- consider the two hypotheses

$$H_0: \quad \mu_A = \mu_B$$
$$H_a: \quad \mu_A \neq \mu_B$$

H_0 is the **NULL HYPOTHESIS**
H_a is the **ALTERNATIVE HYPOTHESIS**

- define the test statistic

$$t = \frac{\overline{x}_A - \overline{x}_B}{s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}}$$

where

$$s^2 = \frac{(n_A - 1)s_A^2 + (n_B - 1)s_B^2}{n_A + n_B - 2}$$

s^2 is the estimate of the common population variance

$$\sigma^2 = \sigma_A^2 = \sigma_B^2$$

Here

$$s^2 = \frac{(10-1)61190.4 + (7-1)198679.6}{10+7-2} = 116186.1$$

so that

$$s = 340.8608.$$

Thus

$$t = \frac{2236.8 - 3023.429}{340.8608\sqrt{\frac{1}{10} + \frac{1}{7}}} = -4.683$$

Now, if the null hypothesis was **true**, so that

$$\mu_A = \mu_B$$

the test statistic t should look like an observation from a

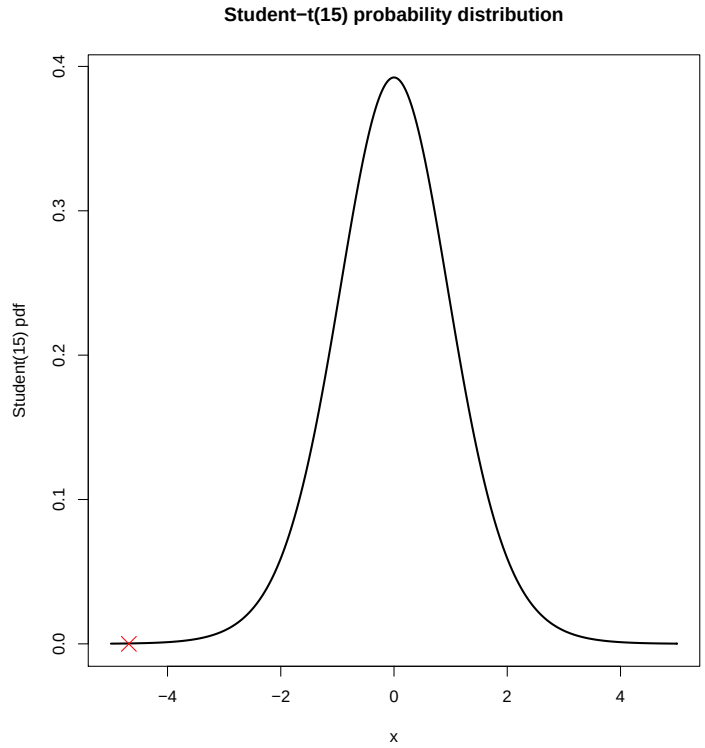
Student-t

distribution with

$$n_A + n_B - 2 = 15$$

"degrees of freedom".

i.e. t should lie somewhere in the "high-probability region" of the Student-t(15) probability distribution



Clearly, in this case, t does not lie in a high probability region.

i.e. we are surprised to see t so far away from zero.

The predicted behaviour of t, under the assumption that H_0 is TRUE, DOES NOT MATCH THE OBSERVED BEHAVIOUR !

Therefore, the assumption that H_0 is true MUST BE INCORRECT and we

REJECT H_0

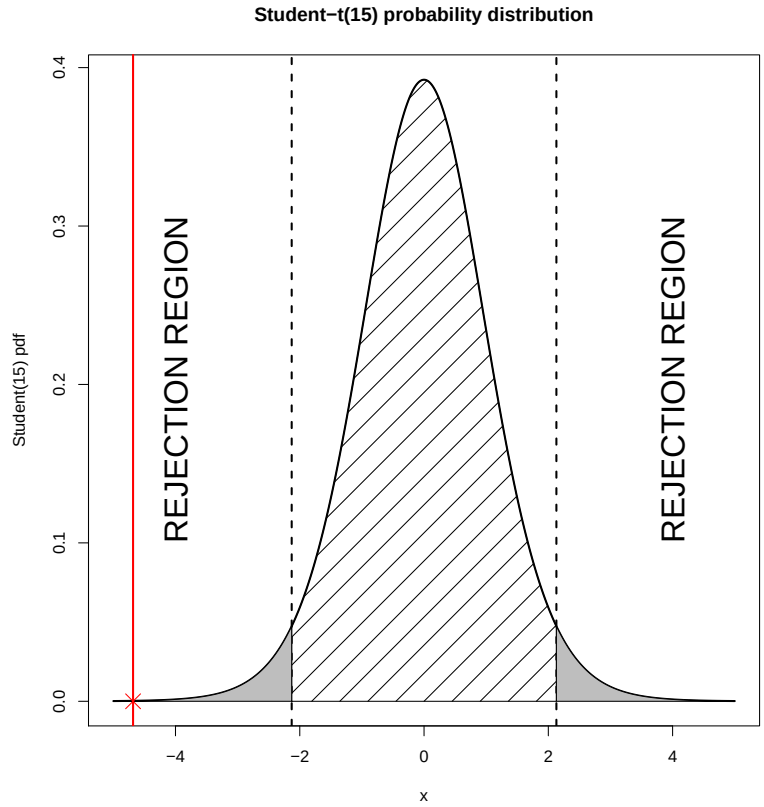
How do we quantify the "statistical significance" ?

Two approaches:

1. Define the "high-probability" region, and reject H_0 if t does not lie in this region.

2. Compute the level of "surprise" at observing t under the assumption that H_0 is TRUE.

For 1: Set *significance level* α, with $0 < \alpha < 1$, and find the central $1 - \alpha$ "high-probability" region, between the two values $-C_R$ and C_R (marked by dotted lines).



If $t < -C_R$ or $t > C_R$, REJECT H_0.

Typically, $\alpha = 0.05$ (or 0.01), so for the Student-t(15) distribution

$$C_R = 2.131 \qquad \text{(or } 2.947\text{)}$$

The regions $(-\infty, -C_R)$ and (C_R, ∞) form the CRITICAL REGION or REJECTION REGION.

If t lies in the critical region, we reject H_0.

For 2: To compute the level of "surprise", we evaluate the probability of observing a "more extreme" test statistic under the assumption that H_0 is TRUE.

Here, this probability is

$$p = 0.00029.$$

This probability is very small, so we are **very surprised** by the observed result.

p is termed the *p-value* or *observed significance level*.

If $p < \alpha = 0.05$ (or 0.01), we reject H_0.

Some questions:

- How do we choose the test statistic ?
- How do we choose α ?
- Why is the distribution of T (and t) a Student-t(15) distribution ?
- How do we compute C_R and p ?

Equal Variances ?

Is the assumption of equal population variances

$$\sigma_A = \sigma_B$$

fair in this case ?

$$\begin{aligned} s_A^2 &= 61190.4 \\ s_B^2 &= 198679.6 \end{aligned}$$

so that

$$\frac{s_A^2}{s_B^2} = 0.3080.$$

Can we test $\sigma_A = \sigma_B$ formally ?

Yes:

$$H_0: \quad \sigma_A = \sigma_B$$
$$H_a: \quad \sigma_A \neq \sigma_B$$

Test statistic is

$$F = \frac{s_A^2}{s_B^2} = 0.3080$$

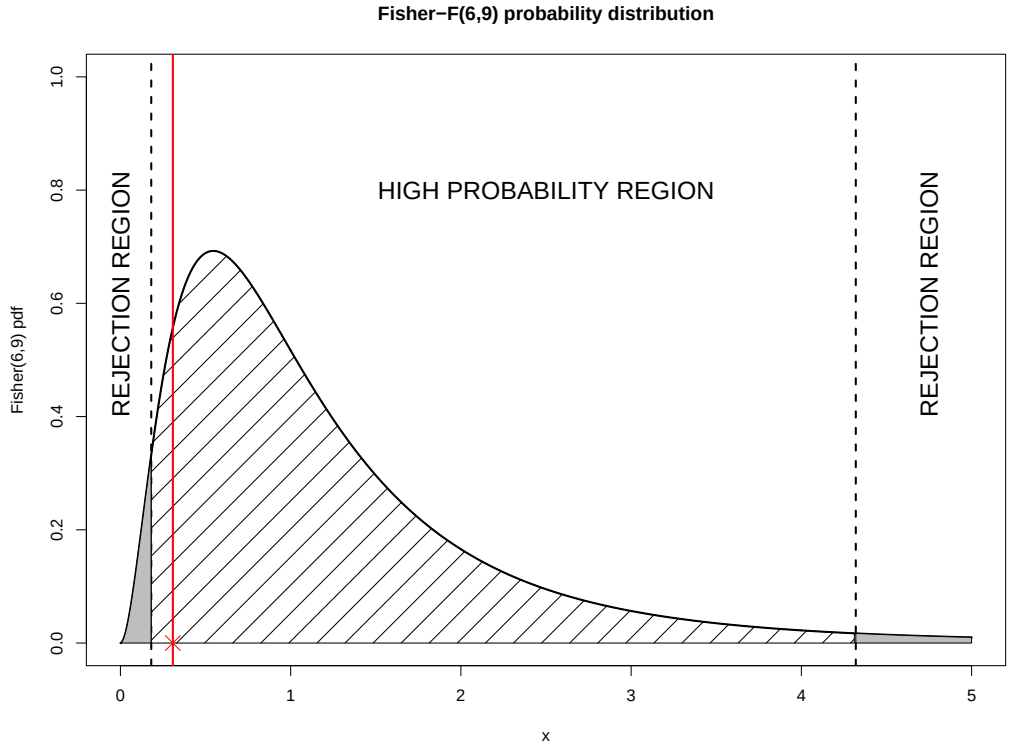
If H_0 is true, F should look like an observation from a

Fisher-F

distribution with

$$(n_A - 1, n_B - 1)$$

"degrees of freedom".



From tables, for $\alpha = 0.05$,

$$C_{R_1} = 0.181 \qquad C_{R_2} = 4.320$$

so the observed value of F **does** lie in the high probability region, and there is no reason to reject H_0 at $\alpha = 0.05$.

Can also compute a 95 % *confidence interval* for $\mu_A - \mu_B$

$$(\overline{x}_A - \overline{x}_B) \pm t_{n_A+n_B-2}(0.975)s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}$$

where

$$t_{n_A+n_B-2}(0.975) = 2.131$$

that is, the 0.975 probability point of the $Student - t(15)$ distribution.

Hence the 95 % confidence interval is

$$(-1144.59, -428.67)$$

- does not contain zero !

NOTE: Significance level α.

$$\begin{aligned} \alpha &= P[H_0 \text{ is rejected, given that } H_0 \text{ is TRUE}] \\ &= P[H_0 \text{ rejected} | H_0 \text{ is TRUE}] \end{aligned}$$

If

- T is the test statistic *random variable*
- $\mathcal{R}$ is the rejection region

then

$$\alpha = P[T \text{ lies in } \mathcal{R} | H_0 \text{ TRUE}] = P[T \in \mathcal{R} | H_0 \text{ TRUE}]$$

that is, α is the probability of committing a

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