

MATH 680: EXERCISE 2

ML ESTIMATION IN THE WEIBULL
DISTRIBUTION

The 1-parameter Weibull distribution

The Weibull density takes the form

$$f_X(x|\alpha) = \alpha x^{\alpha-1} \exp\{-x^\alpha\} \quad x > 0$$

and zero otherwise, for parameter $\alpha > 0$.

This density is used to model time-to-event or survival data.

Based on a random sample $x_1, \dots, x_n$, we seek to estimate α using maximum likelihood.

Likelihood and log-likelihood

We have that

$$\mathcal{L}(\alpha; \mathbf{x}) = \alpha^n \left\{ \prod_{i=1}^n x_i \right\}^{\alpha-1} \exp \left\{ - \sum_{i=1}^n x_i^\alpha \right\}$$

and

$$\ell(\alpha; \mathbf{x}) = n \log \alpha + (\alpha - 1) \sum_{i=1}^n \log x_i - \sum_{i=1}^n x_i^\alpha.$$

The score equation is

$$\dot{\ell}(\alpha; \mathbf{x}) = 0.$$

Score equation and second derivative

$$\dot{\ell}(\alpha; \mathbf{x}) = \frac{n}{\alpha} + \sum_{i=1}^n \log x_i - \sum_{i=1}^n (\log x_i) x_i^{\alpha} = 0$$

$$\ddot{\ell}(\alpha; \mathbf{x}) = -\frac{n}{\alpha^2} - \sum_{i=1}^n (\log x_i)^2 x_i^{\alpha} < 0$$

so therefore the only solution to the score equation gives a maximum of $\ell(\theta; \mathbf{x})$.

Data: $n = 6$

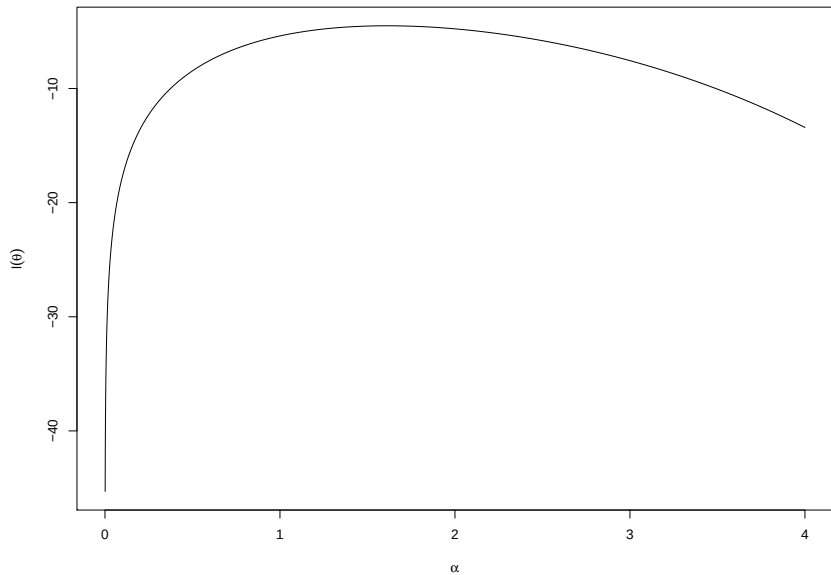
1.168 1.790 0.290 1.322 0.566 0.258

We plot

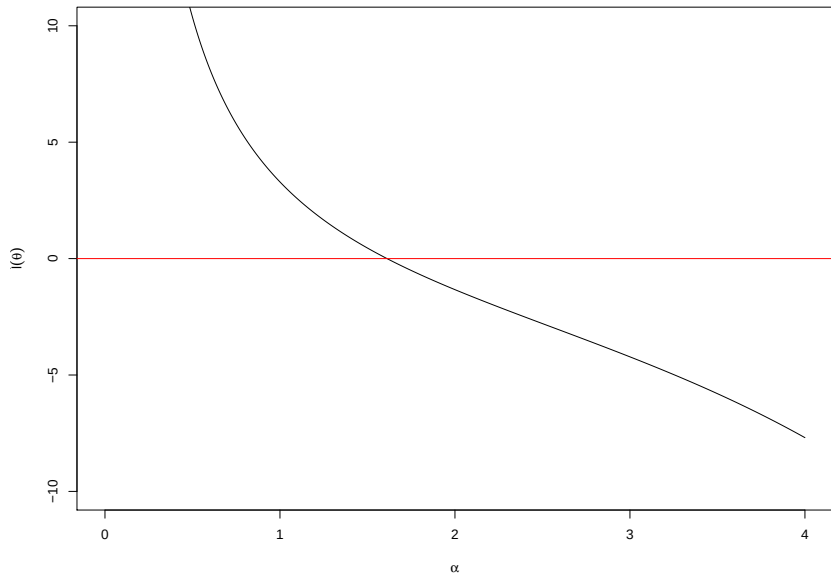
- $\ell(\theta; \mathbf{x})$
- $\dot{\ell}(\theta; \mathbf{x})$
- $\ddot{\ell}(\theta; \mathbf{x})$

for $0 < \theta < 4$.

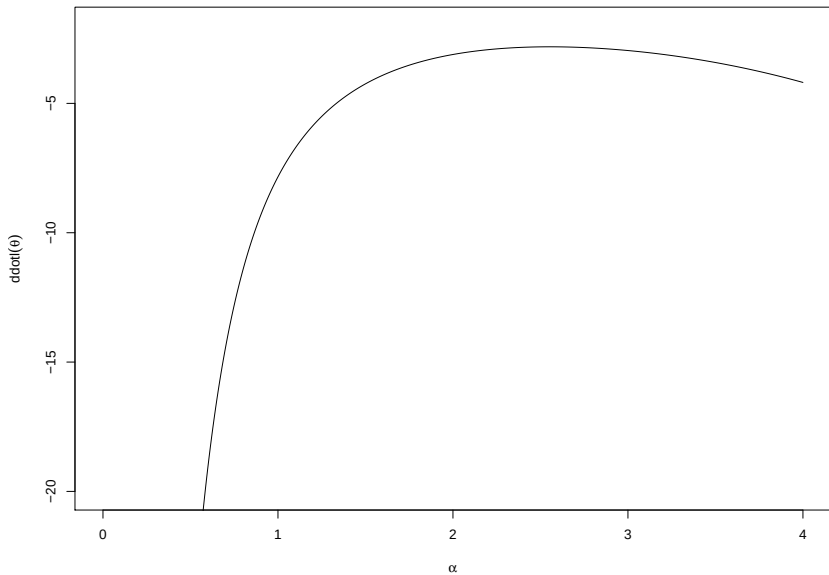
$$\ell(\theta; \mathbf{x})$$



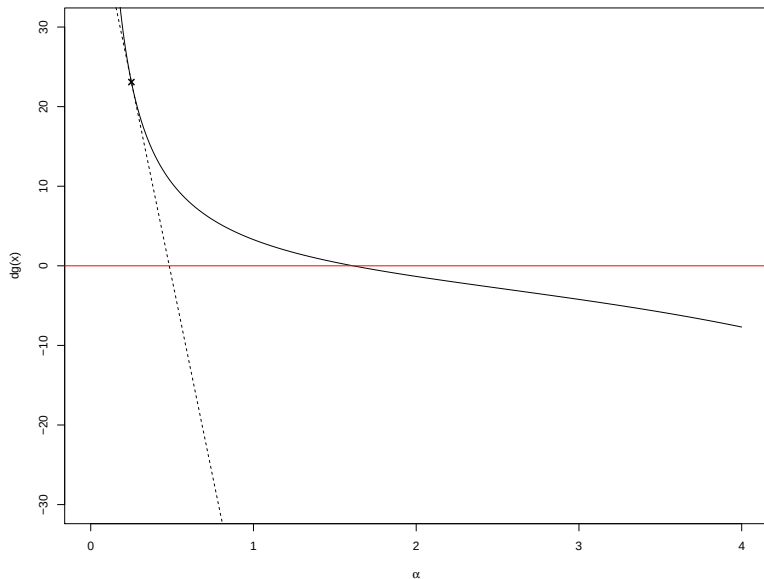
$$\dot{\ell}(\theta; \mathbf{x})$$



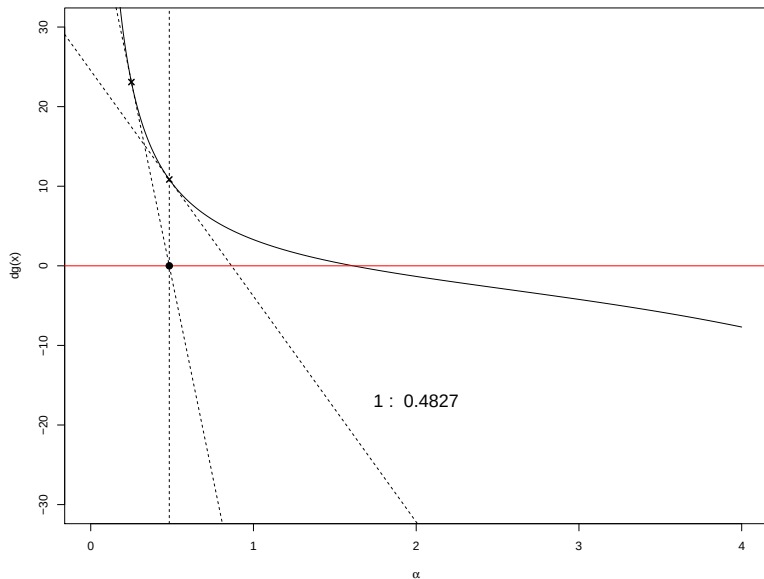
$$\ddot{\ell}(\theta; \mathbf{x})$$



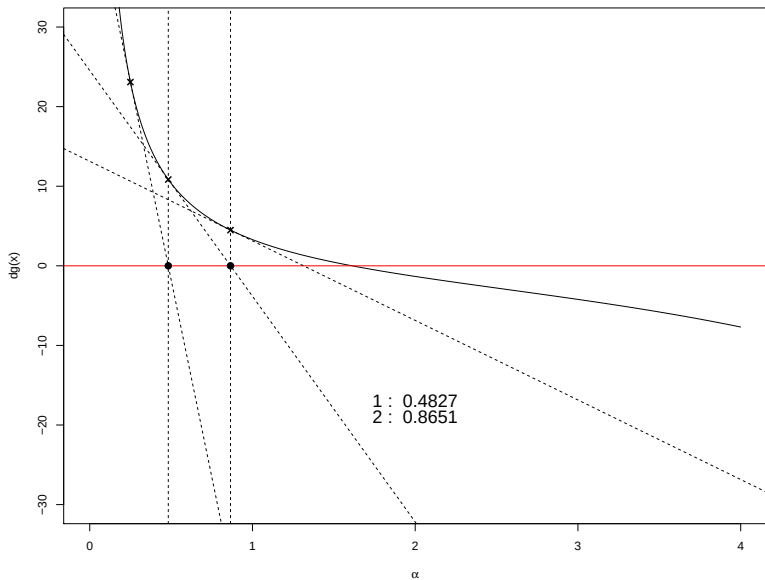
Newton's Method : Starting at $x^{(0)} = 0.25$



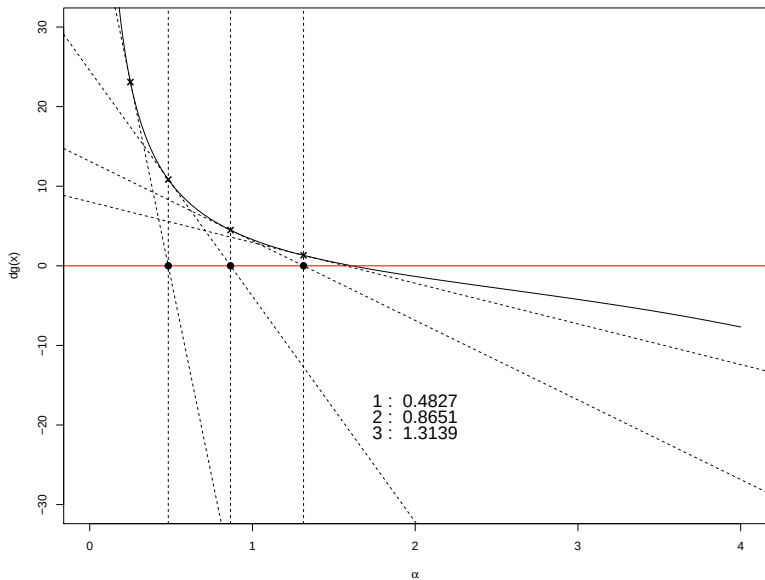
Newton's Method : Starting at $x^{(0)} = 0.25$



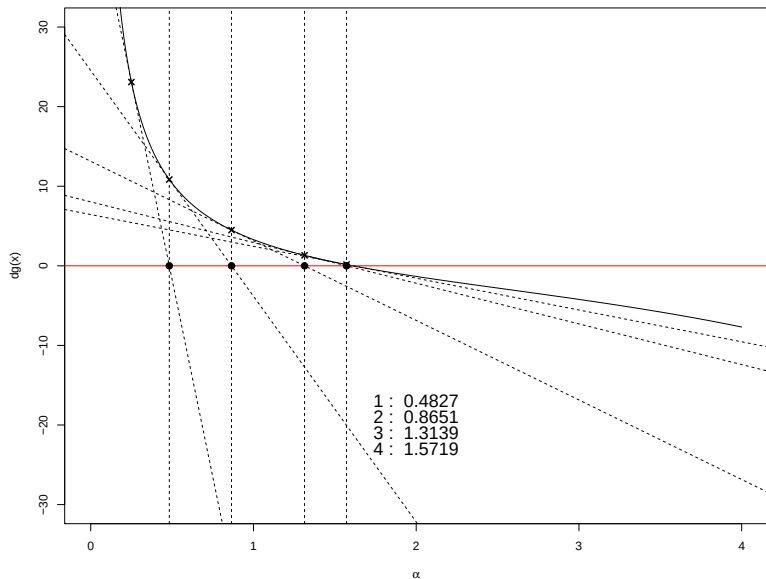
Newton's Method : Starting at $x^{(0)} = 0.25$



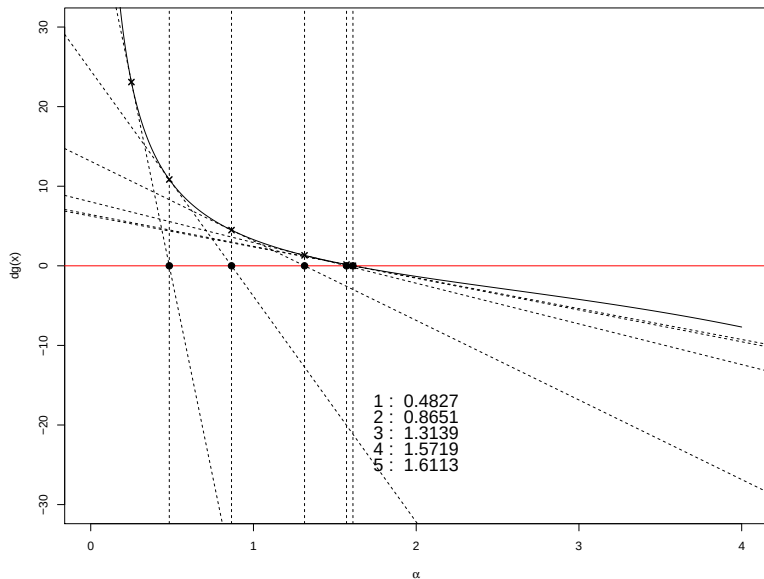
Newton's Method : Starting at $x^{(0)} = 0.25$



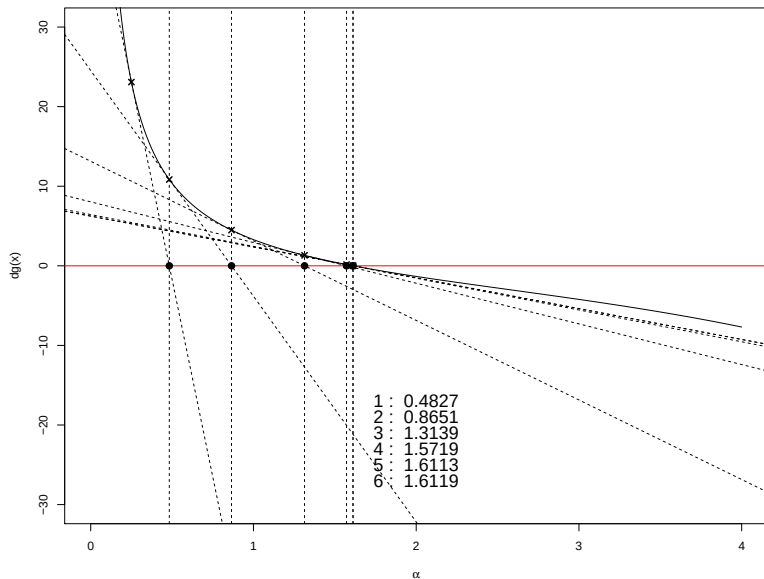
Newton's Method : Starting at $x^{(0)} = 0.25$



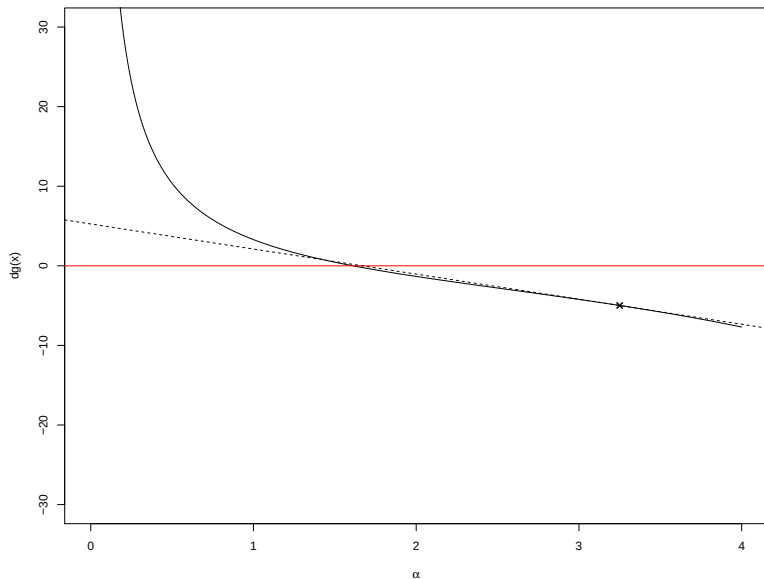
Newton's Method : Starting at $x^{(0)} = 0.25$



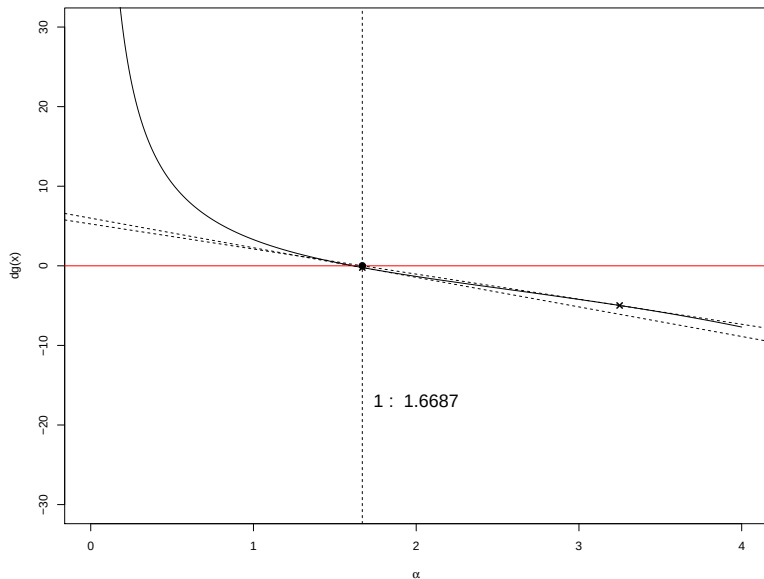
Newton's Method : Starting at $x^{(0)} = 0.25$



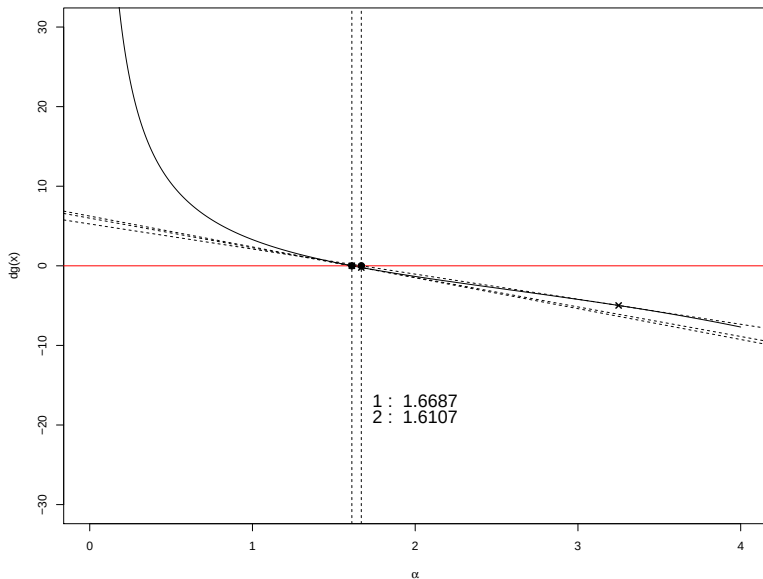
Newton's Method: Starting at $x^{(0)} = 3.25$



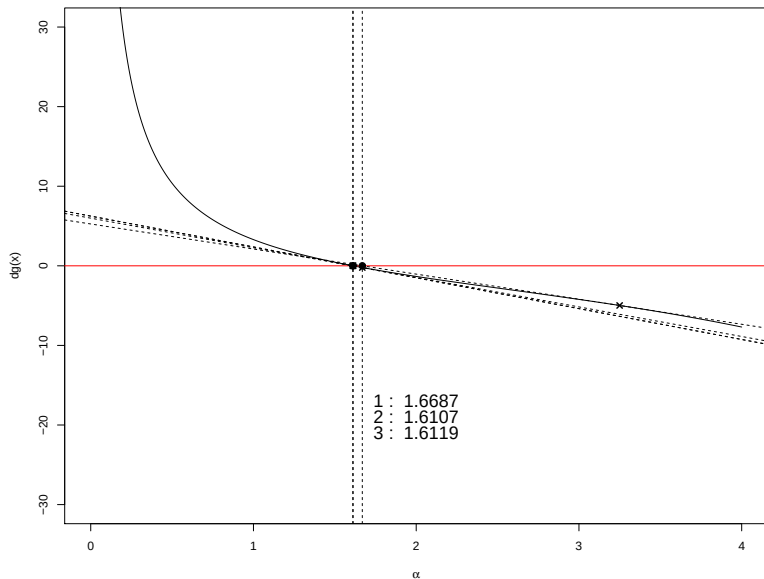
Newton's Method: Starting at $x^{(0)} = 3.25$



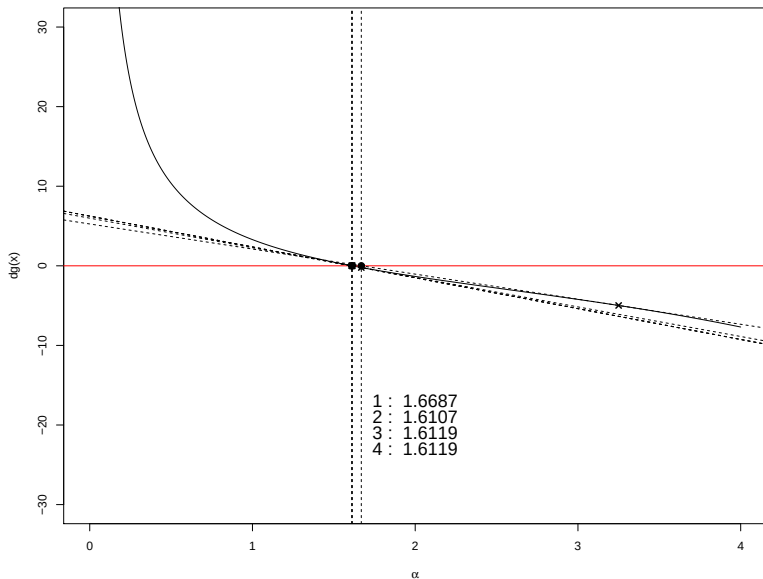
Newton's Method: Starting at $x^{(0)} = 3.25$



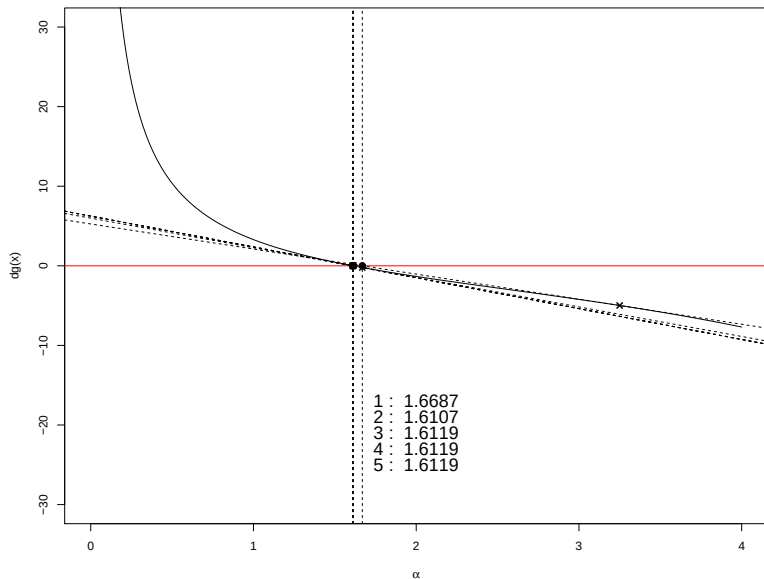
Newton's Method: Starting at $x^{(0)} = 3.25$



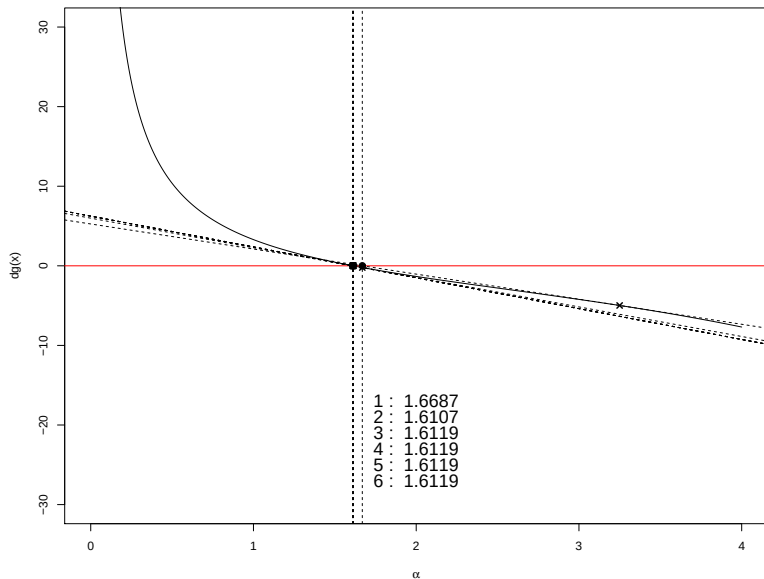
Newton's Method: Starting at $x^{(0)} = 3.25$



Newton's Method: Starting at $x^{(0)} = 3.25$



Newton's Method: Starting at $x^{(0)} = 3.25$



Bisection

With $a^{(0)} = 0.001$, $b^{(0)} = 5.000$:

	iter	a	b	x
1	0	0.001000	5.000000	2.500500
2	1	0.001000	2.500500	1.250750
3	2	1.250750	2.500500	1.875625
4	3	1.250750	1.875625	1.563188
5	4	1.563188	1.875625	1.719406
6	5	1.563188	1.719406	1.641297
7	6	1.563188	1.641297	1.602242
8	7	1.602242	1.641297	1.621770
9	8	1.602242	1.621770	1.612006
10	9	1.602242	1.612006	1.607124
11	10	1.607124	1.612006	1.609565

Bisection

With $a^{(0)} = 0.001$, $b^{(0)} = 5.000$:

	iter	a	b	x
12	11	1.609565	1.612006	1.610785
13	12	1.610785	1.612006	1.611396
14	13	1.611396	1.612006	1.611701
15	14	1.611701	1.612006	1.611853
16	15	1.611853	1.612006	1.611930
17	16	1.611853	1.611930	1.611891
18	17	1.611891	1.611930	1.611911
19	18	1.611891	1.611911	1.611901
20	19	1.611901	1.611911	1.611906
21	20	1.611901	1.611906	1.611903

Bisection

With $a^{(0)} = 1.000$, $b^{(0)} = 3.000$:

	iter	a	b	x
1	0	1.000000	3.000000	2.000000
2	1	1.000000	2.000000	1.500000
3	2	1.500000	2.000000	1.750000
4	3	1.500000	1.750000	1.625000
5	4	1.500000	1.625000	1.562500
6	5	1.562500	1.625000	1.593750
7	6	1.593750	1.625000	1.609375
8	7	1.609375	1.625000	1.617188
9	8	1.609375	1.617188	1.613281
10	9	1.609375	1.613281	1.611328
11	10	1.611328	1.613281	1.612305

Bisection

With $a^{(0)} = 1.000$, $b^{(0)} = 3.000$:

	iter	a	b	x
12	11	1.611328	1.612305	1.611816
13	12	1.611816	1.612305	1.612061
14	13	1.611816	1.612061	1.611938
15	14	1.611816	1.611938	1.611877
16	15	1.611877	1.611938	1.611908
17	16	1.611877	1.611908	1.611893
18	17	1.611893	1.611908	1.611900
19	18	1.611900	1.611908	1.611904
20	19	1.611900	1.611904	1.611902
21	20	1.611900	1.611902	1.611901

Write code to implement

1. Newton's Method
2. Bisection

for this problem.