556: MATHEMATICAL STATISTICS I

DEFINITIONS AND NOTATION FROM REAL ANALYSIS

DEFINITION: Limits of sequences of reals

Sequence $\{a_n\}$ has limit a as $n \longrightarrow \infty$, written

$$\lim_{n\longrightarrow\infty} a_n = a$$

if, for every $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that

$$|a_n - a| < \epsilon$$

for all $n > N$.

DEFINITION: Limits of functions

Let f be a real-valued function of real argument x.

- Limit as $x \longrightarrow \infty$:

$$f(x) \longrightarrow a \quad \text{as} \quad x \longrightarrow \infty$$

  or

$$\lim_{x\longrightarrow\infty} f(x) = a$$

  if, every $\epsilon > 0$, $\exists M = M(\epsilon)$ such that $|f(x) - a| < \epsilon$, $\forall\ x > M$

- Limit as $x \longrightarrow x_0^{\pm}$:

$$f(x) \longrightarrow a \quad \text{as} \quad x \longrightarrow x_0^{\pm}$$

  or

$$\lim_{x\longrightarrow x_0^{\pm}} f(x) = a$$

  if, for all $\epsilon > 0$, $\exists\ \delta$ such that $|f(x) - a| < \epsilon$, $\forall\ x_0 < x < x_0 + \delta$ (or, respectively $x_0 - \delta < x < x_0$).

- Left/Right Limit as $x \longrightarrow x_0$:

$$f(x) \longrightarrow a \quad \text{as} \quad x \longrightarrow x_0$$

  or

$$\lim_{x\longrightarrow x_0} f(x) = a$$

  if

$$\lim_{x\longrightarrow x_0^{+}} f(x) = \lim_{x\longrightarrow x_0^{-}} f(x) = a.$$

DEFINITION: Order Notation (little oh and big oh)

Consider $x \longrightarrow x_0$. Then write

$$f(x) \sim g(x) \quad \text{if} \quad \frac{f(x)}{g(x)} \longrightarrow 1 \quad \text{as} \quad x \longrightarrow x_0$$

$$f(x) = o(g(x)) \quad \text{if} \quad \frac{f(x)}{g(x)} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow x_0$$

$$f(x) = O(g(x)) \quad \text{if} \quad \frac{f(x)}{g(x)} \longrightarrow b \quad \text{as} \quad x \longrightarrow x_0$$

with similar notation for real sequences. For example

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = x + o(x)$$

as $x \longrightarrow 0$.

EXTREMUM LIMITS FOR SEQUENCES

DEFINITION: Supremum and Infimum

A set of real values S is **bounded above (bounded below)** if there exists a real number a (b) such that, for all $x \in S$, $x \leq a$ ($x \geq b$). The quantity a (b) is an **upper bound (lower bound)**. A real value a_L (b_U) is a **least upper bound (greatest lower bound)** if it is an upper bound (a lower bound) of S, and no other upper (lower) bound is smaller (larger) than a_L (b_U). We write

$$a_L = \sup S \qquad b_U = \inf S$$

for the a_L, the **supremum**, and b_U, the **infimum** of S.

If S comprises a sequence of elements $\{x_n\}$, then we can write

$$a_L = \sup_{x_n \in S} x_n \equiv \sup_n x_n \qquad b_U = \inf_{x_n \in S} x_n \equiv \inf_n x_n.$$

A sequence that is both bounded above and bounded below is termed **bounded**.

NOTE : Any bounded, monotone real sequence is **convergent**.

DEFINITION: Limit Superior and Limit Inferior

Suppose that $\{x_n\}$ is a bounded real sequence. Define sequences $\{y_k\}$ and $\{z_k\}$ by

$$y_k = \inf_{n \geq k} x_n \qquad z_k = \sup_{n \geq k} x_n$$

Then $\{y_k\}$ is a bounded non-decreasing sequence and $\{z_k\}$ is a bounded non-increasing sequence, and

$$\lim_{k \to \infty} y_k = \sup_k y_k \qquad \text{and} \qquad \lim_{k \to \infty} z_k = \inf_k z_k.$$

We define the **limit superior** (or **upper** limit, or $\limsup$) and the **limit inferior** (or **lower** limit, or $\liminf$) by

$$\limsup_{n \longrightarrow \infty} x_n = \lim_{k \to \infty} \sup_{n \geq k} x_n = \inf_k \sup_{n \geq k} x_n = \overline{\lim}\, x_n$$

$$\liminf_{n \longrightarrow \infty} x_n = \lim_{k \to \infty} \inf_{n \geq k} x_n = \sup_k \inf_{n \geq k} x_n = \underline{\lim}\, x_n$$

Then we have $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$ and $\lim x_n = x$ if and only if $\underline{\lim}\, x_n = x = \overline{\lim}\, x_n$.