Comparing k Treatments

Suppose

$$\begin{array}{ll} \text{TMT } 1 & \text{has } n_1 \text{ experimental units} \\ \text{TMT } 2 & \text{has } n_2 \text{ experimental units} \\ \vdots & \vdots \\ \text{TMT } k & \text{has } n_k \text{ experimental units} \end{array}$$

Denote by x_{ij} the response for unit j in treatment group i, for $j = 1, \ldots, n_i$ and $i = 1, \ldots, k$.

Let

$$\overline{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}$$

denote the sample mean for treatment i, and

$$s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (x_{ij} - \overline{x}_i)^2$$

denote the sample variance for treatment i.

Now we consider pooling, that is, combining all units into a single group.

Define

- the total sample

$$n = n_1 + \cdots + n_k = \sum_{i=1}^{k} n_i$$

- the overall sample mean

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}$$

- the overall sample variance

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \overline{x})^2$$

Finally, consider the pooled sample variance

$$s_P^2 = \frac{1}{n-k} \sum_{i=1}^{k} (n_i - 1) s_i^2$$

- the extension of the pooled estimate of the population variance in a two-sample t-test.

Using these quantities, we can derive a test statistic for multiple group comparison.

We wish to compare how much variation is due to the

A DIFFERENCE BETWEEN TREATMENTS

and how much is due to

B RANDOM VARIATION WITHIN TREATMENTS

We measure A using the statistic

$$SST = \sum_{i=1}^{k} n_i(\overline{x}_i - \overline{x})^2$$

SST - Sum of Squares for Treatments

We measure B using the statistic

$$\begin{aligned} SSE &= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)^2 \\ &= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)^2 \\ &= (n-k)s_P^2 \end{aligned}$$

SSE - Sum of Squares for Error

NOTE: This measure of random or error variability implicitly assumes that the variability **within** the treatment groups is the **same for each group**. That is, population variances

$$\sigma_1^2, \ldots, \sigma_k^2$$

are equal.

In practice this assumption must be checked.

Finally, we define the test statistic using the mean levels of variability

- MST - Mean Square for Treatments

$$MST = \frac{SST}{k-1} = \frac{1}{k-1}\sum_{i=1}^{k} n_i(\overline{x}_i - \overline{x})^2$$

- MSE - Mean Square for Error

$$MSE = \frac{SSE}{n-k} = \frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \overline{x}_i)^2 = s_P^2$$

Then the test statistic is

$$F = \frac{MST}{MSE} = \frac{\text{Average Variation due to Treatments}}{\text{Average Variation due to Errors}}$$

F large $\Longrightarrow$ Treatments Different !
F small $\Longrightarrow$ Treatments Similar !

The behaviour of F is given by the following Theorem

Theorem: ANOVA F-test to compare k treatments in a Completely Randomized Design.

To test the hypothesis of **equal treatment means**,

$$H_0 \quad : \quad \mu_1 = \mu_2 = \cdots = \mu_k$$
$$H_a \quad : \quad \text{At least two of the } k \text{ treatment means are different}$$

the test statistic is

$$F = \frac{MST}{MSE}$$

If H_0 *is* **TRUE**, *then*

$$F \sim \text{Fisher-}F(k-1, n-k)$$

and the rejection region for a test at significance level α *is the region to the right of the* $1-\alpha$ *probability point of this Fisher-F distribution,* C_R.

NOTE: If

$$SS = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \overline{x})^2$$

is the *overall* or *total* sum of squares, then

$$SS = SST + SSE$$

so we can decompose the overall variation (SS) into the variation due to treatments (SST) and the variation due to the errors (SSE).