MATH 556 - ASSIGNMENT 3
SOLUTIONS

1 (a) (i) This is not an Exponential Family distribution; the support is parameter dependent.

1 MARK

(ii) This is an EF distribution with $k = 1$:

$$f(x|\theta) = \frac{I_{\{1,2,3,\ldots\}}(x)}{x} \frac{-1}{\log(1-\theta)} \exp\{x \log \theta\} = h(x)c(\theta)\exp\{w(\theta)t(x)\}$$

where $$h(x) = \frac{I_{\{1,2,3,\ldots\}}(x)}{x} \qquad c(\theta) = \frac{-1}{\log(1-\theta)} \qquad w(\theta) = \log(\theta) \qquad t(x) = x,$$

so the natural parameter is $\eta = \log(\theta)$.

2 MARKS

(iii) This is an EF distribution with $k = 2$:

$$\begin{aligned} f(x|\phi,\lambda) &= \frac{I_{(0,\infty)}(x)}{(2\pi x^3)^{1/2}} \sqrt{\lambda} e^{\phi} \exp\left\{-\frac{\phi^2}{2\lambda}x - \frac{\lambda}{2}\frac{1}{x}\right\} \\ &= h(x)c(\phi,\lambda)\exp\{w_1(\phi,\lambda)t_1(x) + w_2(\phi,\lambda)t_2(x)\} \end{aligned}$$

where

$$h(x) = \frac{I_{(0,\infty)}(x)}{(2\pi x^3)^{1/2}} \qquad c(\phi,\lambda) = \sqrt{\lambda} e^{\phi}$$

and

$$w_1(\phi,\lambda) = -\frac{\phi^2}{2\lambda} \qquad w_2(\phi,\lambda) = -\frac{\lambda}{2} \qquad t_1(x) = x \qquad t_2(x) = \frac{1}{x},$$

so the natural parameter is $\underset{\sim}{\eta} = (\eta_1, \eta_2)^\mathsf{T}$ where

$$\eta_1 = -\phi^2/2\lambda \qquad \eta_2 = -\lambda/2$$

2 MARKS

In the natural parameterization

$$c^\star(\eta_1,\eta_2) = \sqrt{-2\eta_2}\exp\{2\sqrt{\eta_1\eta_2}\}$$

so, using the results from lectures

$$E_{f_X}[1/X] = E_{f_X}[t_2(X)] = -\frac{\partial}{\partial \eta_2}\log c^\star(\eta_1,\eta_2).$$

We have

$$\log c^\star(\eta_1,\eta_2) = \frac{1}{2}\log(-2\eta_2) + 2\sqrt{\eta_1\eta_2}$$

and hence

$$\begin{aligned} E_{f_X}[1/X] &= -\frac{\partial}{\partial \eta_2}\left\{\frac{1}{2}\log(-2\eta_2) + 2\sqrt{\eta_1\eta_2}\right\} = -\left\{\frac{1}{2}\frac{1}{-2\eta_2}(-2) + 2\sqrt{\frac{\eta_1}{\eta_2}}\frac{1}{2}\right\} \\ &= -\frac{1}{2\eta_2} - \sqrt{\frac{\eta_1}{\eta_2}} = \frac{1}{\lambda} + \frac{\phi}{\lambda} \end{aligned}$$

3 MARKS

(b) (i) We can re-write f_X as

$$f_X(x|\eta) = h(x)\exp\{\eta t(x) - \kappa(\eta)\}$$

where $\kappa(\eta) = -\log c^\star(\eta)$, and by integrating with respect to x, we note that

$$\int h(x)\exp\{\eta t(x)\}\ dx = \exp\{\kappa(\eta)\}$$

for $\eta \in \mathcal{H}$ as given in lectures. Thus, for s in a suitable neighbourhood of zero, we have

$$\begin{aligned} M_T(s) &= E_{f_X}[e^{st(X)}] = \int e^{st(x)} h(x)\exp\{\eta t(x) - \kappa(\eta)\}\ dx \\ &= \exp\{-\kappa(\eta)\}\int h(x)\exp\{t(x)(\eta+s)\}\ dx = \exp\{-\kappa(\eta)\}\exp\{\kappa(\eta+s)\} \end{aligned}$$

as $\eta \in \mathcal{H} \Longrightarrow \eta + s \in \mathcal{H}$ for s small enough, as $\mathcal{H}$ is open. Hence, as $K_T(s) = \log M_T(s)$,

$$K_T(s) = \kappa(\eta+s) - \kappa(\eta)$$

for $s \in (-h, h),$ some $h > 0$ as required.

4 MARKS

(ii) By inspection

$$\ell(x;\eta_1,\eta_2) = (\eta_1 - \eta_2)t(x) - (\kappa(\eta_1) - \kappa(\eta_2))$$

2 MARKS

2 By iterated expectation

$$E_{f_{X_1}}[X_1] = E_{f_M}\left[E_{f_{X_1|M}}[X_1|M=m]\right] = E_{f_M}[M] = \mu$$

and

$$E_{f_{X_1}}[X_1^2] = E_{f_M}\left[E_{f_{X_1|M}}[X_1^2|M=m]\right] = E_{f_M}\left[M^2+\sigma^2\right] = \mu^2+\tau^2+\sigma^2$$

so that

$$Var_{f_{X_1}}[X_1] = E_{f_{X_1}}[X_1^2] - \{E_{f_{X_1}}[X_1]\}^2 = \tau^2+\sigma^2.$$

By symmetry

$$E_{f_{X_2}}[X_2] = \mu \qquad Var_{f_{X_2}}[X_2] = \tau^2+\sigma^2.$$

Now,

$$E_{f_{X_1,X_2}}[X_1X_2] = E_{f_M}\left[E_{f_{X_1,X_2|M}}[X_1X_2|M=m]\right] = E_{f_M}\left[E_{f_{X_1|M}}[X_1|M=m]\times E_{f_{X_2|M}}[X_2|M=m]\right]$$

by conditional independence. Therefore

$$E_{f_{X_1,X_2}}[X_1X_2] = E_{f_M}[M\times M] = E_{f_M}\left[M^2\right] = \mu^2+\tau^2$$

Hence

$$Cov_{f_{X_1,X_2}}[X_1,X_2] = E_{f_{X_1,X_2}}[X_1X_2] - E_{f_{X_1}}[X_1]E_{f_{X_2}}[X_2] = \mu^2+\tau^2-\mu^2 = \tau^2$$

and

$$Corr_{f_{X_1,X_2}}[X_1,X_2] = \frac{Cov_{f_{X_1,X_2}}[X_1,X_2]}{\sqrt{Var_{f_{X_1}}[X_1]Var_{f_{X_2}}[X_2]}} = \frac{\tau^2}{\tau^2+\sigma^2}$$

5 MARKS

X_1 and X_2 are not independent; their covariance is non zero.

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