

MATH 556 - EXERCISES 3

These exercises are not for assessment

1. Suppose that X is a continuous rv with pdf f_X and characteristic function (cf) C_X . Find $C_X(t)$ if

(a)

$$f_X(x) = \frac{1}{2}|x| \exp\{-|x|\} \quad x \in \mathbb{R}.$$

(b)

$$f_X(x) = \exp\{-x - e^{-x}\} \quad x \in \mathbb{R}.$$

(c)

$$f_X(x) = \frac{1}{\cosh(\pi x)} = \frac{2}{e^{-\pi x} + e^{\pi x}} = \sum_{k=0}^{\infty} (-1)^k \exp\{-(2k+1)\pi|x|\} \quad x \in \mathbb{R}.$$

Leave your answer as an infinite sum if necessary.

2. Find $f_X(x)$ if the cf is given by

$$C_X(t) = 1 - |t| \quad -1 < t < 1$$

and zero otherwise.

3. Suppose that random variable Y has cf C_Y . Find the distribution of Y if

(a)

$$C_Y(t) = \frac{2(1 - \cos t)}{t^2} \quad t \in \mathbb{R}.$$

(b)

$$C_Y(t) = \cos(\theta t) \quad t \in \mathbb{R}.$$

for some parameter $\theta > 0$.

4. By considering derivatives at $t = 0$, and the implied moments, assess whether the function

$$C(t) = \frac{1}{1 + t^4}$$

is a valid cf for a pmf or pdf.

5. Prove that if f_X is pdf for a continuous random variable, then

$$|C_X(t)| \longrightarrow 0 \quad \text{as} \quad |t| \longrightarrow \infty.$$

Use the fact that f_X can be approximated to arbitrary accuracy by a step-function; for each $\epsilon > 0$, there exists a step-function $g_\epsilon(x)$ defined as

$$g_\epsilon(x) = \sum_{k=1}^K c_k I_{A_k}(x)$$

where $c_k, k = 1, \dots, K$ are real constants, A_1, \dots, A_k form a partition of \mathbb{R} , such that

$$\int_{-\infty}^{\infty} |f_X(x) - g_\epsilon(x)| dx < \epsilon.$$

The function $I_A(x)$ is an **indicator function** for set A

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

6. Suppose that X_1, \dots, X_n are independent and identically distributed Cauchy rvs each with

$$f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad x \in \mathbb{R} \quad C_X(t) = \exp\{-|t|\} \quad t \in \mathbb{R}.$$

Let continuous random variable Z_n be defined by

$$Z_n = \frac{1}{\bar{X}} = \frac{n}{\sum_{j=1}^n X_j}.$$

Find an expression for $P[|Z_n| \leq c]$ for constant $c > 0$.

7. A probability distribution for rv X is termed *infinitely divisible* if, for all positive integers n , there exists a sequence of independent and identically distributed rvs Z_{n1}, \dots, Z_{nn} such that X and

$$Z_n = \sum_{j=1}^n Z_{nj}$$

have the same distribution, that is, the characteristic function of X can be written

$$C_X(t) = \{C_Z(t)\}^n$$

for some characteristic function C_Z . Show that the *Gamma*(α, β) distribution is infinitely divisible.