

MATH 523 - ASSIGNMENT 1: SOLUTIONS

1 By direct calculation

$$\mathbb{E}_{f_Y}[Y] = \sum_{y=1}^{\infty} y \frac{e^{-\lambda} \lambda^y}{(1 - e^{-\lambda}) y!} = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{y=1}^{\infty} \frac{\lambda^{y-1}}{(y-1)!} = \frac{\lambda}{(1 - e^{-\lambda})}$$

Similarly

$$\mathbb{E}_{f_Y}[Y(Y-1)] = \sum_{y=1}^{\infty} y(y-1) \frac{e^{-\lambda} \lambda^y}{(1 - e^{-\lambda}) y!} = \frac{\lambda^2 e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{y=2}^{\infty} \frac{\lambda^{y-2}}{(y-2)!} = \frac{\lambda^2}{(1 - e^{-\lambda})}$$

so that

$$\begin{aligned} \text{Var}_{f_Y}[Y] &= \mathbb{E}_{f_Y}[Y(Y-1)] + \mathbb{E}_{f_Y}[Y] - \{\mathbb{E}_{f_Y}[Y]\}^2 \\ &= \frac{\lambda^2}{(1 - e^{-\lambda})} + \frac{\lambda}{(1 - e^{-\lambda})} - \left\{ \frac{\lambda}{(1 - e^{-\lambda})} \right\}^2 \\ &= \frac{\lambda(\lambda+1)(1 - e^{-\lambda}) + \lambda^2}{(1 - e^{-\lambda})^2} \end{aligned}$$

4 MARKS

Also

$$f_Y(y) = \mathbb{I}_{\{y \geq 1\}} \exp \left\{ y \log \lambda - (\lambda + \log(1 - e^{-\lambda})) \right\}$$

which is an Exponential Family distribution with canonical parameter $\theta = \log \lambda$, and function $b(\theta) = e^{\theta} + \log(1 - \exp\{-e^{\theta}\})$. The model can also be written

$$f_Y(y) = \mathbb{I}_{\{y \geq 1\}} \exp \left\{ y \log \lambda - \log(e^{\lambda} - 1) \right\}$$

with the equivalent formulation $b(\theta) = \log(\exp\{e^{\theta}\} - 1)$.

2 MARKS

2 (a) As for the Poisson model, $\mathbb{E}[Y_i] = \lambda_i = \mu(x_i\beta)$, the log-likelihood function is given by

$$l(\beta; \mathbf{y}) = \sum_{i=1}^n [-\mu(x_i\beta) + y_i \log \mu(x_i\beta) - \log y_i!]$$

so therefore, taking the partial derivative with respect to β_j , we have for $j = 1 \dots, p$

$$\frac{\partial l(\beta; \mathbf{y})}{\partial \beta_j} = \sum_{i=1}^n \left[-\dot{\mu}_j(x_i\beta) + y_i \frac{\dot{\mu}_j(x_i\beta)}{\mu_j(x_i\beta)} \right] = \sum_{i=1}^n \dot{\mu}_j(x_i\beta) \left[\frac{y_i}{\mu(x_i\beta)} - 1 \right]$$

4 MARKS

(b) The canonical link implies that $\mu(x\beta) = \exp\{x\beta\}$, so that

$$\dot{\mu}_j(x_i\beta) = x_{ij} \exp\{x\beta\} = x_{ij} \mu(x_i\beta)$$

so the estimating equations become

$$\frac{\partial l(\beta; \mathbf{y})}{\partial \beta_j} = \sum_{i=1}^n x_{ij} (y_i - \mu(x_i\beta)) \quad j = 1 \dots, p$$

4 MARKS

(c) The R code and output is given below

```
1 x<-c(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18)
2 y<-c(15,11,14,17,5,11,10,4,8,10,7,9,11,3,6,1,1,4)
3
4 summary(glm(y~x,family=poisson))
5 summary(glm(y~x+I(x^2),family=poisson))
```

Results for the straight-line model:

```
> summary(glm(y~x,family=poisson))
```

Call:

```
glm(formula = y ~ x, family = poisson)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-1.9886	-0.9631	0.1737	0.5131	2.0362

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	2.80316	0.14816	18.920	< 2e-16 ***
x	-0.08377	0.01680	-4.986	6.15e-07 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

Null deviance: 50.843 on 17 degrees of freedom
Residual deviance: 24.570 on 16 degrees of freedom
AIC: 95.825

Number of Fisher Scoring iterations: 5

Results for the quadratic model model:

```
> summary(glm(y~x+I(x^2),family=poisson))
```

Call:

```
glm(formula = y ~ x + I(x^2), family = poisson)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-2.12378	-0.87181	0.04959	0.69928	1.98646

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	2.619013	0.237900	11.009	<2e-16 ***
x	-0.018862	0.066038	-0.286	0.775
I(x^2)	-0.003793	0.003742	-1.014	0.311

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

Null deviance: 50.843 on 17 degrees of freedom
Residual deviance: 23.517 on 15 degrees of freedom
AIC: 96.772

Number of Fisher Scoring iterations: 5

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The R function I is used in the model formula to allow direct incorporation of the x^2 term.