

## 556: MATHEMATICAL STATISTICS I

### THE BOREL-CANTELLI LEMMA

#### DEFINITION Limsup and liminf events

Let  $\{E_n\}$  be a sequence of events in sample space  $\Omega$ . Then

$$E^{(S)} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$$

is the **limsup** event of the infinite sequence; event  $E^{(S)}$  occurs if and only if

- for all  $n \geq 1$ , there exists an  $m \geq n$  such that  $E_m$  occurs.
- infinitely many of the  $E_n$  occur.

Similarly, let

$$E^{(I)} = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m$$

is the **liminf** event of the infinite sequence; event  $E^{(I)}$  occurs if and only if

- there exists  $n \geq 1$ , such that for all  $m \geq n$ ,  $E_m$  occurs.
- only finitely many of the  $E_n$  do not occur.

#### THEOREM The Borel-Cantelli Lemma

Let  $\{E_n\}$  be a sequence of events in sample space  $\Omega$ . Then

(a) If

$$\sum_{n=1}^{\infty} P(E_n) < \infty, \quad \implies \quad P(E^{(S)}) = 0,$$

that is,

$$P[E_n \text{ occurs infinitely often}] = 0.$$

(b) If the events  $\{E_n\}$  are **independent**

$$\sum_{n=1}^{\infty} P(E_n) = \infty \quad \implies \quad P(E^{(S)}) = 1.$$

that is,

$$P[E_n \text{ occurs infinitely often}] = 1.$$

**Note:** This result is useful for assessing almost sure convergence. For a sequence of random variables  $\{X_n\}$  and limit random variable  $X$ , suppose, for  $\epsilon > 0$ , that  $A_n(\epsilon)$  is the event

$$A_n(\epsilon) \equiv \{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}$$

The BC Lemma says

(a) if  $\sum_{n=1}^{\infty} P(A_n(\epsilon)) = \sum_{n=1}^{\infty} P[|X_n - X| > \epsilon] < \infty$  then  $X_n \xrightarrow{a.s.} X$

(b) if  $\sum_{n=1}^{\infty} P(A_n(\epsilon)) = \sum_{n=1}^{\infty} P[|X_n - X| > \epsilon] = \infty$  with the  $X_n$  independent then  $X_n \not\xrightarrow{a.s.} X$

*Proof.* **NOT EXAMINABLE**

(a) Note first that

$$\sum_{n=1}^{\infty} P(E_n) < \infty \implies \lim_{n \rightarrow \infty} \sum_{m=n}^{\infty} P(E_m) = 0.$$

because if the sum on the left-hand side is finite, then the tail-sums on the right-hand side tend to zero as  $n \rightarrow \infty$ . But for every  $n \geq 1$ ,

$$E^{(S)} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m \subseteq \bigcup_{m=n}^{\infty} E_m \quad \therefore \quad P(E^{(S)}) \leq P\left(\bigcup_{m=n}^{\infty} E_m\right) \leq \sum_{m=n}^{\infty} P(E_m). \quad (1)$$

Thus, taking limits as  $n \rightarrow \infty$ , we have that

$$P(E^{(S)}) \leq \lim_{n \rightarrow \infty} \sum_{m=n}^{\infty} P(E_m) = 0.$$

(b) Consider  $N \geq n$ , and the union of events

$$E_{n,N} = \bigcup_{m=n}^N E_m.$$

$E_{n,N}$  corresponds to the collection of sample outcomes that are in *at least one* of the collections corresponding to events  $E_n, \dots, E_N$ . Therefore,  $E'_{n,N}$  is the collection of sample outcomes in  $\Omega$  that are **not in any** of the collections corresponding to events  $E_n, \dots, E_N$ , and hence

$$E'_{n,N} = \bigcap_{m=n}^N E'_m \quad (2)$$

Now,

$$E_{n,N} \subseteq \bigcup_{m=n}^{\infty} E_m \implies P(E_{n,N}) \leq P\left(\bigcup_{m=n}^{\infty} E_m\right)$$

and hence, by assumption and independence,

$$\begin{aligned} 1 - P\left(\bigcup_{m=n}^{\infty} E_m\right) &\leq 1 - P\left(\bigcup_{m=n}^N E_m\right) = 1 - P(E_{n,N}) = P(E'_{n,N}) = P\left(\bigcap_{m=n}^N E'_m\right) = \prod_{m=n}^N P(E'_m) \\ &= \prod_{m=n}^N (1 - P(E_m)) \leq \exp\left\{-\sum_{m=n}^N P(E_m)\right\}, \end{aligned}$$

as  $1 - x \leq \exp\{-x\}$  for  $0 < x < 1$ . Now, taking the limit of both sides as  $N \rightarrow \infty$ , for fixed  $n$ ,

$$1 - P\left(\bigcup_{m=n}^{\infty} E_m\right) \leq \lim_{N \rightarrow \infty} \exp\left\{-\sum_{m=n}^N P(E_m)\right\} = 0$$

as, by assumption  $\sum_{n=1}^{\infty} P(E_n) = \infty$ . Thus, for each  $n$ , we have that

$$P\left(\bigcup_{m=n}^{\infty} E_m\right) = 1 \quad \therefore \quad \lim_{n \rightarrow \infty} P\left(\bigcup_{m=n}^{\infty} E_m\right) = 1.$$

But the sequence of events  $\{A_n\}$  defined for  $n \geq 1$  by

$$A_n = \bigcup_{m=n}^{\infty} E_m$$

is monotone non-increasing, and hence, by continuity,

$$P\left(\lim_{n \rightarrow \infty} A_n\right) = \lim_{n \rightarrow \infty} P(A_n). \quad (3)$$

From (4), we have that the right hand side of equation (5) is equal to 1, and, by definition,

$$\lim_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m. \quad (4)$$

Hence, combining (4), (5) and (6) we have finally that

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m\right) = 1 \quad \implies \quad P\left(E^{(S)}\right) = 1.$$

■

### Interpretation and Implications

The Borel-Cantelli result is concerned with the calculation of the probability of the limsup event  $E^{(S)}$  occurring for general infinite sequences of events  $\{E_n\}$ . From previous discussion, we have seen that  $E^{(S)}$  corresponds to the collection of sample outcomes in  $\Omega$  that are in **infinitely many** of the  $E_n$  collections. Alternately,  $E^{(S)}$  occurs if and only if **infinitely many**  $\{E_n\}$  occur. The Borel-Cantelli result tells us conditions under which  $P(E^{(S)}) = 0$  or 1.

**EXAMPLE :** Consider the event  $E$  defined by

“ $E$  occurs” = “run of  $100^{100}$  Heads occurs in an infinite sequence of independent coin tosses”

We wish to calculate  $P(E)$ , and proceed as follows; consider the infinite sequence of events  $\{E_n\}$  defined by

“ $E_n$  occurs” = “run of  $100^{100}$  Heads occurs in the  $n$ th block of  $100^{100}$  coin tosses”

Then  $\{E_n\}$  are independent events, and

$$P(E_n) = \frac{1}{2^{100^{100}}} > 0 \implies \sum_{n=1}^{\infty} P(E_n) = \infty,$$

and hence by part (b) of the Borel-Cantelli result,

$$P\left(E^{(S)}\right) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m\right) = 1$$

so that the probability that infinitely many of the  $\{E_n\}$  occur is 1. But, crucially,

$$E^{(S)} \subseteq E \implies P(E) = 1.$$

Therefore the probability that  $E$  occurs, that is that a run of  $100^{100}$  Heads occurs in an infinite sequence of independent coin tosses, is 1.