

7. Spectral Analysis of Time Series

The *spectral* or *frequency domain* representation of time series $\{X_t\}$ focusses on stationary processes, and constructs an alternative view to the time domain representation studied previously.

The focus on time domain methods is paramount in statistics because observed data are typically collected *sequentially* in time. However, in the context of stationary processes, the non-negative definite autocovariance function admits a spectral decomposition, and this decomposition can facilitate alternative inference approaches.

7.1 Spectral Representation

The acvf of a (zero-mean) stationary process $\{X_t\}$

$$\gamma_X(h) = \mathbb{E}[X_t X_{t+h}]$$

is a symmetric, non-negative definite function that has a unique representation in terms of “*frequency contributions*”.

The Spectral Density Function

Let $\{X_t\}$ stationary and zero-mean, with acvf $\gamma_X(\cdot)$, with

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty.$$

Then the *spectral density function* (SDF) of $\{X_t\}$ is the function $f_X(\cdot)$ defined by

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \gamma_X(h) \quad \lambda \in \mathbb{R}$$

Note that $f_X(\lambda)$ is well-defined, as $|e^{-i\lambda h}| = 1$, so

$$\sum_{h=-\infty}^{\infty} |e^{-i\lambda h} \gamma_X(h)| \leq \sum_{h=-\infty}^{\infty} |e^{-i\lambda h}| |\gamma_X(h)| = \sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty.$$

and $f_X(\lambda)$ is periodic, we can restrict attention to $\lambda \in (-\pi, \pi]$ 560

The Spectral Density Function

(i) $f_X(\lambda) = f_X(-\lambda)$ and

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \cos(\lambda h) \gamma_X(h)$$

(ii) $f_X(\lambda) \geq 0$, $-\pi < \lambda \leq \pi$.

(iii) Inversion formula:

$$\begin{aligned} \int_{-\pi}^{\pi} e^{i h \lambda} f_X(\lambda) d\lambda &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{k=-\infty}^{\infty} e^{i \lambda (h-k)} \gamma_X(k) \right\} d\lambda \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_X(k) \left\{ \int_{-\pi}^{\pi} e^{i \lambda (h-k)} d\lambda \right\} = \gamma_X(h) \end{aligned}$$

The Spectral Density Function

Note: The summability condition

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty$$

is not a necessary condition for the existence of a SDF. There is a 1-1 correspondence between acvf's and spectral density functions; if

- (i) $f_X(\lambda) \geq 0$, for all $\lambda \in (0, \pi]$.
- (ii) For $h = 0, \pm 1, \pm 2, \dots$, if

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda) d\lambda$$

then f_X is the SDF $\{X_t\}$.

The Spectral Density Function

A real-valued function f defined on $(-\pi, \pi]$ is the SDF of a stationary process if and only if

- (i) f is even: $f(\lambda) = f(-\lambda)$ for all λ .
- (ii) $f(\lambda) > 0$
- (iii)

$$\int_{-\pi}^{\pi} f(\lambda) d\lambda < \infty$$

The Spectral Density Function

Suppose f satisfies (i)-(iii), then

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{i h \lambda} f_X(\lambda) d\lambda$$

is even. Let $\mathbf{a} \in \mathbb{R}^n$. Then

$$\begin{aligned} \mathbf{a}^T \Gamma \mathbf{a} &= \sum_{r=1}^n \sum_{s=1}^n a_r \gamma(r-s) a_s = \int_{-\pi}^{\pi} \sum_{r=1}^n \sum_{s=1}^n a_r a_s e^{i \lambda (r-s)} f(\lambda) d\lambda \\ &= \int_{-\pi}^{\pi} \left| \sum_{r=1}^n a_r e^{i \lambda r} \right|^2 f(\lambda) d\lambda \geq 0 \end{aligned}$$

so $\gamma(\cdot)$ is non-negative definite, and hence the acvf for a stationary process.

The Spectral Density Function

Note: f_X is the *Fourier transform* of γ_X , γ_X is the *inverse Fourier transform*.

The properties of the SDF f_X are similar to a pdf, **symmetric about zero**, with the integrability relaxed to

$$\int_{-\pi}^{\pi} f_X(\lambda) d\lambda = C < \infty$$

rather than having $C = 1$.

Spectral Representation Theorem

Theorem: A function $\gamma(\cdot), \gamma : \mathbb{Z} \longrightarrow \mathbb{R}$, is an autocovariance function if and only if there exists a right continuous, non-decreasing bounded function F on $[-\pi, \pi]$ with $F(-\pi) = 0$, and for all $h \in \mathbb{Z}$

$$\gamma(h) = \int_{(-\pi, \pi]} e^{ih\lambda} dF(\lambda).$$

If the corresponding series $\{\gamma(h)\}$ is real-valued, then F satisfies

$$\int_{(a, b]} dF(x) = \int_{(-b, -a]} dF(x)$$

Spectral Representation Theorem

F is a *generalized distribution function* as $F(\lambda)/F(\pi)$ is a distribution function on $[-\pi, \pi]$.

F is termed the *spectral distribution function*; it can be purely discrete, absolutely continuous, or mixed. If F is absolutely continuous

$$F(\lambda) = \int_{-\pi}^{\lambda} f(x) dx \quad \lambda \text{ in } (-\pi, \pi]$$

then f is the *spectral density function*.

Example

Example (Harmonic process)

Suppose

$$X_t = A \cos(\omega t) + B \sin(\omega t)$$

where A and B are uncorrelated, with

$$\mathbb{E}[A] = \mathbb{E}[B] = 0 \qquad \mathbb{V}\text{ar}[A] = \mathbb{V}\text{ar}[B] = 1$$

Then by a previous result

$$\gamma_X(h) = \cos(\omega h) \qquad h \in \mathbb{Z}$$

Example

Example (Harmonic process)

Now

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| = \sum_{h=-\infty}^{\infty} |\cos(\omega h)| = \infty$$

However, let

$$F_X(\lambda) = \begin{cases} 0 & \lambda < -\omega \\ \frac{1}{2} & -\omega \leq \lambda < \omega \\ 1 & \omega \leq \lambda \end{cases}$$

Example

Example (Harmonic process)

Then

$$f_X(\lambda) = \begin{cases} \frac{1}{2} & \lambda = -\omega \\ \frac{1}{2} & \lambda = \omega \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_{(-\pi, \pi)} e^{i h \lambda} dF_X(\lambda) = \frac{1}{2} e^{-i h \omega} + \frac{1}{2} e^{i h \omega} = \cos(h\omega)$$

Example: Extension

Example (Superposition of harmonic processes)

Suppose

$$X_t = \sum_{j=1}^k [A_j \cos(\omega_j t) + B_j \sin(\omega_j t)]$$

where $A_1, \dots, A_k$ and $B_1, \dots, B_k$ are uncorrelated, with

$$\mathbb{E}[A_j] = \mathbb{E}[B_j] = 0 \qquad \mathbb{V}ar[A_j] = \mathbb{V}ar[B_j] = \sigma_j^2$$

and where

$$0 < \omega_1 < \dots < \omega_k < \pi.$$

Then

$$\gamma_X(h) = \sum_{j=1}^k \sigma_j^2 \cos(\omega_j h) \qquad h \in \mathbb{Z}.$$

Example: Extension

Example (Superposition of harmonic processes)

Then

$$F_X(\lambda) = \sum_{j=1}^k \sigma_j^2 F_j(\lambda)$$

where

$$F_j(\lambda) = \begin{cases} 0 & \lambda < -\omega_j \\ \frac{1}{2} & -\omega_j \leq \lambda < \omega_j \\ 1 & \omega_j \leq \lambda \end{cases}$$

Note that

$$\text{Var}[X_t] = \gamma_X(0) = \sum_{j=1}^k \sigma_j^2$$

Spectral decomposition

This result shows that the variance of $\{X_t\}$ can be decomposed into the variances contributed by the harmonic components at different frequencies

$$\omega_1, \dots, \omega_k$$

In the limit as $k \rightarrow \infty$

$$X_t \stackrel{\text{m.s.}}{=} \int_{(-\pi, \pi]} e^{it\lambda} dZ(\lambda)$$

where $\{Z(\lambda) : -\pi < \lambda \leq \pi\}$ is a non-decreasing stochastic process.

In general, Z is complex-valued, with uncorrelated increments.

Spectral decomposition

For finite k ,

$$dZ(\lambda) = \begin{cases} \frac{A_j + \iota B_j}{2} & \lambda = -\omega_j, j = 1, \dots, k \\ \frac{A_j - \iota B_j}{2} & \lambda = \omega_j, j = 1, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

Here

$$\mathbb{E}[dZ(\lambda)\overline{dZ(\lambda)}] = \mathbb{E}[(A_j^2 + B_j^2)/2] = \frac{\sigma_j^2}{2}$$

for $\lambda = \pm\omega_j, j = 1, \dots, k$ and zero otherwise.

Example: White noise process

Example ($X_t \sim \text{WN}(0, \sigma^2)$)

If $X_t \sim \text{WN}(0, \sigma^2)$, so that $\gamma_X(0) = \sigma^2$, $\gamma_X(h) = 0$ for all $h \neq 0$.
Then, for $-\pi < \lambda \leq \pi$,

$$\begin{aligned} f_X(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-\iota h \lambda} \gamma_X(h) \\ &= \frac{1}{2\pi} [\cos(0) - \iota \sin(0)] \sigma^2 \\ &= \frac{\sigma^2}{2\pi} \end{aligned}$$

Spectral decomposition

Recap: we have

$$X_t = \int_{(-\pi, \pi]} e^{it\lambda} dZ(\lambda) = \int_{(-\pi, \pi]} [\cos(t\lambda) + i \sin(t\lambda)] dZ(\lambda)$$

where $Z(\lambda)$ is a random process on $(-\pi, \pi]$. By symmetry, we need only consider $Z(\lambda)$ for $0 \leq \lambda \leq \pi$.

λ is an *angular frequency*:

- ▶ For a time series with dominant *high frequency* components, $f_X(\lambda)$ is large when λ is near π , and the series exhibits *rapid variation*.
- ▶ For a time series with dominant *low frequency* components, $f_X(\lambda)$ is large when λ is near zero, and the series exhibits *slow variation*.

Spectral decomposition

Often we use the frequency λ^* ,

$$\lambda^* = \frac{\lambda}{2\pi} \quad \text{so that} \quad -\frac{1}{2} < \lambda^* \leq \frac{1}{2}$$

so that

$$f_X^*(\lambda^*) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i2\pi\lambda^*h} \gamma_X(h)$$

and

$$\gamma_X(h) = \int_{-1/2}^{1/2} e^{i2\pi\lambda^*h} f_X^*(\lambda^*) d\lambda^*$$

7.2 Spectral Representation for ARMA processes

A linear process is the output of a *time-invariant linear filter* acting on a white noise process: if $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, then the linear filter Ψ , is determined by coefficients $\{\psi_k\}$, and

$$X_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}.$$

7.2 Spectral Representation for ARMA processes

Let $\{X_t\}$ be a stationary time series with mean zero and sdf $f_X(\cdot)$. Suppose $\Psi \equiv \{\psi_j : j = 0, \pm 1, \pm 2, \dots\}$ is a linear filter with

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

Then $\{Y_t\}$ defined by

$$Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j}$$

is stationary, zero mean, with sdf

$$f_Y(\lambda) = |\Psi(e^{-i\lambda})|^2 f_X(\lambda) = \Psi(e^{-i\lambda}) \overline{\Psi(e^{-i\lambda})} f_X(\lambda)$$

where

$$\Psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j \quad z \in \mathbb{C}.$$

7.2 Spectral Representation for ARMA processes

This result follows by the previous result for acvfs

$$\gamma_Y(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_X(h + k - j)$$

and the representation result for $\gamma_X(\cdot)$ via $f_X(\cdot)$.

$\Psi(e^{-i\lambda})$ is the *transfer function*.

$|\Psi(e^{-i\lambda})|^2$ is the *power transfer function*.

The transfer function defines the action of the filter on the process $\{X_t\}$.

Low Pass filter

Example (Low Pass filter)

A *low pass* filter removes high frequency components. To remove frequency components beyond threshold ω_C , we would require a transfer function taking the value 1 for $-\omega_C \leq \lambda \leq \omega_C$, that is,

$$\Psi(e^{-\iota\lambda}) = \begin{cases} 1 & -\omega_C \leq \lambda \leq \omega_C \\ 0 & \text{otherwise} \end{cases}$$

Term this filter the ideal low pass filter.

Low Pass filter

Example (Low Pass filter)

Let

$$\psi_j = \frac{1}{2\pi} \int_{-\omega_C}^{\omega_C} e^{ij\lambda} d\lambda = \begin{cases} \frac{\omega_C}{\pi} & j = 0 \\ \frac{\sin(j\omega_C)}{j\pi} & j \neq 0 \end{cases}$$

and define

$$\Psi_q(e^{-i\lambda}) = \sum_{j=-q}^q \psi_j e^{-ij\lambda}$$

for integer q . This filter is a finite Fourier series approximation to the ideal low pass filter.

Example ($X_t \sim \text{ARMA}(p, q)$)

Suppose $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, and define $\{X_t\}$ by

$$\Phi(B)X_t = \Theta(B)Z_t.$$

Thus

$$f_Z(\lambda) = \frac{\sigma^2}{2\pi}$$

and as $X_t = [\Phi(B)]^{-1}\Theta(B)Z_t$, we have

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left| \frac{\Theta(e^{-i\lambda})}{\Phi(e^{-i\lambda})} \right|^2 = \frac{\sigma^2}{2\pi} \frac{|\Theta(e^{-i\lambda})|^2}{|\Phi(e^{-i\lambda})|^2}$$

by the general result.

7.3 Model-free estimation

The sdf can be estimated from time-domain samples. Let $x_{1:n}$ be the time-domain data, and let

$$\omega_k = \frac{2\pi k}{n}$$

for k equal to

$$-\left\lfloor \frac{n-1}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor$$

The set $\{\omega_k\}$ are termed the *Fourier frequencies*.

7.3 Model-free estimation

For each k , let the $n \times 1$ vector $\mathbf{e}_k$ defined by

$$\mathbf{e}_k = \frac{1}{\sqrt{n}}(e^{i\omega_k}, e^{2i\omega_k}, \dots, e^{ni\omega_k})^\top$$

so that

$$\bar{\mathbf{e}}_j^\top \mathbf{e}_k = 0$$

unless $j = k$, in which case $\bar{\mathbf{e}}_k^\top \mathbf{e}_k = 1$.

7.3 Model-free estimation

$\{e_1, \dots, e_n\}$ form an orthonormal basis for $\mathbb{C}^n$, that is, for any n -dimensional complex number x , we have

$$x = \sum_{k=-\lfloor (n-1)/2 \rfloor}^{\lfloor n/2 \rfloor} a_k e_k$$

where

$$a_k = \bar{e}_k^T x = \frac{1}{\sqrt{n}} \sum_{t=1}^n x_t e^{-\iota t \omega_k}$$

The vector $(a_1, \dots, a_n)^T$ is the *discrete Fourier transform* of x . Note that

$$x_t = \sum_{k=-\lfloor (n-1)/2 \rfloor}^{\lfloor n/2 \rfloor} a_k [\cos(\omega_k t) + \iota \sin(\omega_k t)]$$

is the discrete version of the spectral representation theorem.

The Periodogram

The *periodogram*, $\hat{I}_n(\lambda)$ is defined by

$$\hat{I}_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^n x_t e^{-it\lambda} \right|^2 \quad -\pi < \lambda \leq \pi.$$

Note that $\hat{I}_n(\omega_k) = |a_k|^2$, and

$$\bar{\mathbf{x}}^T \mathbf{x} = \sum_{k=-\lfloor (n-1)/2 \rfloor}^{\lfloor n/2 \rfloor} |a_k|^2 = \sum_{k=-\lfloor (n-1)/2 \rfloor}^{\lfloor n/2 \rfloor} \hat{I}_n(\omega_k)$$

which gives a decomposition of the total variation in $\mathbf{x}$ via components at the ω_k .

The Periodogram

Note that

$$\begin{aligned}\hat{I}_n(\omega_k) &= \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n x_t e^{-\iota t \omega_k} \right) \left(\frac{1}{\sqrt{n}} \sum_{s=1}^n x_s e^{-\iota s \omega_k} \right) \\&= \frac{1}{n} \sum_{t=1}^n \sum_{s=1}^n x_t x_s e^{-\iota(t-s)\omega_k} \\&= \frac{1}{n} \sum_{t=1}^n \sum_{s=1}^n (x_t - \bar{x})(x_s - \bar{x}) e^{-\iota(t-s)\omega_k} \\&= \sum_{h=-(n-1)}^{n-1} \hat{\gamma}_n(h) e^{-\iota h \omega_k} \quad (h = s - t)\end{aligned}$$

as $\sum_{t=1}^n e^{-\iota t \omega_k} = 0$ if $\omega_k \neq 0$.

The Periodogram

Thus $\hat{I}_n(\lambda)$ is a sample-based estimate of $2\pi f_X(\lambda)$; it is **not** consistent, however. For $0 < \lambda_1 < \dots < \lambda_m < \pi$,

$$(\hat{I}_n(\lambda_1), \dots, \hat{I}_n(\lambda_m))^T \xrightarrow{d} (V_1, \dots, V_m)^T$$

as $n \rightarrow \infty$, where

$$V_j \sim \text{Exponential}(1/(2\pi f_X(\lambda_j))).$$

That is, as $n \rightarrow \infty$

$$\mathbb{E}[\hat{I}_n(\lambda_j)] \rightarrow 2\pi f_X(\lambda_j)$$

so $\hat{I}_n(\lambda_j)$ is asymptotically unbiased, but $\hat{I}_n(\lambda_1)$ converges in distribution, not in probability or almost surely.

The Periodogram

To obtain a consistent estimator, a local averaging process is used: define

$$\hat{f}_n(\lambda) = \frac{1}{2\pi} \sum_{|j| \leq m_n} w_n(j) \hat{I}_n(g(n, \lambda) + 2\pi j/n)$$

where *bandwidths* m_n satisfy $m_n \rightarrow \infty$, $m_n/n \rightarrow 0$ as $n \rightarrow \infty$, and weights $w_n(j)$ satisfy $w_n(j) = w_n(-j) \geq 0$,

$$\sum_{|j| \leq m_n} w_n(j) = 1$$

and $w_n^2(j) \rightarrow 0$ as $n \rightarrow \infty$.

Example: $m_n = \sqrt{n}$, $w_n(j) = 1/(2m_n + 1)$ for $|j| \leq m_n$.

7.4 Likelihood-based estimation

For parametric models such as ARMA models with Gaussian errors, likelihood-based inference can be carried out using a large sample approximation to the time domain likelihood known as the *Whittle likelihood*.

Recall that for the periodogram, for any λ , as $n \rightarrow \infty$

$$\hat{I}_n(\lambda) \xrightarrow{d} V \sim \text{Exponential}(1/(2\pi f_X(\lambda)))$$

If $\lambda_1, \dots, \lambda_n$ are chosen to be the Fourier frequencies, then

$$V_1, \dots, V_n$$

are approximately independent.

7.4 Likelihood-based estimation

The Whittle likelihood is then

$$\mathcal{L}(\theta) = \prod_j \frac{1}{2\pi f_x(\omega_j; \theta)} \exp \left\{ -\frac{I_n(\omega_j)}{2\pi f_x(\omega_j; \theta)} \right\}$$

where $\omega_j = 2\pi j/n$, and

$$j \in \left\{ -\left\lfloor \frac{n-1}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\}$$

This (asymptotic) likelihood can then be maximized to estimate θ .

7.5 Long memory models

Long memory time series models exhibit slow decay in autocorrelation. For ARMA models, the decay in acf is exponentially fast, that is, there exists $r > 1$ such that as $h \rightarrow \infty$,

$$r^h \rho_X(h) \rightarrow 0$$

Let $\{X_t\}$ be a stationary process such that for some $\alpha \in (0, 1)$ and constant $c > 0$ such that

$$\lim_{k \rightarrow \infty} \frac{\rho_X(k)}{ck^{-\alpha}} = 1$$

Then $\{X_t\}$ is a *long memory* or *long range dependent* process.

Fractional Integration

Recall the $\text{ARIMA}(p, d, q)$ model, for $d = 1, 2, \dots$,

$$(1 - B)^d \Phi(B) X_t = \Theta(B) Z_t$$

Now suppose that $-0.5 < d < 0.5$ is a non-integer value. We have

$$(1 - B)^d = \sum_{j=0}^{\infty} \pi_j B_j$$

where, for $j = 1, 2, \dots$

$$\pi_j = \prod_{k=1}^j \left(\frac{k-1-d}{k} \right) = \frac{\Gamma(d+1)}{\Gamma(j+1)\Gamma(d-j+1)}$$

Fractional Integration

Let $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, and $W_t = (1 - B)^d Z_t$, so that

$$\Phi(B)X_t = \Theta(B)W_t$$

gives an ARMA(p, q) model driven by $\{W_t\}$.

$\{W_t\}$ is *fractionally integrated white noise*. It is a stationary process with

$$\gamma_W(0) = \frac{\sigma^2 \Gamma(1 - 2d)}{\{\Gamma(1 - d)\}^2}$$

and

$$\rho_W(h) = \frac{\Gamma(h + d)\Gamma(1 - d)}{\Gamma(h - d + 1)\Gamma(d)} = \prod_{k=1}^h \left(\frac{k - 1 - d}{k} \right)$$

The acvf of $\{X_t\}$ can be computed using the usual approach

$$\gamma_X(h) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_j \psi_k \gamma_W(h + j - k)$$

where $\{\psi_j\}$ are the linear process coefficients corresponding to the ARMA process with polynomials $\Phi(B)$ and $\Theta(B)$.

Spectral density for Fractional ARIMA

If $\{X_t\} \sim \text{ARIMA}(p, d, q)$ with $-1/2 < d < 1/2$, then

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\Theta(e^{-i\lambda})|^2}{|\Phi(e^{-i\lambda})|^2} \frac{1}{|1 - e^{-i\lambda}|^{2d}}$$

If $p = q = 0$,

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 - e^{-i\lambda}|^{2d}}$$

and as $\lambda \rightarrow 0$

$$f_X(\lambda) \approx \frac{\sigma^2}{2\pi|\lambda|^{2d}}$$

that is, the sdf is unbounded at zero.

Fractional White Noise

If $\{W_t\} \sim \text{ARIMA}(0, d, 0)$, that is,

$$W_t = (1 - B)^d Z_t$$

then

- $\text{MA}(\infty)$ representation

$$W_t = \sum_{k=0}^{\infty} \psi_k Z_{t-k} \quad \psi_k = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)}$$

- $\text{AR}(\infty)$ representation

$$Z_t = \sum_{k=0}^{\infty} \pi_k W_{t-k} \quad \pi_k = \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)}$$

Fractional White Noise

For large k

- MA(∞) representation

$$\psi_k \approx \frac{k^{d-1}}{\Gamma(d)}$$

- AR(∞) representation

$$\pi_k \approx \frac{k^{-d-1}}{\Gamma(-d)}$$

For $0 < d < 1/2$, if $\{X_t\} \sim \text{ARIMA}(p, d, q)$,

$$(1 - B)^d \Phi(B) X_t = \Theta(B) Z_t$$

then, for large h

$$\gamma_X(h) \approx C|h|^{2d-1}$$