RANDOMIZED COMPLETE BLOCK DESIGNS WITH BALANCED REPLICATION

Consider a **randomized block design** (RBD) with k treatments and b blocks, and r **replications**, giving $n = rbk$ observations in total. Let x_{ijt} be the tth replicated observation in the (i, j)th treatment/block combination.

- sample mean for **treatment** i

$$\overline{x}_i = \frac{1}{br} \sum_{j=1}^{b} \sum_{t=1}^{r} x_{ijt} \qquad i = 1, \ldots, k$$

- sample mean for **block** j

$$\overline{x_j^{(\mathrm{B})}} = \frac{1}{kr} \sum_{i=1}^{k} \sum_{t=1}^{r} x_{ijt} \qquad j = 1, \ldots, b$$

- sample mean for replicates in (i, j)th **treatment/block** combination

$$\overline{x}_{ij} = \frac{1}{r} \sum_{t=1}^{r} x_{ijt} \qquad i = 1, \ldots, k, \ j = 1, \ldots, b$$

- overall sample mean

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{b} \sum_{t=1}^{r} x_{ijt}$$

- Sum of Squares for Treatments (SST)

$$\mathrm{SST} = \sum_{i=1}^{k} br(\overline{x}_i - \overline{x})^2$$

- Sum of Squares for Blocks (SSB)

$$\mathrm{SSB} = \sum_{j=1}^{b} kr(\overline{x_j^{(\mathrm{B})}} - \overline{x})^2$$

- Sum of Squares for Interaction (SSI)

$$\mathrm{SSI} = \sum_{i=1}^{k} \sum_{j=1}^{b} r(\overline{x}_{ij} - \overline{x}_i - \overline{x_j^{(\mathrm{B})}} + \overline{x})^2$$

- Overall Sum of Squares (SS)

$$\mathrm{SS} = \sum_{i=1}^{k} \sum_{j=1}^{b} \sum_{t=1}^{r} (x_{ijt} - \overline{x})^2$$

The following decomposition holds

$$\mathrm{SS} = \mathrm{SST} + \mathrm{SSB} + \mathrm{SSI} + \mathrm{SSE} \qquad \therefore \qquad \mathrm{SSE} = \mathrm{SS} - \mathrm{SST} - \mathrm{SSB} - \mathrm{SSI}$$

Define

$$\mathrm{MST} = \frac{\mathrm{SST}}{k-1} \qquad \mathrm{MSB} = \frac{\mathrm{SSB}}{b-1} \qquad \mathrm{MSI} = \frac{\mathrm{SSI}}{(k-1)(b-1)}$$

and

$$\mathrm{MSE} = \frac{\mathrm{SSE}}{n - bk}$$

HYPOTHESIS TESTING

- For testing for a **TREATMENT** effect, use

$$F = \frac{\text{MST}}{\text{MSE}}$$

Under the assumption of **NO TREATMENT EFFECT**, then

$$F \sim \text{Fisher-F}(k-1, n-bk)$$

which defines the rejection region and p-value in the usual way.

- For testing for a **BLOCK** effect, use

$$F = \frac{\text{MSB}}{\text{MSE}}$$

Under the assumption of **NO BLOCK EFFECT**, then

$$F \sim \text{Fisher-F}(b-1, n-bk)$$

- For testing for an **INTERACTION**, use

$$F = \frac{\text{MSI}}{\text{MSE}}$$

Under the assumption of **NO INTERACTION**, then

$$F \sim \text{Fisher-F}((k-1)(b-1), n-bk)$$

RANDOMIZED COMPLETE BLOCK DESIGNS WITH BALANCED REPLICATION: EXAMPLE

Data: Measurements were made on the lifetimes of batteries (in hours) for three battery types constructed from different materials, to investigate the effect of operating temperature on lifetime. It was believed before the experiment that the battery types were likely to behave differently in the experiment.

The **response variable** is lifetime. The single **factor** is the *temperature* and there are $k = 3$ **factor levels**:

1. 15 Celsius
2. 70 Celsius
3. 125 Celsius

The *material* types determine the $b = 3$ **blocks**

1. Lead
2. Acetate
3. Nickel Cadmium

$r = 4$ replicate measurements were made, so that

$$n = 3 \times 3 \times 4 = 36$$

data were obtained in total.

The data observed in the study were as follows:

	Block		
Treatment	Lead	Acetate	Nickel Cadmium
15	130,155,74,180	150,188,159,126	138,119,168,160
70	34,40,80,75	126,122,106,115	174,120,150,139
120	20,70,82,58	25,70,58,45	96,104,82,60

Using SPSS, the following ANOVA table was obtained; see the related SPSS screens at

`www.math.mcgill.ca/~dstephens/204/Handouts/Math204-SPSS-RBDANOVAREP-Screens.pdf`

Tests of Between-Subjects Effects

Dependent Variable: Battery Life (hr)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	59154.000[a]	8	7394.250	11.103	.000
Intercept	398792.250	1	398792.250	598.829	.000
temp	39083.167	2	19541.583	29.344	.000
material	10633.167	2	5316.583	7.983	.002
temp * material	9437.667	4	2359.417	3.543	.019
Error	17980.750	27	665.954		
Total	475927.000	36			
Corrected Total	77134.750	35			

a. R Squared = .767 (Adjusted R Squared = .698)

There is a **significant difference** between **blocks** (row 4, material, $F = 7.983$, p-value=0.002), a **significant difference** between **treatments** (row 3, temp, $F = 29.344$, p-value< 0.001), and also a significant interaction (row 5, temp*material, $F = 3.543$, p-value=0.019), Levene's test reveals that there is no evidence to suspect that the population variances are different:

Levene's Test of Equality of Error Variances[a]

Dependent Variable: Battery Life (hr)

F	df1	df2	Sig.
1.059	8	27	.420

Tests the null hypothesis that the error variance of the dependent variable is equal across groups.

a. Design: Intercept+temp+material+temp * material

The means plots also indicate some significant interaction.