Parameterization of Factor Predictor Models

What the models and the parameters mean

Example

For example: $M_1 = 4, M_2 = 3$.

- Factor X_1: levels $1, 2, \ldots, M_1$
  - indices will be $j = 0, 1, \ldots, M_1 - 1$
- Factor X_2: levels $1, 2, \ldots, M_2$
  - indices will be $l = 0, 1, \ldots, M_2 - 1$

Example (cont.)

Most complicated possible model: Main Effects plus Interaction

$$1 + X_1 + X_2 + X_1 : X_2$$

(or equivalently $X_1 + X_2 + X_1 : X_2$) that is, we have

- a baseline mean: β_0
- a contrast for each non-baseline level of Factor X_1: β^{C}_{1j}
- a contrast for each non-baseline level of Factor X_2: β^{C}_{2l}
- an interaction that modifies the effect of changing levels of Factor X_1 at each level of Factor X_2: β^{C}_{12jl}

Example

The modelled mean is therefore

$$\beta_0 + \underbrace{\sum_{j=1}^{M_1-1} \beta^{C}_{1j} \mathbb{1}_j(x_{i1})}_{\text{main effect of } X_1} + \underbrace{\sum_{l=1}^{M_2-1} \beta^{C}_{2l} \mathbb{1}_l(x_{i2})}_{\text{main effect of } X_2} + \underbrace{\sum_{j=1}^{M_1-1} \sum_{l=1}^{M_2-1} \beta^{C}_{12jl} \mathbb{1}_j(x_{i1}) \mathbb{1}_l(x_{i2})}_{\text{interaction}} .$$

For any i, the intercept β_0 is always present, but there is **at most** one contribution from each of the three summations.

Two-way table: 4×3

Factor X_2

Factor X_1	0	1	2
0			
1			
2			
3			

Null Model : Baseline Mean Only

Factor X_2

Factor X_1	0	1	2
0	β_0	β_0	β_0
1	β_0	β_0	β_0
2	β_0	β_0	β_0
3	β_0	β_0	β_0

Null Model: cell entries are modelled means for data for each factor level combination.

Effect of Factor X_1 only

Factor X_2

Factor X_1	0	1	2
0	β_0	β_0	β_0
1	$\beta_0 + \beta_{11}^C$	$\beta_0 + \beta_{11}^C$	$\beta_0 + \beta_{11}^C$
2	$\beta_0 + \beta_{12}^C$	$\beta_0 + \beta_{12}^C$	$\beta_0 + \beta_{12}^C$
3	$\beta_0 + \beta_{13}^C$	$\beta_0 + \beta_{13}^C$	$\beta_0 + \beta_{13}^C$

Main Effect Only: X_1

Effect of Factor X_2 only

Factor X_2

Factor X_1	0	1	2
0	β_0	$\beta_0 + \beta_{21}^{C}$	$\beta_0 + \beta_{22}^{C}$
1	β_0	$\beta_0 + \beta_{21}^{C}$	$\beta_0 + \beta_{22}^{C}$
2	β_0	$\beta_0 + \beta_{21}^{C}$	$\beta_0 + \beta_{22}^{C}$
3	β_0	$\beta_0 + \beta_{21}^{C}$	$\beta_0 + \beta_{22}^{C}$

Main Effect Only: X_2

Effect of Factor X_1 plus Effect of Factor X_2

Factor X_2

Factor X_1		0	1	2
	0	β_0	$\beta_0 + \beta^C_{21}$	$\beta_0 + \beta^C_{22}$
	1	$\beta_0 + \beta^C_{11}$	$\beta_0 + \beta^C_{11} + \beta^C_{21}$	$\beta_0 + \beta^C_{11} + \beta^C_{22}$
	2	$\beta_0 + \beta^C_{12}$	$\beta_0 + \beta^C_{12} + \beta^C_{21}$	$\beta_0 + \beta^C_{12} + \beta^C_{22}$
	3	$\beta_0 + \beta^C_{13}$	$\beta_0 + \beta^C_{13} + \beta^C_{21}$	$\beta_0 + \beta^C_{13} + \beta^C_{22}$

Main Effects Only: $X_1 + X_2$

Main effects plus Interaction between A and B

Factor X_2

Factor X_1	0	1	2
0	β_0	$\beta_0 + \beta^C_{21}$	$\beta_0 + \beta^C_{22}$
1	$\beta_0 + \beta^C_{11}$	$\beta_0 + \beta^C_{11} + \beta^C_{21} + \beta^C_{1211}$	$\beta_0 + \beta^C_{11} + \beta^C_{22} + \beta^C_{1212}$
2	$\beta_0 + \beta^C_{12}$	$\beta_0 + \beta^C_{12} + \beta^C_{21} + \beta^C_{1221}$	$\beta_0 + \beta^C_{12} + \beta^C_{22} + \beta^C_{1222}$
3	$\beta_0 + \beta^C_{13}$	$\beta_0 + \beta^C_{13} + \beta^C_{21} + \beta^C_{1231}$	$\beta_0 + \beta^C_{13} + \beta^C_{22} + \beta^C_{1232}$

Main Effects Plus Interaction: $X_1 + X_2 + X_1 : X_2$.

‘Illegal’ models

Q. Why are the following models

- $X_1 : X_2$
- $X_1 + X_1 : X_2$
- $X_2 + X_1 : X_2$

not considered ?

A. Because they make specific and perhaps **unrealistic** assumptions about the data, and they imply that the levels of the factors are **not arbitrarily labelled**.

Therefore, although it is possible *in general* to fit such models, it is no longer possible to talk of the “effect of Factor X_1” etc.

‘Illegal’ models

Recall the definition of interaction:

- Variation in the effect of changing levels of one factor at the different levels of the other factor.
- For example, the effect on the response mean of moving from level 1 to level 2 for Factor X_2 is **different** at different levels of Factor X_1.

Consider the model

$$X_1 : X_2$$

this model implies that all parameters apart from the **baseline** and the **interaction** parameters are zero.

Interaction between X_1 and X_2 only

Factor X_2

Factor X_1	0	1	2
0	β_0	$\beta_0 \quad + 0$	$\beta_0 \quad + 0$
1	$\beta_0 + 0$	$\beta_0 + 0 + 0 + \beta^C_{1211}$	$\beta_0 + 0 + 0 + \beta^C_{1212}$
2	$\beta_0 + 0$	$\beta_0 + 0 + 0 + \beta^C_{1221}$	$\beta_0 + 0 + 0 + \beta^C_{1222}$
3	$\beta_0 + 0$	$\beta_0 + 0 + 0 + \beta^C_{1231}$	$\beta_0 + 0 + 0 + \beta^C_{1232}$

- for Factor X_1, Level 1 ($j = 0$): the effect of moving from Level 2 ($l = 1$) to Level 1 ($l = 0$) of factor X_2 is **zero**
- for Factor X_1, Level 2 ($j = 1$): the effect of moving from Level 2 ($l = 1$) to Level 1 ($l = 0$) of factor X_2 is β^C_{1211}.

Therefore, there is a **fundamental difference** between the way that we regard the levels of Factor X_1.

Main Effects of Factor X_1 Plus Interaction: $X_1 + X_1 : X_2$.

Factor X_2

Factor X_1	0	1	2
0	β_0	$\beta_0 + 0$	$\beta_0 + 0$
1	$\beta_0 + \beta^C_{11}$	$\beta_0 + \beta^C_{11} + 0 + \beta^C_{1211}$	$\beta_0 + \beta^C_{11} + 0 + \beta^C_{1212}$
2	$\beta_0 + \beta^C_{12}$	$\beta_0 + \beta^C_{12} + 0 + \beta^C_{1221}$	$\beta_0 + \beta^C_{12} + 0 + \beta^C_{1222}$
3	$\beta_0 + \beta^C_{13}$	$\beta_0 + \beta^C_{13} + 0 + \beta^C_{1231}$	$\beta_0 + \beta^C_{13} + 0 + \beta^C_{1232}$

- for Factor X_1, Level 1 ($j = 0$): the effect of moving from Level 2 ($l = 1$) to Level 1 ($l = 0$) of factor X_2 is **zero**
- for Factor X_1, Level 2 ($j = 1$): the effect of moving from Level 2 ($l = 1$) to Level 1 ($l = 0$) of factor X_2 is β^C_{1211}.

Therefore, there is a **fundamental difference** between the way that we regard the levels of Factor X_1. If we rearrange the labels of the levels of Factor X_1, **we may get a different result.**