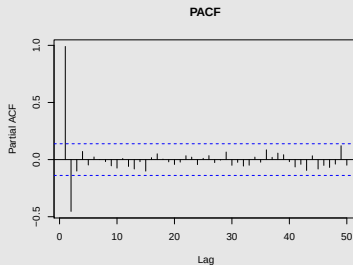
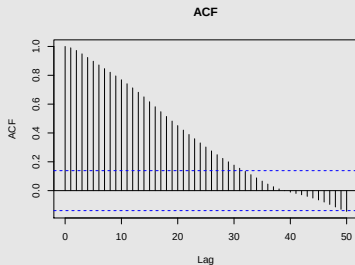
4.1 ARIMA Models

Example (ARIMA(1, 1, 0) vs ARIMA(2, 0, 0))



4.1 ARIMA Models

Example (ARIMA(1, 1, 0) vs ARIMA(2, 0, 0))



4.1 ARIMA Models

R Code:

```
1 n<-200;set.seed(164664)
2 r1<-0.6;r2<-0.99
3 y1<-arima.sim(n,model=list(ar=c(r1+r2,-r1*r2),order=c(2,0,0)))
4 y2<-arima.sim(n,model=list(ar=c(r1),order=c(1,1,0)))
5 plot(y1,xlab=expression(t),type="n",ylim=range(c(y1,y2)))
6 lines(y1);lines(y2,col="red")
7
8 par(mfrow=c(2,2))
9 acf(y1,main="ACF",lag.max=50)
10 pacf(y1,main="PACF",lag.max=50)
11 acf(y2,main="ACF",lag.max=50)
12 pacf(y2,main="PACF",lag.max=50)
13 par(mfrow=c(1,1))
```

4.1 ARIMA Models

Consider the AR(2) model with complex reciprocal roots

$$(1 - \xi B)(1 - \bar{\xi} B)X_t = Z_t$$

and the corresponding AR polynomial

$$\Phi(z) = (1 - \xi z)(1 - \bar{\xi} z) = 1 - 2\operatorname{Re}(\xi)z + |\xi|^2 z^2$$

Suppose that $\xi = \exp\{-i\omega\}/r$, where $r > 0$ and $-\pi < \omega < \pi$.
Then

$$\Phi(z) = 1 - \left(\frac{2}{r} \cos \omega\right) z + \left(\frac{1}{r^2}\right) z^2.$$

4.1 ARIMA Models

It follows that

$$\rho_X(h) = \left(\frac{1}{r}\right)^h \frac{\sin(h\omega + \lambda)}{\sin \lambda} \quad h = 0, \pm 1, \pm 2, \dots$$

where

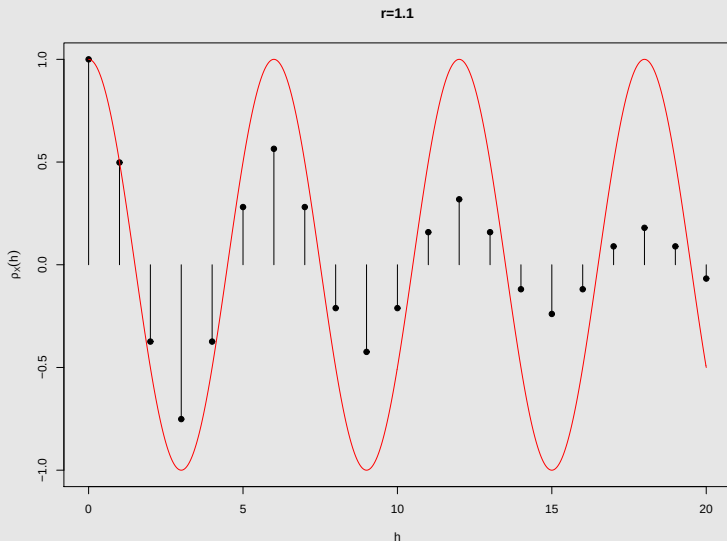
$$\lambda = \arctan \left[\left(\frac{r^2 + 1}{r^2 - 1} \right) \tan \omega \right].$$

In the limit as $r \downarrow 1$

$$\rho_X(h) \longrightarrow \cos(h\omega).$$

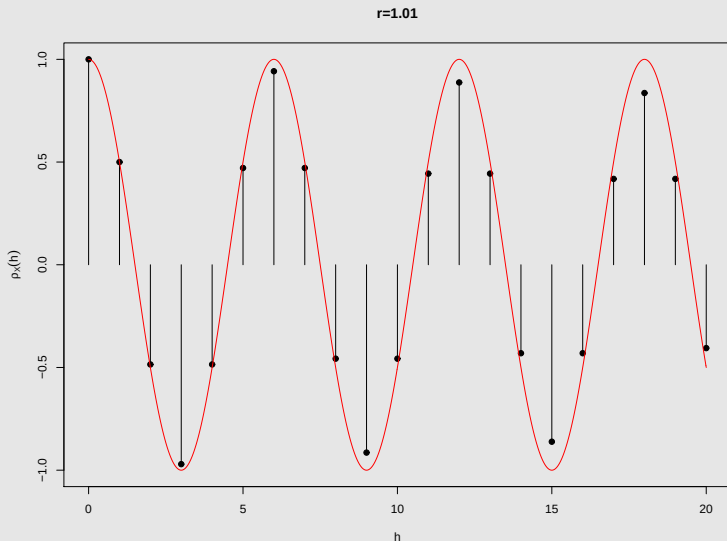
4.1 ARIMA Models

Example ($r = 1.1, \omega = \pi/3$)



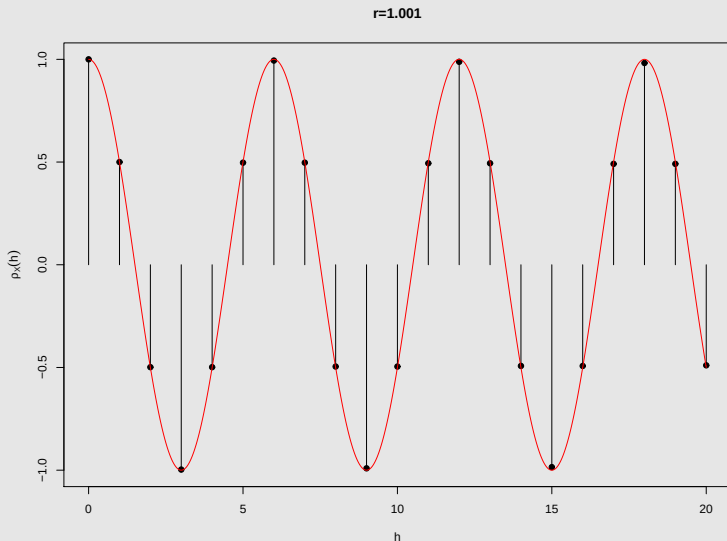
4.1 ARIMA Models

Example ($r = 1.01, \omega = \pi/3$)



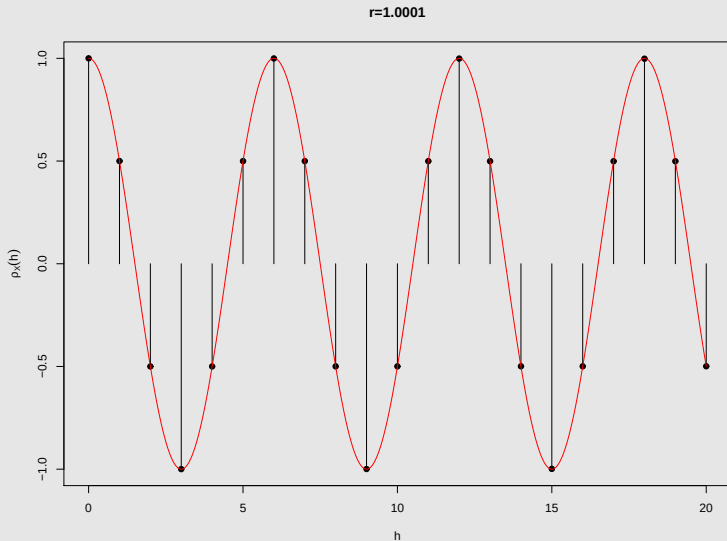
4.1 ARIMA Models

Example ($r = 1.001, \omega = \pi/3$)



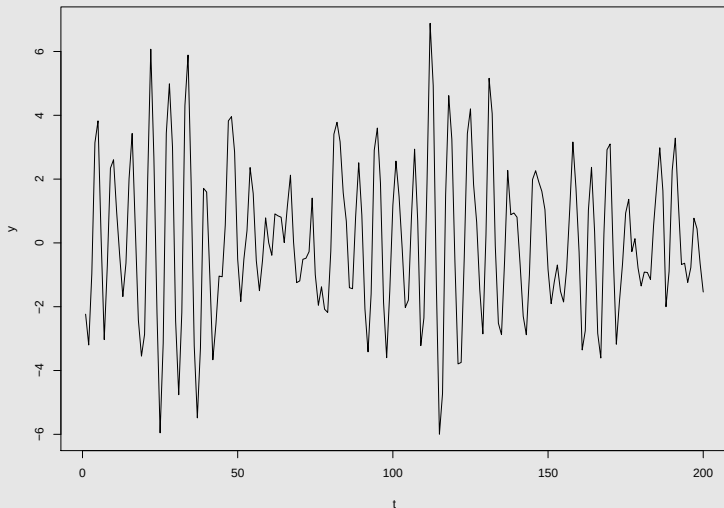
4.1 ARIMA Models

Example ($r = 1.0001, \omega = \pi/3$)



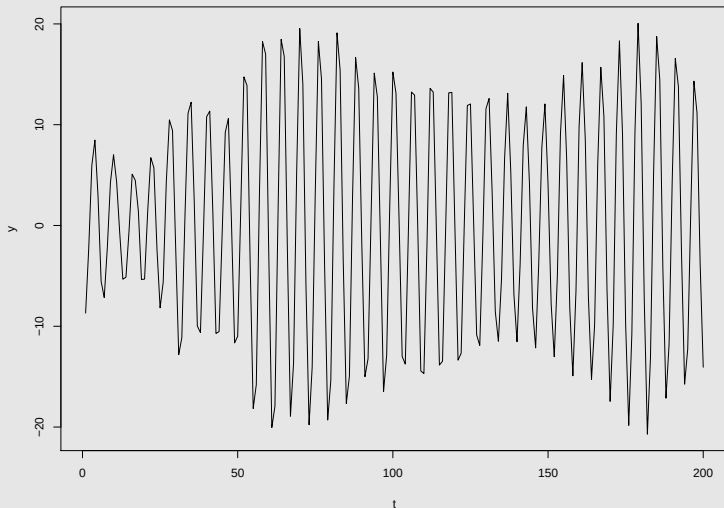
4.1 ARIMA Models

Example ($r = 1.1, \omega = \pi/3$)



4.1 ARIMA Models

Example ($r = 1.001, \omega = \pi/3$)



4.2 Unit Roots

The presence of a *unit root* of the AR polynomial equation

$$\Phi(z) = 0$$

(that is, a solution where $|z| = 1$) renders the resulting AR or ARMA process non-stationary.

Therefore it is crucial to be able to identify processes where a unit root is present, and to distinguish this from a root near to, but not equal to, one in modulus.

4.2 Unit Roots

The Dickey-Fuller Test

Suppose $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, and

$$X_t = \phi X_t + Z_t$$

with $|\phi| < 1$. Then, by standard asymptotic theory, ML or moment-based estimators $\hat{\phi}$ satisfy

$$\sqrt{n}(\hat{\phi} - \phi) \xrightarrow{d} \mathcal{N}(0, 1 - \phi^2)$$

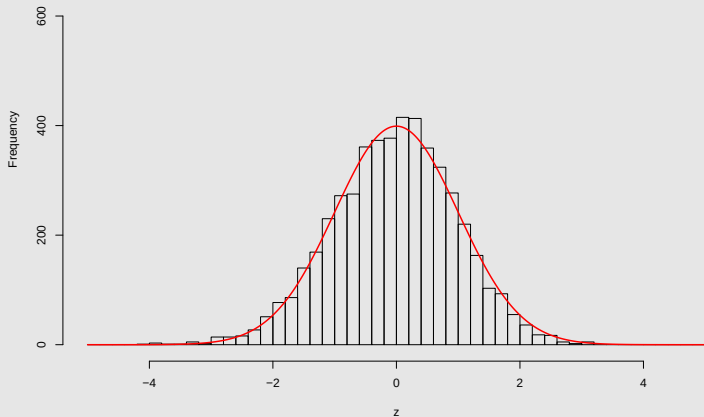
as $n \rightarrow \infty$. We may study the distribution of

$$Z = \frac{\sqrt{n}(\hat{\phi} - \phi)}{\sqrt{1 - \phi^2}}$$

If $|\phi| = 1$, the asymptotic result does not hold.

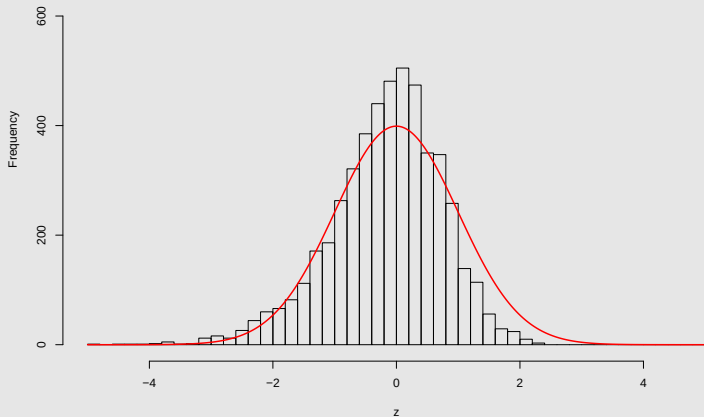
4.2 Unit Roots

Example ($\phi = 0.9$: Distribution of $\hat{\phi}$ vs $\mathcal{N}(0, 1)$)



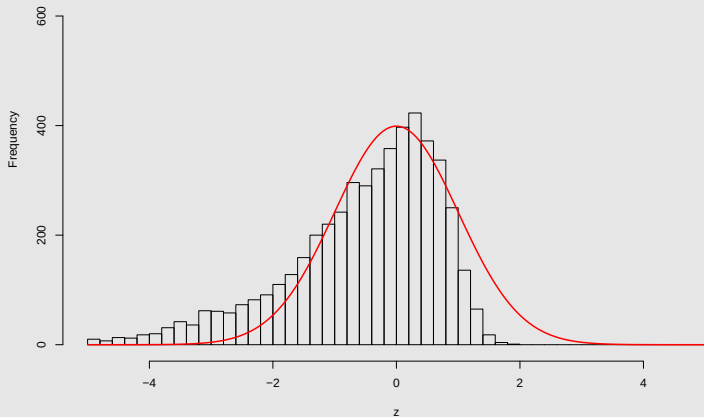
4.2 Unit Roots

Example ($\phi = 0.99$: Distribution of $\hat{\phi}$ vs $\mathcal{N}(0, 1)$)



4.2 Unit Roots

Example ($\phi = 0.999$: Distribution of $\hat{\phi}$ vs $\mathcal{N}(0, 1)$)



4.2 Unit Roots

The Dickey-Fuller Test

Suppose again $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, and

$$X_t = \mu + \phi(X_t - \mu) + Z_t.$$

Then

$$\begin{aligned}(1 - B)X_t = X_t - X_{t-1} &= \mu(1 - \phi) + (\phi - 1)X_{t-1} + Z_t \\ &= \phi_0^* + \phi_1^*X_{t-1} + Z_t\end{aligned}$$

where

$$\phi_0^* = \mu(1 - \phi) \qquad \phi_1^* = \phi - 1.$$

can be estimated by OLS.

4.2 Unit Roots

To test the hypothesis

$$H_0 : \phi_1^* = 0$$

which is equivalent to

$$H_0 : \phi_1 = 1$$

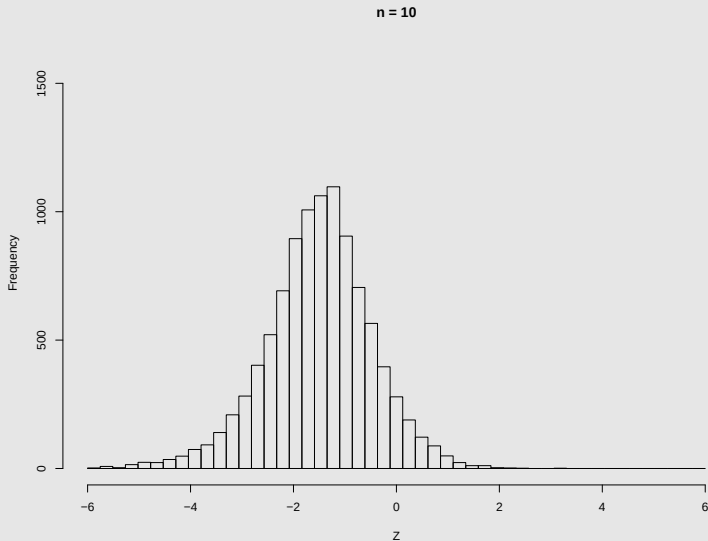
the test statistic

$$t = \frac{\hat{\phi}_1^*}{\text{e.s.e}(\hat{\phi}_1^*)}$$

can be used. However, if H_0 is **true**, and $\{X_t\}$ is non-stationary, the null distribution is non-standard.

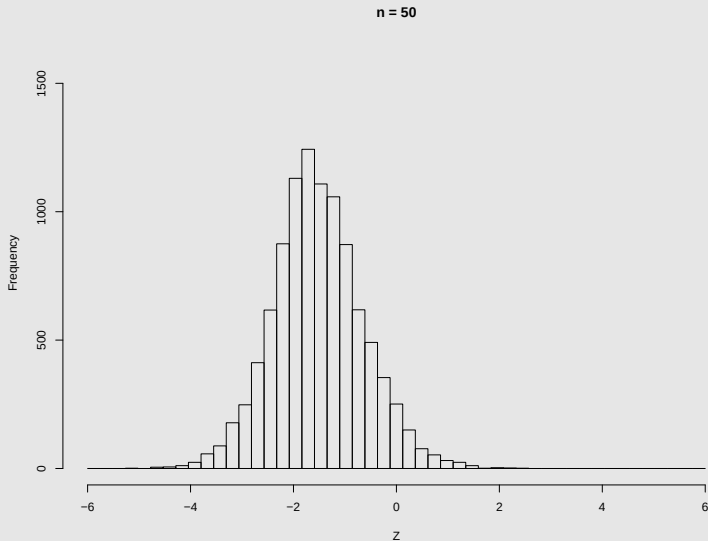
4.2 Unit Roots

Example (Dickey-Fuller Statistic: $n = 10$)



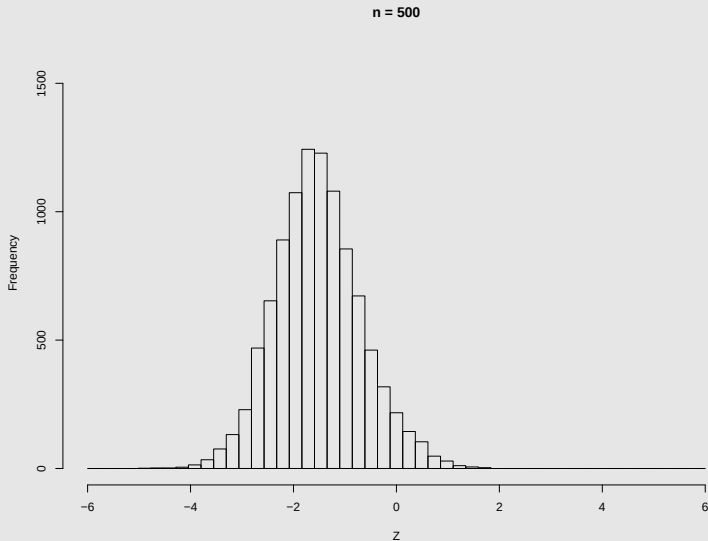
4.2 Unit Roots

Example (Dickey-Fuller Statistic: $n = 50$)



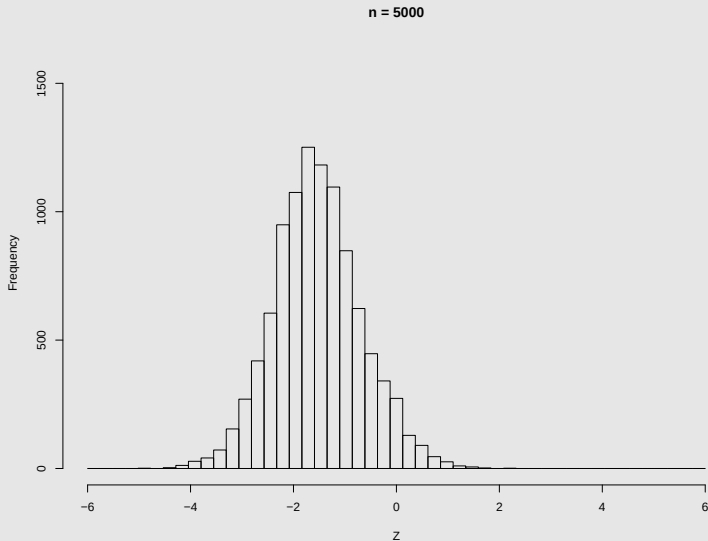
4.2 Unit Roots

Example (Dickey-Fuller Statistic: $n = 500$)



4.2 Unit Roots

Example (Dickey-Fuller Statistic: $n = 5000$)



4.2 Unit Roots

The Augmented Dickey-Fuller Test

The test can be extended to $AR(p)$ processes: if

$$X_t - \mu = \sum_{j=1}^p \phi_j (X_{t-j} - \mu) + Z_t$$

then for $p \geq 2$,

$$(1 - B)X_t = \phi_0^* + \phi_1^* X_{t-1} + \sum_{j=2}^p \phi_j^* (X_{t-j+1} - X_{t-j}) + Z_t$$

where

$$\phi_0^* = \mu(1 - \phi_1 - \phi_2 - \cdots - \phi_p)$$

and

$$\phi_1^* = \sum_{j=1}^p \phi_j - 1 \qquad \phi_j^* = - \sum_{k=j}^p \phi_k \qquad j = 2, \dots, p.$$

4.2 Unit Roots

If $\{X_t\} \sim \text{AR}(p)$ with one unit root, then

$$(1 - B)X_t \sim \text{AR}(p - 1)$$

therefore a unit root test can be carried out by examining the hypothesis

$$H_0 : \phi_1^* = 0$$

by fitting a regression of $(1 - B)X_t$ on

$$1, X_{t-1}, X_{t-1} - X_{t-2}, \dots, X_{t-p+1} - X_{t-p}$$

The null distribution of the test statistic changes with p .

4.2 Unit Roots

R Code: The library `tseries` contains the function `adf.test`

```
1 library(tseries)
2 x <- rnorm(1000)
3 adf.test(x,k=0)
4 adf.test(cumsum(x),k=0)
```

The series of length n , x , is a white noise process, but the vector constructed by `cumsum(x)` is a random walk.

The lag order, p , can be specified by using a `k=p+1` argument to the function.

4.2 Unit Roots

```
1 > adf.test(x,k=0)
2           Augmented Dickey-Fuller Test
3
4 data:  x
5 Dickey-Fuller = -31.9595, Lag order = 0, p-value = 0.01
6 alternative hypothesis: stationary
7
8 Warning message:
9 In adf.test(x, k = 0) : p-value smaller than printed p-value
10
11
12 > adf.test(cumsum(x),k=0)
13           Augmented Dickey-Fuller Test
14
15 data:  cumsum(x)
16 Dickey-Fuller = -2.1115, Lag order = 0, p-value = 0.5311
17 alternative hypothesis: stationary
```

4.2 Unit Roots

Example (S&P 500: Oct 03, 2001 - Oct 03, 2011)



4.2 Unit Roots

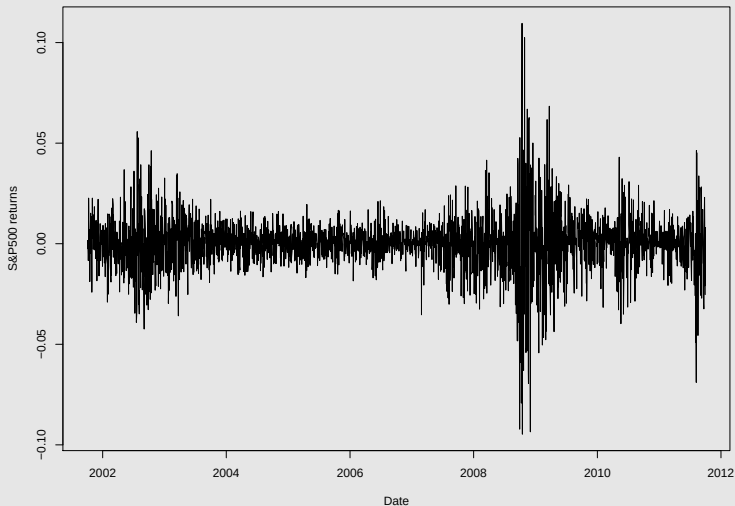
Example (S&P 500: Oct 03, 2001 - Oct 03, 2011)

```
1 > adf.test(sp500$Close,k=0)
2
3       Augmented Dickey-Fuller Test
4
5 data:  sp500$Close
6 Dickey-Fuller = -2.0111, Lag order = 0, p-value = 0.5736
7 alternative hypothesis: stationary
```

Even for higher lag orders, null hypothesis would not be rejected.

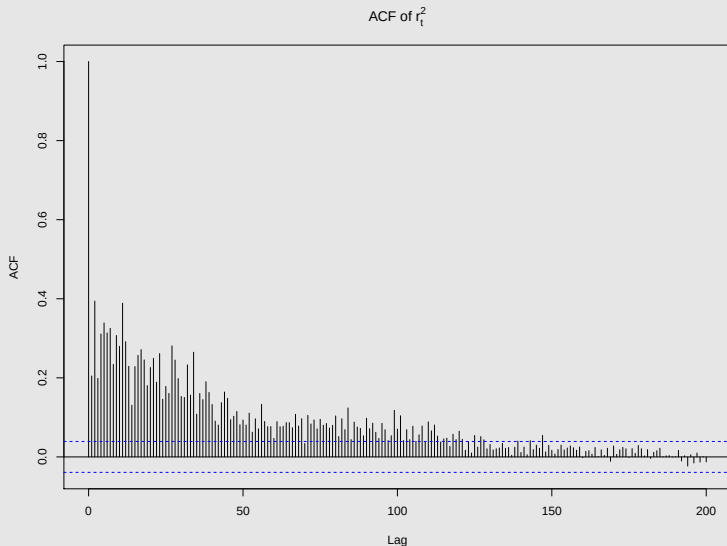
4.2 Unit Roots

Example (S&P 500: log returns $r_t = \log(s_{t+1}/s_t)$)



4.2 Unit Roots

Example (S&P 500: ACF of r_t^2)



4.2 Unit Roots

Unit Roots in MA processes.

If $\{X_t\} \sim \text{ARMA}(p, q)$ is causal and invertible, then

$$Y_t = (1 - B)X_t$$

is an $\text{ARMA}(p, q + 1)$ process. This follows as

$$\Phi(B)X_t = \Theta(B)Z_t$$

with if $\{X_t\} \sim \text{WN}(0, \sigma^2)$, then

$$\Phi(B)Y_t = \Phi(B)(1 - B)X_t = (1 - B)\Theta(B)Z_t.$$

4.2 Unit Roots

The MA polynomial for $\{Y_t\}$ is

$$\Theta_Y(z) = (1 - z)\Theta(z)$$

Thus a test for a unit root in the MA polynomial for $\{Y_t\}$ is equivalent to a test for over-differencing of a stationary, causal and invertible ARMA process.

The asymptotic null distribution of the (likelihood-ratio) test is non-standard in this setting.

4.3 Forecasting ARIMA

For the general ARIMA(p, d, q) model

$$(1 - B)^d \Phi(B) X_t = \Theta(B) Z_t$$

we perform first forecasting of

$$(1 - B)^d X_t = Y_t$$

where $\{Y_t\}$ is a causal ARMA(p, q) process. Then

$$X_t = Y_t - \sum_{j=1}^d \binom{d}{j} (-1)^j X_{t-j}$$

for $t > 0$.

4.3 Forecasting ARIMA

Therefore, for $h = 1, 2, \dots$, given $X_{1:n}$, we have that

$$\hat{P}_n X_{n+h} = \hat{P}_n Y_{n+h} - \sum_{j=1}^d \binom{d}{j} (-1)^j \hat{P}_n X_{n+h-j}$$

for which

- $\hat{P}_n Y_{n+h}$ is computed using the methods devised earlier for stationary processes
- $\hat{P}_n X_{n+h-j}$ is computed recursively, noting that

$$\hat{P}_n X_{n+1-j} = X_{n+1-j}$$

for $j \geq 1$.

4.4 Seasonal ARIMA Models

Recall the lag- s difference operator, $B^s, s \geq 1$,

$$B^s X_t = X_t - X_{t-s}$$

A *seasonal ARIMA model* utilizes this operator: for example

$$(1 - B^s)\Phi(B)X_t = \Theta(B)Z_t$$

applies a seasonal differencing with period s .

4.4 Seasonal ARIMA Models

If d and D are non-negative integers, then $\{X_t\}$ is a seasonal ARIMA (SARIMA) process with period s

$$\{X_t\} \sim \text{ARIMA}(p, d, q) \times (P, D, Q)_s$$

of

$$Y_t = (1 - B)^d (1 - B^s)^D X_t$$

is a causal ARMA process determined by

$$\Phi(B)\Phi_S(B^s)Y_t = \Theta(B)\Theta_S(B^s)Z_t$$

where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$.

4.4 Seasonal ARIMA Models

In this formulation

$$\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$$

$$\Phi_S(z) = 1 - \phi_1^{(S)} z - \cdots - \phi_P^{(S)} z^P$$

$$\Theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$$

$$\Theta_S(z) = 1 + \theta_1^{(S)} z + \cdots + \theta_Q^{(S)} z^Q$$

Note that

$$Y_t = (1 - B)^d (1 - B^S)^D X_t$$

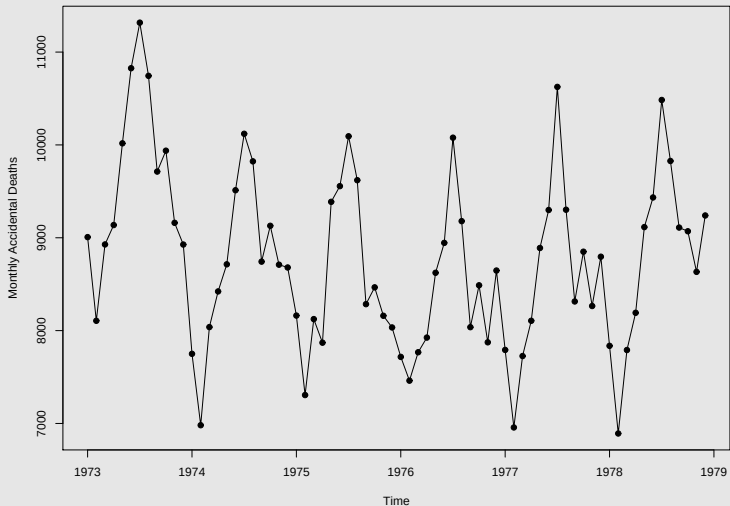
is an ARMA($p + sP, q + sQ$) process with AR and MA polynomials

$$\Phi(B)\Phi_S(B^S)$$

$$\Theta(B)\Theta_S(B^S)$$

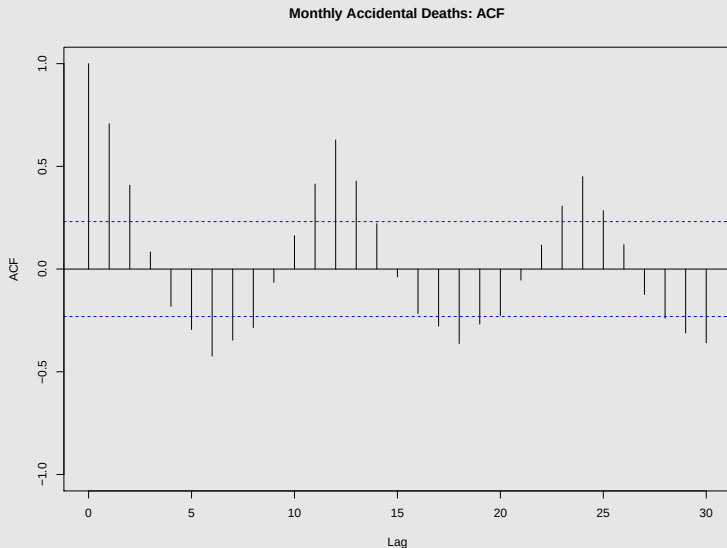
4.4 Seasonal ARIMA Models

Example (US Accidental Deaths: Jan 1973 - Dec 1978)



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4.4 Seasonal ARIMA Models

Example (US Accidental Deaths: Jan 1973 - Dec 1978)

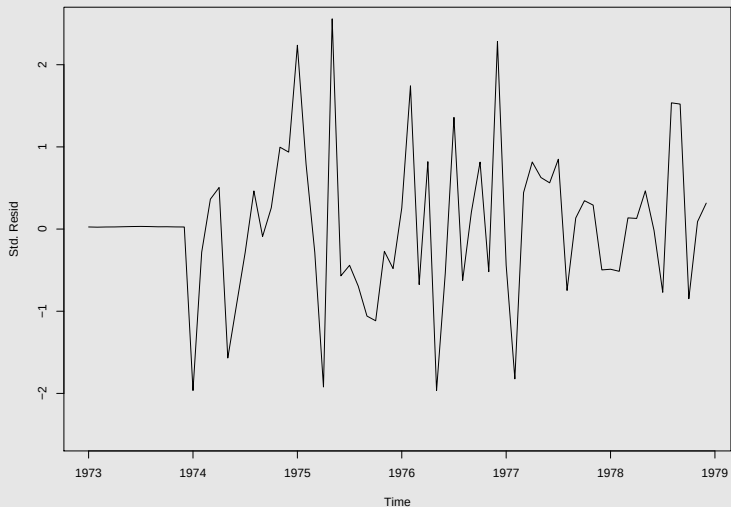
Consider two SARIMA models:

- A. $\text{ARIMA}(1, 0, 1) \times (0, 1, 0)_{12}$: This model fits an $\text{ARMA}(1, 1)$ model to the seasonally-differenced data with $s = 12$.
- B. $\text{ARIMA}(1, 0, 1) \times (1, 0, 0)_{12}$ This model fits a SARIMA model with $p = q = 1, P = 1, Q = 0$, and $s = 12$.

Model A posits a non-stationary model, and requires first differencing at lag 12. Model B posits a stationary model.

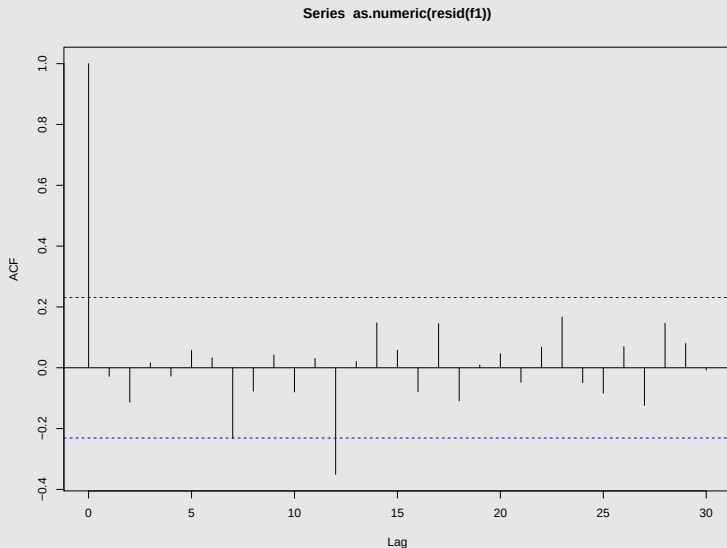
4.4 Seasonal ARIMA Models

Example (Model A: Residuals)



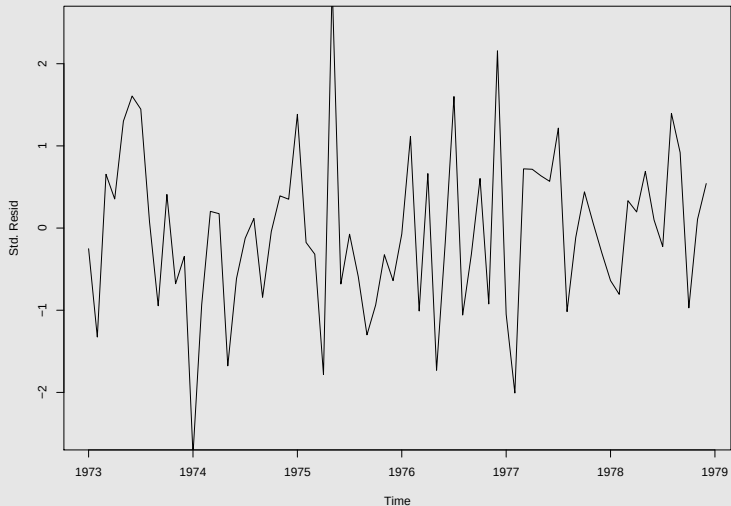
4.4 Seasonal ARIMA Models

Example (Model A: Residuals ACF)



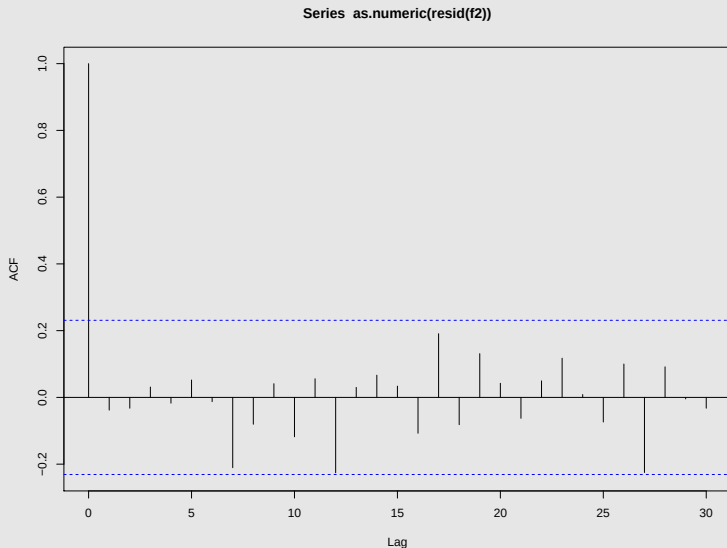
4.4 Seasonal ARIMA Models

Example (Model B: Residuals)



4.4 Seasonal ARIMA Models

Example (Model B: Residuals ACF)



4.4 Seasonal ARIMA Models

R Code:

```
1 > f2<-arima(USAccDeaths, order = c(1,0,1),
2           seasonal = list(order=c(1,0,0),period=12))
3 > f2
4
5 Call: arima(x = USAccDeaths, order = c(1, 0, 1),
6           seasonal = list(order = c(1, 0, 0), period = 12))
7
8 Coefficients:
9           ar1           ma1           sar1  intercept
10          0.8364        -0.1849         0.8724    9313.3561
11 s.e.      0.0874         0.1651         0.0491     884.3092
12
13 sigma^2 estimated as 121721:  log likelihood = -532.85,  aic = 1075.7
```

The estimate of $\phi_1^{(s)}$ is 0.8724(0.0491).

4.5 Non-stationarity in Mean

Non-stationarity in mean arises when there is a time-varying or trend structure, as described in earlier sections. Previously OLS was used to estimate parameters in a parametric trend, m_t ; OLS assumes that the residual errors are uncorrelated.

More efficient estimation can be used to take into account the correlation present if the residual error process is stationary, but exhibits autocorrelation.

4.5 Non-stationarity in Mean

Suppose

$$Y_t = m_t + W_t = \mathbf{x}_t^\top \beta + W_t$$

where

- $\mathbf{x}_t$ is a vector of (exogenous) covariates
- β is a vector of (fixed effects) coefficients
- $\{W_t\} \sim \text{ARMA}(p, q)$, with parameters (ϕ, θ)

We wish to estimate

$$\beta \quad (\phi_1, \dots, \phi_p)^\top \quad (\theta_1, \dots, \theta_q)^\top$$

4.5 Non-stationarity in Mean

Suppose

$$\text{Cov}[W_{1:n}] = \mathbb{E}[W_{1:n} W_{1:n}^T] = \Gamma_n(\phi, \theta).$$

If (ϕ, θ) are known, then we may estimate β using *Weighted Least Squares* (WLS)

$$\hat{\beta} = (\mathbf{X}^T \Gamma_n(\phi, \theta)^{-1} \mathbf{X})^{-1} \mathbf{X}^T \Gamma_n(\phi, \theta)^{-1} \mathbf{Y}$$

with variance

$$(\mathbf{X}^T \Gamma_n(\phi, \theta)^{-1} \mathbf{X})^{-1}$$

4.5 Non-stationarity in Mean

This suggests an iterative procedure

- ▶ Initialize $\phi^{(0)}, \theta^{(0)}$
- ▶ Compute $\beta^{(0)}$ using WLS setting $(\phi, \theta) = (\phi^{(0)}, \theta^{(0)})$.
- ▶ Compute the residuals

$$\hat{w}_t = y_t - \mathbf{x}_t^\top \beta^{(0)}$$

and then fit an ARMA(p, q) model to $\{\hat{W}_t\}$ to update estimates $(\phi^{(1)}, \theta^{(1)})$.

- ▶ Continue iteration until $(\beta^{(l)}, \phi^{(l)}, \theta^{(l)})$ ceases to change.