# MATH 557 - EXERCISES 2
# SOLUTIONS

1 Let $X \sim Binomial(n, \theta)$ for $0 < \theta < 1$.

(a) By standard results for expectations

$$\mathrm{E}_{f_{T|\theta}}[T|\theta] = \frac{1}{n}\mathrm{E}_{f_{X|\theta}}[X|\theta] = \frac{n\theta}{n} = \theta$$

(b) Suppose that $U(X)$ is an unbiased estimator of $\tau(\theta)$, so that

$$\mathrm{E}_{f_{U|\theta}}[U|\theta] = \sum_{x=0}^{n} U(x)\binom{n}{x}\theta^x(1-\theta)^{n-x} = \frac{1}{\theta}$$

Clearly $U(x)$ must be finite on $0 \le x \le n$, otherwise the expectation is not finite, in which case $U(X)$ is biased for any $\theta$. But also, for $0 < \theta < 1$

$$\sum_{x=0}^{n} U(x)\binom{n}{x}\theta^x(1-\theta)^{n-x} \le \sum_{x=0}^{n} U(x)\binom{n}{x} = M(n) < \infty$$

so therefore the expectation of $U(X)$ is bounded above. Therefore, if $\theta < 1/M(n)$, the expectation cannot attain $1/\theta$, so the estimator is not unbiased; no unbiased estimator exists.

2 (a) The ML estimator of $\theta$ is $M(\underset{\sim}{X}) = X_{(n)} = \max\{X_1, \ldots, X_n\}$ which has pdf

$$f_{M|\theta}(m|\theta) = \frac{nm^{n-1}}{\theta^n} \qquad 0 < m < \theta$$

which has expectation

$$\mathrm{E}_{f_{M|\theta}}[M|\theta] = \int_0^\theta \frac{nm^n}{\theta^n}\, dm = \frac{n}{n+1}\theta$$

so therefore $T = (n+1)M/n$ is unbiased for $\theta$.

(b) The variance/MSE of $T(\underset{\sim}{X})$ is

$$\begin{aligned}
\mathrm{Var}_{f_{T|\theta}}[T|\theta] = \mathrm{E}_{f_{T|\theta}}[T^2|\theta] - \theta^2 &= \left(\frac{n+1}{n}\right)^2 \mathrm{E}_{f_{M|\theta}}[M^2|\theta] - \theta^2 \\
&= \left(\frac{n+1}{n}\right)^2 \int_0^\theta m^2 \frac{nm^{n-1}}{\theta^n}\, dm - \theta^2 \\
&= \left(\frac{n+1}{n}\right)^2 \left(\frac{n+2}{n}\right)\theta^2 - \theta^2
\end{aligned}$$

(c) The Cramér-Rao bound is

$$B(\theta) = \frac{\left(\frac{d}{d\theta}\left\{\mathrm{E}_{f_{T|\theta}}[T]\right\}\right)^2}{\mathrm{E}_{f_{\underset{\sim}{X}|\theta}}\left[S(\underset{\sim}{X};\theta)^2\right]} = \frac{(\dot{\tau}(\theta))^2}{\mathrm{E}_{f_{\underset{\sim}{X}|\theta}}\left[n^2/\theta^2\right]} = \frac{1}{n^2/\theta^2} = \frac{\theta^2}{n^2}$$

3 (a) The likelihood, log-likelihood and derivative are

$$\begin{aligned} L(\theta|\underset{\sim}{x}) &= \theta^n \left(\prod_{i=1}^n x_i\right)^{\theta-1} \qquad \theta > 0 \\ l(\theta|\underset{\sim}{x}) &= n\log\theta + (\theta-1)\sum_{i=1}^n \log x_i \\ \dot{l}(\theta|\underset{\sim}{x}) &= \frac{n}{\theta} + \sum_{i=1}^n \log x_i \end{aligned}$$

and hence

$$\widehat{\theta}_n(\underset{\sim}{X}) = -\frac{n}{\sum_{i=1}^n \log X_i} = \frac{n}{T(\underset{\sim}{X})} \quad \text{say.}$$

(b) If $X \sim \text{Beta}(\theta, 1)$ then if $Y = -\log X$, we have

$$\begin{aligned} F_{Y|\theta}(y|\theta) &= \Pr[Y \le y|\theta] = \Pr[-\log X \le y|\theta] = \Pr[X \ge e^{-y}|\theta] = 1 - F_{X|\theta}(e^{-y}|\theta) \\ f_{Y|\theta}(y|\theta) &= e^{-y} f_{X|\theta}(e^{-y}|\theta) = e^{-y}\theta(e^{-y})^{\theta-1} = \theta e^{-\theta y} \qquad y > 0 \end{aligned}$$

so $Y \sim$ Exponential$(\theta)$ and $T(\underset{\sim}{X}) \sim$ Gamma$(n, \theta)$. Now for $r = 1, 2, \ldots$

$$\text{E}_{f_{T|\theta}}[T^{-r}] = \int_0^\infty \frac{1}{t^r}\frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)}\frac{\Gamma(n-r)}{\theta^{n-r}} = \frac{\theta^r}{(n-1)(n-2)\ldots(n-r+1)}.$$

Hence

$$\text{E}_{f_{T|\theta}}[T^{-1}] = \frac{\theta}{(n-1)} \qquad \text{E}_{f_{T|\theta}}[T^{-2}] = \frac{\theta^2}{(n-1)(n-2)}$$

so that

$$\text{Var}_{f_{T|\theta}}[T^{-1}] = \frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2} = \frac{\theta^2}{(n-1)^2(n-2)}$$

Thus the expectation and variance of $\widehat{\theta}_n(\underset{\sim}{X})$ are

$$\frac{n\theta}{(n-1)} \qquad \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

so $\widehat{\theta}_n(\underset{\sim}{X})$ is not unbiased for $\theta$.

(c) Writing $T_1(\underset{\sim}{X}) = \widehat{\theta}_n(\underset{\sim}{X})$, the Cramér-Rao bound is

$$\begin{aligned} B(\theta) &= \frac{\left(\frac{d}{d\theta}\left\{\text{E}_{f_{T_1|\theta}}[T_1]\right\}\right)^2}{\text{E}_{f_{\underset{\sim}{X}|\theta}}\left[S(\underset{\sim}{X};\theta)^2\right]} = \frac{(n/(n-1))^2}{\text{E}_{f_{T|\theta}}\left[(n/\theta - T)^2\right]} = \frac{\left(\frac{n}{n-1}\right)^2}{\frac{n^2}{\theta^2} - 2\frac{n}{\theta}\text{E}_{f_{T|\theta}}[T] + \text{E}_{f_{T|\theta}}[T^2]} \\ &= \frac{\left(\frac{n}{n-1}\right)^2}{\frac{n^2}{\theta^2} - 2\frac{n}{\theta}\frac{n}{\theta} + \frac{n(n+1)}{\theta^2}} = \frac{n\theta^2}{(n-1)^2} \end{aligned}$$

Thus the Cramér-Rao bound is not met here despite the fact that this is an Exponential Family distribution. The Exponential Family does yield unbiased estimators of parameters, but only in specific parameterizations relating to the natural parameterization. In this case, we have

$$f_{X|\theta}(x|\theta) = \theta \exp\{(\theta - 1)\log x\}$$

so that, by properties of the score function

$$E_{f_{X|\theta}}[-\log X|\theta] = \frac{1}{\theta} = \tau(\theta)$$

and the result yields that

$$T_2(\underset{\sim}{X}) = -\frac{1}{n}\sum_{i=1}^{n} \log X_i$$

is an unbiased estimator of $\tau(\theta)$. As

$$T_2(\underset{\sim}{X}) \sim \text{Gamma}(n, n\theta)$$

we have that

$$\text{Var}_{f_{T_2|\theta}}[T_2|\theta] = \frac{1}{n\theta^2}$$

for which the Cramér-Rao bound is

$$B(\theta) = \frac{\left(\frac{d}{d\theta}\left\{\text{E}_{f_{T_2|\theta}}[T_2]\right\}\right)^2}{\text{E}_{f_{\underset{\sim}{X}|\theta}}\left[S(\underset{\sim}{X};\theta)^2\right]} = \frac{(\dot{\tau}(\theta))^2}{\text{E}_{f_{T|\theta}}\left[(n/\theta - nT_2)^2\right]} = \frac{1/\theta^4}{n^2\text{Var}_{f_{T|\theta}}[T_2]} = \frac{1}{n\theta^2}$$

so the bound is attained.

4 The likelihood, log-likelihood and derivative are

$$\begin{aligned}
L(\theta|\underset{\sim}{x}) &= \left(\frac{1}{2\pi\theta}\right)^{n/2} \exp\left\{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2\right\} \theta > 0 \\
l(\theta|\underset{\sim}{x}) &= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\theta - \frac{1}{2\theta}\sum_{i=1}^{n} x_i^2 \\
\dot{l}(\theta|\underset{\sim}{x}) &= -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2
\end{aligned}$$

and hence

$$\widehat{\theta}_n(\underset{\sim}{X}) = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

As $X_i^2/\theta \sim \chi_1^2 \equiv \text{Gamma}(1/2, 1/2)$, so that

$$\widehat{\theta}_n(\underset{\sim}{X}) = \frac{\theta}{n}\sum_{i=1}^{n}(X_i^2/\theta) \sim \text{Gamma}(n/2, n/2\theta)$$

so $\widehat{\theta}_n(\underset{\sim}{X})$ has expectation $\theta$ and variance $2\theta^2/n$. In this problem, the Cramér-Rao bound is

$$B(\theta) = \frac{\left(\frac{d}{d\theta}\left\{\mathrm{E}_{f_{T_2|\theta}}[T_2]\right\}\right)^2}{\mathrm{E}_{f_{\underset{\sim}{X}|\theta}}\left[S(\underset{\sim}{X};\theta)^2\right]} = \frac{1}{\mathrm{E}_{f_{T|\theta}}\left[-\Psi(\underset{\sim}{X};\theta)\right]} = \frac{1}{\mathrm{E}_{f_{T|\theta}}\left[\frac{n}{2\theta^2} + \frac{1}{\theta^3}\sum_{i=1}^{n} X_i^2\right]}$$

$$= \frac{1}{-\frac{n}{2\theta^2} + \frac{n}{\theta^2}} = \frac{2\theta^2}{n}$$

so the bound is attained by the ML estimator, and $\widehat{\theta}_n(\underset{\sim}{X})$ is the best unbiased estimator.

5 (a) We have that

$$\begin{aligned}\mathcal{I}(\theta) = \mathrm{E}_{f_{\underset{\sim}{X}|\theta}}\left[S(X;\theta)^2\right] &= \int \left\{\frac{\partial}{\partial\theta}\log f_{X|\theta}(x|\theta)\right\}^2 f_{X|\theta}(x|\theta)\,dx \\ &= \int \left\{\frac{\dot{f}(x-\theta)}{f(x-\theta)}\right\}^2 f(x-\theta)\,dx \\ &= \int \left\{\frac{\dot{f}(x)}{f(x)}\right\}^2 f(x)\,dx\end{aligned}$$

after making the change of variables $x \longrightarrow x - \theta$. Hence $\mathcal{I}(\theta)$ is constant in $\theta$.

(b) For the Double Exponential model

$$\begin{aligned}\mathcal{I}(\theta) &= \int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial\theta}\log f_{X|\theta}(x|\theta)\right\}^2 f_{X|\theta}(x|\theta)\,dx = \int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial\theta} - |x-\theta|\right\}^2 \frac{1}{2}e^{-|x-\theta|}\,dx \\ &= \int_{-\infty}^{\theta}\left\{\frac{\partial}{\partial\theta}(x-\theta)\right\}^2 \frac{1}{2}e^{(x-\theta)}\,dx + \int_{\theta}^{\infty}\left\{\frac{\partial}{\partial\theta} - (x-\theta)\right\}^2 \frac{1}{2}e^{-(x-\theta)}\,dx \\ &= \int_{-\infty}^{\theta}\frac{1}{2}e^{(x-\theta)}\,dx + \int_{\theta}^{\infty}\frac{1}{2}e^{-(x-\theta)}\,dx = 1\end{aligned}$$

Note that the lack of differentiability of the pdf at $\theta$ does not render this computation invalid, as this is a single point in a continuous distribution and hence has probability zero, that is, the function is differentiable *almost everywhere*.

(c) For the Logistic model

$$\begin{aligned}\mathcal{I}(\theta) &= \int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial\theta}\log f_{X|\theta}(x|\theta)\right\}^2 f_{X|\theta}(x|\theta)\,dx = \int_{-\infty}^{\infty}\left\{-\frac{\partial^2}{\partial\theta^2}\log f_{X|\theta}(x|\theta)\right\} f_{X|\theta}(x|\theta)\,dx \\ &= \int_{-\infty}^{\infty}\frac{2}{(1+e^{-(x-\theta)})^2}\frac{e^{-(x-\theta)}}{(1+e^{-(x-\theta)})^2}\,dx \\ &= \int_{-\infty}^{\infty}\frac{2e^{-x}}{(1+e^{-x})^4}\,dx \qquad (x-\theta \longrightarrow x) \\ &= \left[\frac{2}{3(1+e^{-x})^3}\right]_{-\infty}^{\infty} = \frac{2}{3}\end{aligned}$$

6 For this pdf, the corresponding expectation is

$$\begin{aligned} \mathrm{E}_{f_{X|\theta}}[X|\theta] = \int_0^\infty x \frac{3\theta^3}{(x+\theta)^4}\, dx &= \left[\frac{-x\theta^3}{(x+\theta)^3}\right]_0^\infty + \int_0^\infty \frac{\theta^3}{(x+\theta)^3}\, dx \\ &= 0 + \left[-\frac{\theta^3}{2(x+\theta)^2}\right]_0^\infty = \frac{\theta}{2} \end{aligned}$$

so therefore

$$T(\underset{\sim}{X}) = \frac{2}{n}\sum_{i=1}^n X_i$$

is an unbiased estimator of $\theta$. By a similar calculation

$$\begin{aligned} \mathrm{E}_{f_{X|\theta}}[X^2|\theta] = \int_0^\infty x^2 \frac{3\theta^3}{(x+\theta)^4}\, dx &= \left[\frac{-x^2\theta^3}{(x+\theta)^3}\right]_0^\infty + \int_0^\infty \frac{2x\theta^3}{(x+\theta)^3}\, dx \\ &= 0 + \left[-\frac{x\theta^3}{(x+\theta)^2}\right]_0^\infty + \int_0^\infty \frac{\theta^3}{(x+\theta)^2}\, dx \\ &= 0 + \left[-\frac{\theta^3}{(x+\theta)}\right]_0^\infty\, dx = \theta^2 \end{aligned}$$

so therefore

$$\mathrm{Var}_{f_{X|\theta}}[X|\theta] = \theta^2 - \frac{\theta^2}{4} = \frac{3\theta^2}{4}$$

and hence

$$\mathrm{Var}_{f_{T|\theta}}[T|\theta] = \frac{3\theta^2}{n}$$