

Bias Correction in Non-Differentiable Estimating Equations for Optimal Dynamic Regimes

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Asymptotic bias calculation for two intervals It is useful in the ensuing discussion to express the blip as $\gamma_j(\bar{L}_j, \bar{A}_j; \psi_j) = A_j g_j(\bar{L}_j, \bar{A}_{j-1}; \psi_j)$. Using this form, the optimal rule – and the definition of exceptional laws – can be written in terms of the function $g_j(\bar{L}_j, \bar{A}_{j-1}; \psi_j)$. For binary treatment, the optimal rule is defined by the indicator $I[g_j(\bar{L}_j, \bar{A}_{j-1}; \psi_j) > 0]$. Exceptional laws exist when $P(g_j(\bar{L}_j, \bar{A}_{j-1}; \psi_j) = 0) > 0$ and $g_j(\bar{L}_j, \bar{A}_{j-1}; \psi_j)$ depends on at least one component of $\bar{L}_j, \bar{A}_{j-1}$. Robins (2004, p. 315) calculates the asymptotic bias of $\hat{\psi}_1(\psi_2)$ found using equation (??) for linear blips in the special case where exceptional laws occur when $L_j = A_{j1} = 1$. We generalize the result here, and provide the asymptotic bias for (??) in Appendix A:

Suppose a linear blip where $g_2(\bar{L}_2, A_1; \psi_2)$ depends on at least one of L_1, L_2, A_1 so that $g_2(\bar{L}_2, A_1; \psi_2) = 0$ whenever $\sum_{j=0}^{|\psi_2|-1} \psi_{2j} \cdot X_2 = 0$ where $|\psi_2|$ is the dimension of the vector ψ_2 and X_2 denotes the matrix of covariates (in our simulations, $X_2 = (1, L_2, A_1, L_2 A_1)$). Let $\psi_1^\dagger, \psi_2^\dagger$ denote the true values of ψ_1 and ψ_2 and let $\Delta_1^g(\hat{\psi}_2, \psi_2^\dagger) = U_1^*(\psi_1, \hat{\psi}_2) - U_1^*(\psi_1, \psi_2^\dagger)$. As indicated by the notation, $\Delta_1^g(\hat{\psi}_2, \psi_2^\dagger)$ does not depend on ψ_1 when blips are linear. For linear blip functions,

$$\begin{aligned}
\Delta_1^g(\hat{\psi}_2, \psi_2^\dagger) &= I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) = 0] \{S_1(A_1) - E[S_1(A_1)|L_1]\} \cdot \\
&\quad \left(I[g_2(\bar{L}_2, A_1; \hat{\psi}_2) > 0] - A_2 \right) \left(g_2(\bar{L}_2, A_1; \hat{\psi}_2) - g_2(\bar{L}_2, A_1; \psi_2^\dagger) \right) \\
&\quad + I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) \neq 0] \{S_1(A_1) - E[S_1(A_1)|L_1]\} \cdot \\
&\quad \left\{ \left(I[g_2(\bar{L}_2, A_1; \hat{\psi}_2) > 0] - A_2 \right) g_2(\bar{L}_2, A_1; \hat{\psi}_2) - \right. \\
&\quad \left. \left(I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) > 0] - A_2 \right) g_2(\bar{L}_2, A_1; \psi_2^\dagger) \right\} \\
&= I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) = 0] \{S_1(A_1) - E[S_1(A_1)|L_1]\} \cdot \\
&\quad \left(I[(\hat{\psi}_2 - \psi_2^\dagger) \cdot X_2 > 0] - A_2 \right) (\hat{\psi}_2 - \psi_2^\dagger) \cdot X_2 + \\
&\quad I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) \neq 0] \{S_1(A_1) - E[S_1(A_1)|L_1]\} \cdot \\
&\quad \left\{ \left(I[g_2(\bar{L}_2, A_1; \hat{\psi}_2) > 0] - A_2 \right) g_2(\bar{L}_2, A_1; \hat{\psi}_2) - \right. \\
&\quad \left. \left(I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) > 0] - A_2 \right) g_2(\bar{L}_2, A_1; \psi_2^\dagger) \right\}, \tag{1}
\end{aligned}$$

where $(\hat{\psi}_2 - \psi_2^\dagger) \cdot X_2 = g_2(\bar{L}_2, A_1; \hat{\psi}_2) - g_2(\bar{L}_2, A_1; \psi_2^\dagger)$. The second equality follows since we may add a zero (in the form of $g_2(\bar{L}_2, A_1; \psi_2^\dagger)$) inside the LHS of the inequality in the indicator $I[g_2(\bar{L}_2, A_1; \hat{\psi}_2) > 0]$. Since $\hat{\psi}_2$ is regular, a Taylor expansion gives

$$\begin{aligned} (\hat{\psi}_2 - \psi_2^\dagger) &= - \left(E \left[\frac{\partial}{\partial \psi_2} U_2^*(\psi_2^\dagger) \right] \right)^{-1} U_2^*(\psi_2^\dagger) + o_p(n^{-1/2}) \\ &= (IF_{20}, IF_{21}, \dots, IF_{2, |\psi_2| - 1})^T + o_p(n^{-1/2}) = IF_2 + o_p(n^{-1/2}), \end{aligned}$$

where IF_{2j} denotes the influence function for $- \left(E \left[\frac{\partial}{\partial \psi_{2j}} U_2^*(\psi_2^\dagger) \right] \right)^{-1} U_2^*(\psi_2^\dagger)$ and IF_2 is the vector of such functions. This can be substituted into $\Delta_1^g(\hat{\psi}_2, \psi_2^\dagger)$, giving

$$\begin{aligned} \sqrt{n} \mathbb{P}_n \Delta_1^g(\hat{\psi}_2, \psi_2^\dagger) &= E \{ I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) = 0] \{ S_1(A_1) - E[S_1(A_1)|L_1] \} \cdot \\ &\quad (I[\sqrt{n} \mathbb{P}_n(IF_2 \cdot X_2) > 0] - A_2) \sqrt{n} \mathbb{P}_n(IF_2 \cdot X_2) \} + \\ &\quad \{ I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) \neq 0] \{ S_1(A_1) - E[S_1(A_1)|L_1] \} \cdot \\ &\quad (I[g_2(\bar{L}_2, A_1; \hat{\psi}_2) > 0] - A_2) \sqrt{n} \mathbb{P}_n(IF_2 \cdot X_2) \} + o_p(1). \end{aligned} \quad (2)$$

It can be shown that $g_2(\bar{L}_2, A_1; \hat{\psi}_2)$ and $g_2(\bar{L}_2, A_1; \psi_2^\dagger)$ have the same sign with probability converging to 1, so we follow ? in replacing $I[g_2(\bar{L}_2, A_1; \hat{\psi}_2) > 0]$ with $I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) > 0]$. The equality of equations (1) and (2) then follows by noting that when blips are linear, $g_2(\bar{L}_2, A_1; \hat{\psi}_2) - g_2(\bar{L}_2, A_1; \psi_2^\dagger) = \hat{\psi}_2 \cdot X_2 - \psi_2^\dagger \cdot X_2$.

The vector of influence functions scaled by covariates, $IF_2 \cdot X_2$, has mean zero. Let Z_{2+} denote $\sum_{j=0}^{|\psi_2| - 1} IF_{2j} X_{2j}$. A further Taylor expansion gives

$$\begin{aligned} (\hat{\psi}_1 - \psi_1^\dagger) &= - \left(E \left[\frac{\partial}{\partial \psi_1} U_1^*(\psi_1^\dagger, \psi_2^\dagger) \right] \right)^{-1} U_1^*(\psi_1^\dagger, \hat{\psi}_2) + o_p(n^{-1/2}) \\ &= - \left(E \left[\frac{\partial}{\partial \psi_1} U_1^*(\psi_1^\dagger, \psi_2^\dagger) \right] \right)^{-1} (U_1^*(\psi_1^\dagger, \psi_2^\dagger) + \Delta_1^g(\hat{\psi}_2, \psi_2^\dagger)) + o_p(n^{-1/2}). \end{aligned}$$

From this, we find that the asymptotic bias of $\hat{\psi}_1$ equals

$$-\left(E\left[\frac{\partial}{\partial\psi_1}U_1^*(\psi_1^\dagger, \psi_2^\dagger)\right]\right)^{-1} E\left\{I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) = 0] \{S_1(A_1) - E[S_1(A_1)|L_1]\} (I[Z_{2+} > 0] - A_2) Z_{2+} + I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) \neq 0] \{S_1(A_1) - E[S_1(A_1)|L_1]\} (I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) > 0] - A_2) (IF_2 \cdot X_2)\right\}.$$

As noted above, $IF_2 \cdot X_2$ has mean zero and hence $Z_{2+} = \sum_{j=0}^{|\psi_2|-1} IF_{2j} X_{2j}$ has mean zero also ($I[Z_{2+} > 0] Z_{2+}$ does not). Thus, the above expression can be further reduced to

$$-\left(E\left[\frac{\partial}{\partial\psi_1}U_1^*(\psi_1^\dagger, \psi_2^\dagger)\right]\right)^{-1} E\{I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) = 0] \{S_1(A_1) - E[S_1(A_1)|L_1]\} I[Z_{2+} > 0] Z_{2+}\}.$$

When optimal regimes are unique, $I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) = 0] = 0$ and so the asymptotic bias is zero. When optimal regimes are not unique, $I[g_2(\bar{L}_2, A_1; \psi_2^\dagger) = 0] = 1$ and the g-estimates have bounded asymptotic bias at exceptional laws equal to

$$-\left(E\left[\frac{\partial}{\partial\psi_1}U_1^*(\psi_1^\dagger, \psi_2^\dagger)\right]\right)^{-1} E\{S_1(A_1) - E[S_1(A_1)|L_1]\} \cdot E[Z_{2+}|Z_{2+} > 0]P[Z_{2+} > 0].$$

For more than two intervals, the asymptotic bias calculation is more complex. Consider a three-interval problem where non-unique optimal rules exist at the last interval. The asymptotic bias at the second (middle) interval proceed as above, so that the asymptotic bias in $\hat{\psi}_2$ is

$$-\left(E\left[\frac{\partial}{\partial\psi_2}U_2^*(\psi_1, \psi_2^\dagger, \psi_3^\dagger)\right]\right)^{-1} E\{S_2(A_2) - E[S_2(A_2)|L_1, A_1, L_2]\} \cdot E[Z_{3+}|Z_{3+} > 0]P[Z_{3+} > 0].$$

Note that in the recursive estimation setting, $U_2^*(\psi_1, \psi_2, \psi_3) = U_2^*(\psi_2, \psi_3)$. The asymptotic bias calculation at the first interval can be found by examining $\Delta_1^g(\hat{\psi}_2, \psi_2^\dagger, \hat{\psi}_3, \psi_3^\dagger) = U_1^*(\psi_1, \hat{\psi}_2, \hat{\psi}_3) - U_1^*(\psi_1, \psi_2^\dagger, \psi_3^\dagger)$. This function is more complex than that derived above because it requires plug-in estimates of both the regular estimator ψ_3 and the asymptotically biased, non-regular estimator ψ_2 .