CAUSAL MEDIATION ANALYSIS

Statistical methods for causal *mediation* analysis can also be formulated using potential outcomes: where z and m will represent potential levels of treatment Z and mediator M respectively, with *potential outcome*

$$Y(\mathsf{z}, \mathsf{m})$$

and *potential mediator* value in light of $Z = \mathsf{z}$

$$M(\mathsf{z}).$$

In general, both Z and M can take arbitrary values, but we most commonly consider the case when

- Z is binary $\{0, 1\}$
- M is
  - binary,
  - discrete, or
  - continuous.

In the binary case we have that

$$M = (1 - Z)M(0) + ZM(1)$$

and, if M is binary say,

$$Y = (1 - Z)(1 - M)\, Y(0, 0) + (1 - Z)M\, Y(0, 1) + Z(1 - M)\, Y(1, 0) + Z\, M\, Y(1, 1)$$

etc. Then following Imai et al. (*Statistical Science*, 2010)

- the *total causal effect* (TCE) of treatment is based on

  $$Y(\mathsf{z}, M(\mathsf{z}))$$

  i.e. set $Z = \mathsf{z}$ and observe the consequence.

  For example

  $$\tau = \mathbb{E}[Y(1, M(1)) - Y(0, M(0))].$$

- the *causal mediation effect* (CME) is based on

  $$Y(\mathsf{z}, M(\mathsf{z}^*))$$

  i.e. set $Z = \mathsf{z}$ and observe the consequence of changing the mediator as if $Z = \mathsf{z}^*$

  For example

  $$\delta(\mathsf{z}) = \mathbb{E}[Y(\mathsf{z}, M(1)) - Y(\mathsf{z}, M(0))] \qquad \mathsf{z} = 0, 1.$$

- the *direct effect* (DE) of treatment is based on

  $$Y(\mathsf{z}^*, M(\mathsf{z}))$$

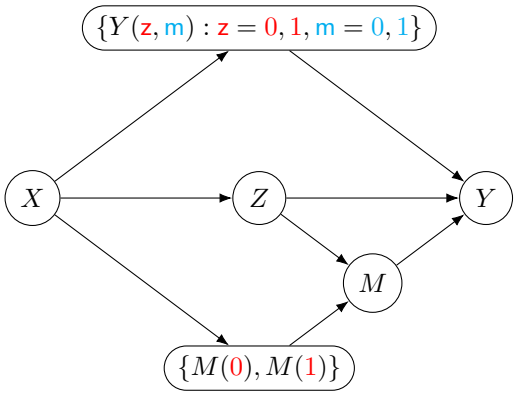
  i.e. set the mediator as if $Z = \mathsf{z}$, and observe the consequence of changing treatment according to z^*.

  For example

  $$\zeta(\mathsf{z}) = \mathbb{E}[Y(1, M(\mathsf{z})) - Y(0, M(\mathsf{z}))] \qquad \mathsf{z} = 0, 1.$$

Terminology:

		Imai et al.	Pearl	Robins
Total effect	τ	ATCE	Total	Total
Indirect effect	$\delta(\mathsf{z})$	ACME	Natural	Pure/Total
Direct effect	$\zeta(\mathsf{z})$	ADE	Natural	Pure/Total



with

$$\text{Pure} : \mathsf{z} = 0 \qquad \text{Total} : \mathsf{z} = 1.$$

That is,

$$\mathbb{E}[Y(1, M(1)) - Y(0, M(0))] = \mathbb{E}[Y(1, M(1)) - Y(1, M(0))] + \mathbb{E}[Y(1, M(0)) - Y(0, M(0))]$$

so that

$$\tau = \delta(1) + \zeta(0) = \delta(0) + \zeta(1).$$

The DAG encompasses two ignorability assumptions

(i) $\{Y(\mathsf{z}^*, \mathsf{m}), M(\mathsf{z})\} \perp\!\!\!\perp Z \mid X = x$;

(ii) $Y(\mathsf{z}^*, \mathsf{m}) \perp\!\!\!\perp M(\mathsf{z}) \mid Z = \mathsf{z}, X = x$;

for all z, z^*, m and x, and relies upon two positivity assumptions

$$0 < \Pr[Z = z | X = x] < 1 \qquad \text{and} \qquad 0 < \Pr[M(z) = m | X = x, Z = z] < 1.$$

Imai et al (2010) gives the following identities:

$$\delta(\mathsf{z}) = \iiint y f_{Y|M,X,Z}(y|m, x, \mathsf{z}) f_{M|X,Z}(m|x, 1) f_X(x)\, dy\, dm\, dx$$
$$- \iiint y f_{Y|M,X,Z}(y|m, x, \mathsf{z}) f_{M|X,Z}(m|x, 0) f_X(x)\, dy\, dm\, dx$$

$$\zeta(\mathsf{z}) = \iiint y f_{Y|M,X,Z}(y|m, x, 1) f_{M|X,Z}(m|x, \mathsf{z}) f_X(x)\, dy\, dm\, dx$$
$$- \iiint y f_{Y|M,X,Z}(y|m, x, 0) f_{M|X,Z}(m|x, \mathsf{z}) f_X(x)\, dy\, dm\, dx$$

These quantities can be computed by considering

$$\mathbb{E}[Y(\mathsf{z}, M(\mathsf{z}^*))] = \iiint y f_{Y|M,X,Z}(y|m, x, \mathsf{z}) f_{M|X,Z}(m|x, \mathsf{z}^*) f_X(x)\, dy\, dm\, dx.$$

The challenge in estimating these quantities using moment-based methods is the mis-match in the terms

$$f_{Y|M,X,Z}(y|m, x, \mathsf{z}) f_{M|X,Z}(m|x, \mathsf{z}^*)$$

whenever $\mathsf{z} \neq \mathsf{z}^*$. Model-based estimation is possible via the mean-model

$$\mu(m, x, z) = \int y\, f_{Y|M,X,Z}(y|m, x, z)\, dy$$

and the mediator model

$$f_{M|X,Z}(m|x, z).$$

is possible.

Example:

- $(X_1, X_2, X_3)^\top \sim \text{Normal}_3(\mu, \Sigma)$ with

$$\mu = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 0.25 & 0.00 & 0.00 \\ 0.00 & 0.50 & 0.00 \\ 0.00 & 0.00 & 0.75 \end{bmatrix} \begin{bmatrix} 1.0 & 0.9 & 0.0 \\ 0.9 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} 0.25 & 0.00 & 0.00 \\ 0.00 & 0.50 & 0.00 \\ 0.00 & 0.00 & 0.75 \end{bmatrix}$$

- $Z|X = x \sim Bernoulli(e(x;\alpha))$, with

$$e(x;\alpha) = \frac{\exp\{\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_1 x_2\}}{1 + \exp\{\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_1 x_2\}}$$

  with $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)^\top = (11.5, -0.2, 2.7, 2, 2)^\top$.

- $M|X = x, Z = z \sim Bernoulli(g(x,z;\gamma))$, with

$$g(x,z;\gamma) = \frac{\exp\{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \gamma_4 x_1 x_2 + \gamma_5 z\}}{1 + \exp\{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \gamma_4 x_1 x_2 \gamma_5 z\}}$$

  with $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)^\top = (11.5, -0.2, 2.7, 2, 2, 2)^\top$.

- $Y|X = x, M = m, Z = z \sim \text{Normal}(\mu(x,m,z;\beta,\phi,\psi), \sigma^2)$, with

  $\mu(x,z) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \phi m + z(\psi_0 + \psi_1 x_1 + \psi_2 x_2 + \psi_3 x_1 x_2) = \mu_0(x;\beta) + \phi m + z\mu_2(x;\psi)$

  say, with $\sigma^2 = 0.1^2$, and

$$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)^\top = (10, -2.0, 1.2, 0.6)^\top \qquad \phi = 2 \qquad \psi = (\psi_0, \psi_1, \psi_2, \psi_3)^\top = (1, 1, 1, 1)^\top$$

  In this model, X_1 and X_2 are confounders, and X_3 is a predictor of Z alone (and therefore is an *instrument*); X_3 is independent of (X_1, X_2).

```r
#Data simulation
library(mvnfast);library(xtable)
set.seed(23987)
n<-5000
D<-diag(c(0.25,0.5,0.75))
Mu<-c(1,-2,-1); Sigma<-D %*% (matrix(c(1,0.9,0.0,0.9,1,0.0,0,0,1),3,3) %*% D)
X<-rmvn(n,mu=Mu,Sigma)
al<-c(11.5,-0.2,2.7,2,2);gam<-c(al,2)
Xal<-cbind(1,X[,1],X[,2],X[,3],X[,1]*X[,2])
expit<-function(x){return(1/(1+exp(-x)))}
ps.true<-expit(Xal %*% al)
Z<-rbinom(n,1,ps.true)
Xgam<-cbind(1,X[,1],X[,2],X[,3],X[,1]*X[,2],Z)
ps.med.true<-expit(Xgam %*% gam)
M<-rbinom(n,1,ps.med.true)

be<-c(10,-2.0,1.2,0.6)
Xb<-cbind(1,X[,1],X[,2],X[,1]*X[,2])
mu0<- Xb %*% be
psi<-c(1,1,1,1)
phi<-2
mu1<-M*phi
Xp<-cbind(1,X[,1],X[,2],X[,1]*X[,2])
mu2<- (Xp %*% psi)
eta<-mu0 + mu1 + Z * mu2 ; sig<-0.1
Y<-rnorm(n,eta,sig)
X1<-X[,1];X2<-X[,2];X3<-X[,3]
true.vals<-c(be,psi,phi)
eX<-as.vector(Z-ps.true)
```

The causal estimands of interest can be expressed via the quantities

$$\mu(\mathsf{z}, \mathsf{z}^*) = \mathbb{E}[Y(\mathsf{z}, M(\mathsf{z}^*))] = \iiint y f_{Y|M,X,Z}(y|m,x,\mathsf{z}) f_{M|X,Z}(m|x,\mathsf{z}^*) f_X(x)\, dy\, dm\, dx$$

with

$$\delta(\mathsf{z}) = \mu(\mathsf{z}, 1) - \mu(\mathsf{z}, 0) \qquad \zeta(\mathsf{z}) = \mu(1, \mathsf{z}) - \mu(0, \mathsf{z})$$

so in this case

$$\begin{aligned}\mu(\mathsf{z}, \mathsf{z}^*) &= \mathbb{E}^{\varepsilon}_{X,M}\left[\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + \phi M(\mathsf{z}^*) + \mathsf{z}(\psi_0 + \psi_1 X_1 + \psi_2 X_2 + \psi_3 X_1 X_2)\right] \\ &= \beta_0 + \beta_1 \mathbb{E}[X_1] + \beta_2 \mathbb{E}[X_2] + \beta_3 \mathbb{E}[X_1 X_2] + \phi \mathbb{E}[g(X, \mathsf{z}^*; \gamma)] + \mathsf{z}(\psi_0 + \psi_1 \mathbb{E}[X_1] + \psi_2 \mathbb{E}[X_2] + \psi_3 \mathbb{E}[X_1 X_2])\end{aligned}$$

Here we have that

$$\mathbb{E}[XX^\top] = \begin{bmatrix} \mathbb{E}[X_1^2] & \mathbb{E}[X_1X_2] & \mathbb{E}[X_1X_3] \\ \mathbb{E}[X_1X_2] & \mathbb{E}[X_2^2] & \mathbb{E}[X_2X_3] \\ \mathbb{E}[X_1X_3] & \mathbb{E}[X_2X_2] & \mathbb{E}[X_3^2] \end{bmatrix} = \Sigma + \mu\mu^\top = \begin{bmatrix} 0.0625 & 0.1125 & 0 \\ 0.1125 & 0.25 & 0 \\ 0 & 0 & 0.5625 \end{bmatrix} + \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

which yields

```
(X.M<-Sigma + Mu %*% t(Mu))

##          [,1]    [,2]    [,3]
## [1,]  1.0625 -1.8875 -1.0000
## [2,] -1.8875  4.2500  2.0000
## [3,] -1.0000  2.0000  1.5625
```

The quantity $\mathbb{E}[g(X, \mathsf{z}^*; \gamma)]$ cannot be computed analytically, but can be computed approximately using Monte Carlo.

```
#Compute using Monte Carlo
N<-100000
X0<-rmvn(N,mu=Mu,Sigma)
Xal<-cbind(1,X0[,1],X0[,2],X0[,3],X0[,1]*X0[,2])
Xgam0<-cbind(Xal,0)
Xgam1<-cbind(Xal,1)
E0<-mean(expit(Xgam0 %*% gam))
E1<-mean(expit(Xgam1 %*% gam))
c(E0,E1)

+ [1] 0.5114032 0.7802692
```

Therefore $\mu(0,0), \mu(1,0), \mu(0,1)$ and $\mu(1,1)$ are

```
treat.free<-sum(be*c(1,Mu[1],Mu[2],X.M[1,2]))
treat.contr<-sum(psi*c(1,Mu[1],Mu[2],X.M[1,2]))
mu00<-treat.free+phi*E0
mu10<-mu00 + treat.contr
mu01<-treat.free+phi*E1
mu11<-mu01 + treat.contr
c(mu00,mu01,mu10,mu11)

## [1] 5.490306 6.028038 3.602806 4.140538
```

and hence the causal quantities of interest are

$$\delta(1) = \mu(1,1) - \mu(1,0) = 4.1405383 - 3.6028064 = 0.5377319$$

$$\delta(0) = \mu(0,1) - \mu(0,0) = 6.0280383 - 5.4903064 = 0.5377319$$

$$\zeta(1) = \mu(1,1) - \mu(0,1) = 4.1405383 - 6.0280383 = -1.8875$$

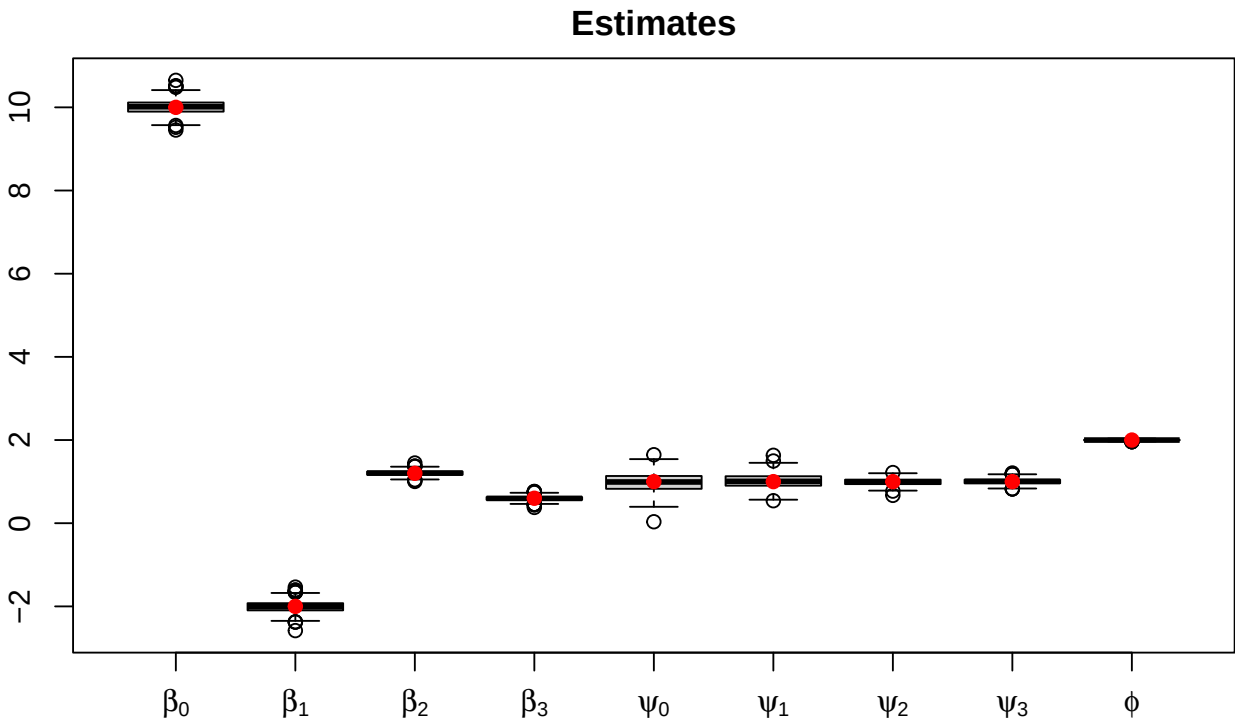
$$\zeta(0) = \mu(1,0) - \mu(0,0) = 3.6028064 - 5.4903064 = -1.8875$$

```r
ps.true<-expit(Xal %*% al)
Z<-rbinom(N,1,ps.true)
Xgam<-cbind(1,X[,1],X[,2],X[,3],X[,1]*X[,2],Z)
ps.med.true<-expit(Xgam %*% gam)
M<-rbinom(N,1,ps.med.true)
table(Z,M)/N
```

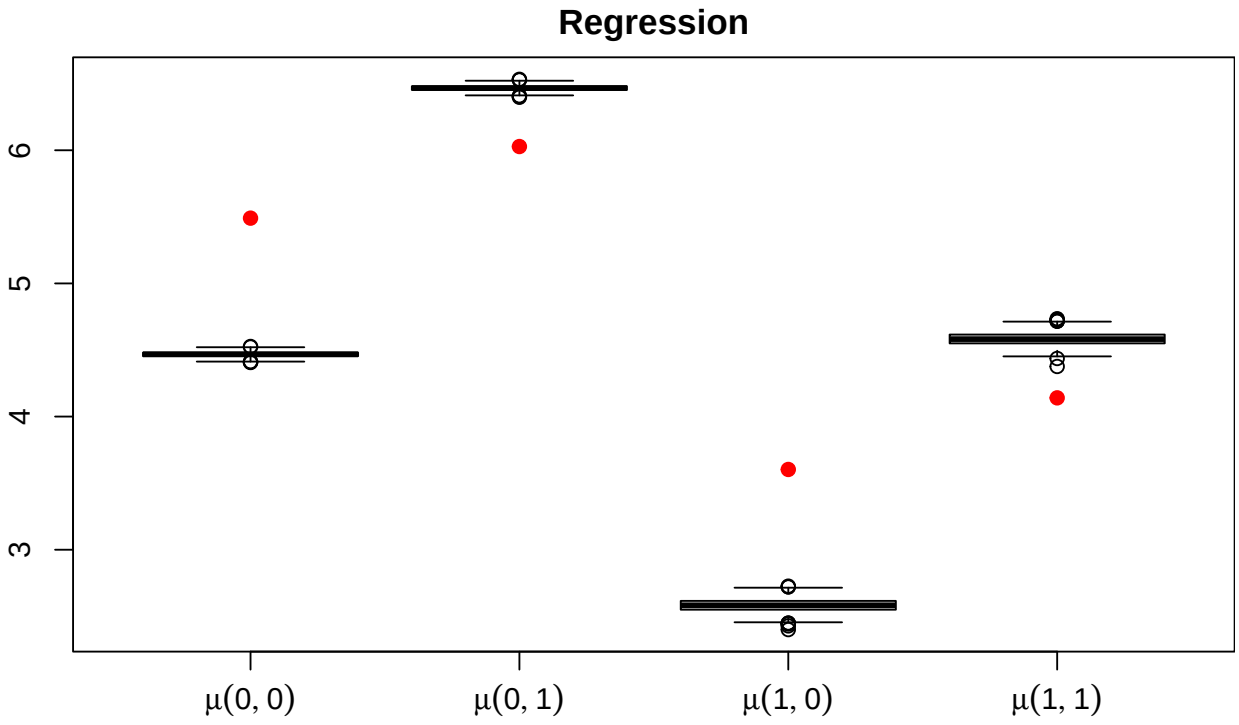
```
##    M
## Z         0       1
##   0 0.24159 0.24638
##   1 0.11464 0.39739
```

```r
set.seed(1234)
n<-500
nreps<-1000
ests.mat<-matrix(0,nrow=nreps,ncol=9);pred.mat<-matrix(0,nrow=nreps,ncol=4)
for(irep in 1:nreps){
    X<-rmvn(n,mu=Mu,Sigma)
    Xal<-cbind(1,X[,1],X[,2],X[,3],X[,1]*X[,2])
    ps.true<-expit(Xal %*% al)
    Z<-rbinom(n,1,ps.true)
    Xgam<-cbind(1,X[,1],X[,2],X[,3],X[,1]*X[,2],Z)
    ps.med.true<-expit(Xgam %*% gam)
    M<-rbinom(n,1,ps.med.true)
    Xb<-cbind(1,X[,1],X[,2],X[,1]*X[,2])
    mu0<- Xb %*% be
    Xm<-M
    mu1<-Xm*phi
    Xp<-cbind(1,X[,1],X[,2],X[,1]*X[,2])
    mu2<- (Xp %*% psi)
    eta<-mu0 + mu1 + Z * mu2 ; sig<-0.1
    Y<-rnorm(n,eta,sig)
    Xd<-Xp[,-1]
    fit0<-lm(Y~Xd*Z+M)
    ests.mat[irep,]<-coef(fit0)[c(1,2,3,4,5,7,8,9,6)]
    p00<-mean(predict(fit0,newdata=data.frame(Xd=Xd,Z=0,M=0)))
    p01<-mean(predict(fit0,newdata=data.frame(Xd=Xd,Z=0,M=1)))
    p10<-mean(predict(fit0,newdata=data.frame(Xd=Xd,Z=1,M=0)))
    p11<-mean(predict(fit0,newdata=data.frame(Xd=Xd,Z=1,M=1)))
    pred.mat[irep,]<-c(p00,p01,p10,p11)
}
```

```r
par(mar=c(4,4,2,0))
pnames<-c(expression(beta[0]),expression(beta[1]),expression(beta[2]),expression(beta[3]),
          expression(psi[0]),expression(psi[1]),expression(psi[2]),expression(psi[3]),
          expression(phi))
boxplot(ests.mat,names=pnames,main='Estimates')
points(1:9,true.vals,col='red',pch=19)
```

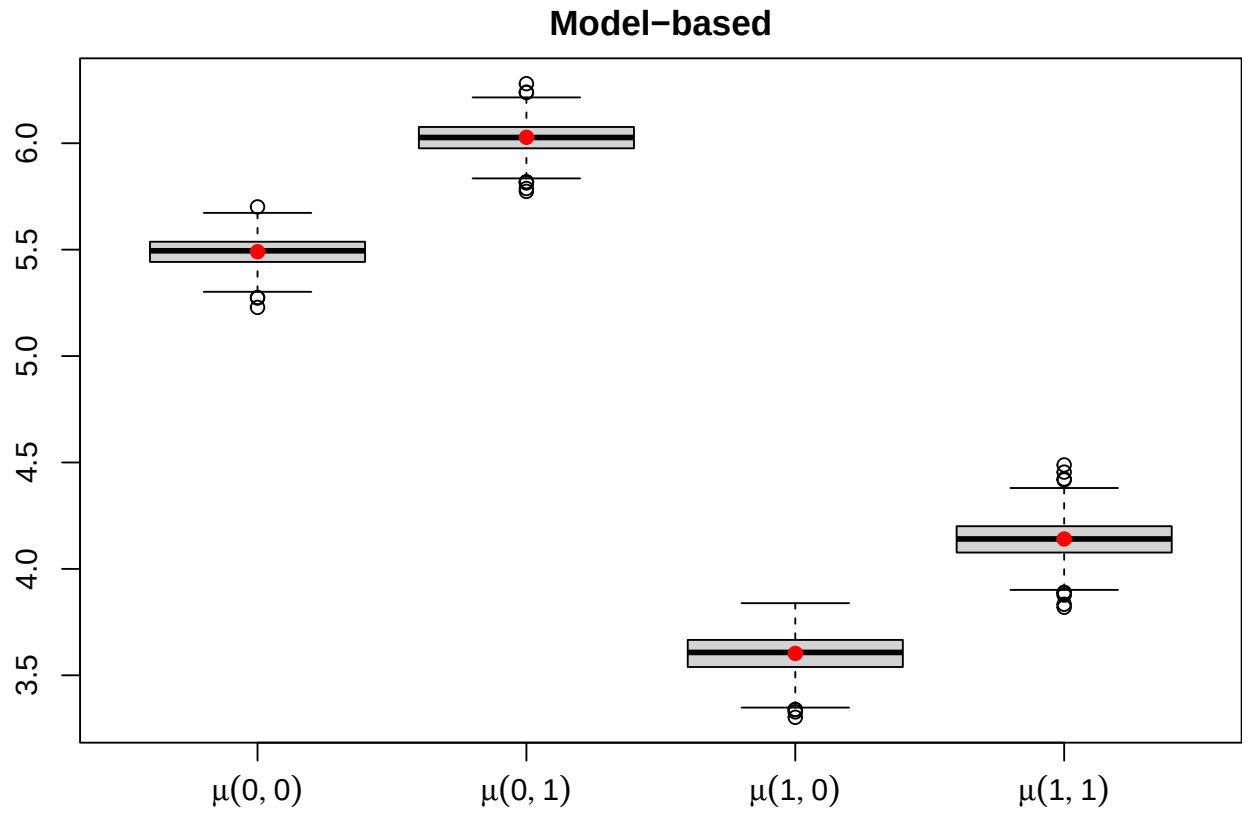



```
mnames<-c(expression(mu(0,0)),expression(mu(0,1)),expression(mu(1,0)),expression(mu(1,1)))
boxplot(pred.mat,names=mnames,main='Regression')
points(1:4,c(mu00,mu01,mu10,mu11),col='red',pch=19)
```



```r
set.seed(1234)
n<-500
N<-10000
nreps<-1000
ests.mat.reg<-matrix(0,nrow=nreps,ncol=9)
pred.mat.reg<-matrix(0,nrow=nreps,ncol=4)
for(irep in 1:nreps){
    X<-rmvn(n,mu=Mu,Sigma)
    Xal<-cbind(1,X[,1],X[,2],X[,3],X[,1]*X[,2])
    ps.true<-expit(Xal %*% al)
    Z<-rbinom(n,1,ps.true)
    Xgam<-cbind(1,X[,1],X[,2],X[,3],X[,1]*X[,2],Z)
    ps.med.true<-expit(Xgam %*% gam)
    M<-rbinom(n,1,ps.med.true)
    Xb<-Xp<-cbind(1,X[,1],X[,2],X[,1]*X[,2])
    mu0<- Xb %*% be
    mu1<-M*phi
    mu2<- (Xp %*% psi)
    eta<-mu0 + mu1 + Z * mu2 ; sig<-0.1
    Y<-rnorm(n,eta,sig)

    X1<-X[,1]
    X2<-X[,2]
    X3<-X[,3]
    X12<-X1*X2

    fit0<-glm(M~-1+Xgam,family=binomial)
    fit1<-lm(Y~(X1+X2+X12)*Z+M)

    iboot<-sample(1:n,size=N,rep=T)
    X1<-X1[iboot]
    X2<-X2[iboot]
    X3<-X3[iboot]
    X12<-X1*X2

    M0<-rbinom(N,1,expit(cbind(1,X1,X2,X3,X12,0) %*% coef(fit0)))
    M1<-rbinom(N,1,expit(cbind(1,X1,X2,X3,X12,1) %*% coef(fit0)))

    d00<-data.frame(X1,X2,X12,Z=0,M=M0)
    d01<-data.frame(X1,X2,X12,Z=0,M=M1)
    d10<-data.frame(X1,X2,X12,Z=1,M=M0)
    d11<-data.frame(X1,X2,X12,Z=1,M=M1)

    sig<-summary(fit1)$sigma
    Y00<-rnorm(N)*sig + predict(fit1,newdata=d00)
    Y01<-rnorm(N)*sig + predict(fit1,newdata=d01)
    Y10<-rnorm(N)*sig + predict(fit1,newdata=d10)
    Y11<-rnorm(N)*sig + predict(fit1,newdata=d11)
    p00<-mean(Y00)
    p10<-mean(Y10)
    p01<-mean(Y01)
    p11<-mean(Y11)
    pred.mat.reg[irep,]<-c(p00,p01,p10,p11)
}
```

```r
par(mar=c(4,4,2,0))
mnames<-c(expression(mu(0,0)),expression(mu(0,1)),expression(mu(1,0)),expression(mu(1,1)))
boxplot(pred.mat.reg,names=mnames,main='Model-based')
points(1:4,c(mu00,mu01,mu10,mu11),col='red',pch=19)
```



Reweighting: note that we may rewrite

$$\mu(\mathsf{z},\mathsf{z}^*) = \mathbb{E}[Y(\mathsf{z}, M(\mathsf{z}^*))] = \iiint y f_{Y|M,X,Z}(y|m,x,\mathsf{z}) f_{M|X,Z}(m|x,\mathsf{z}^*) f_X(x)\, dy\, dm\, dx$$

$$= \iiint y f_{Y|M,X,Z}(y|m,x,\mathsf{z}) \frac{f_{M|X,Z}(m|x,\mathsf{z}^*)}{f_{M|X,Z}(m|x,\mathsf{z})} f_{M|X,Z}(m|x,\mathsf{z}) f_X(x)\, dy\, dm\, dx$$

which suggests the inverse weighting estimator

$$\widehat{\mu}(\mathsf{z},\mathsf{z}^*) = \sum_{i=1}^n W_i(\mathsf{z},\mathsf{z}^*) Y_i$$

where

$$W_i(\mathsf{z},\mathsf{z}^*) = \frac{w_i(\mathsf{z},\mathsf{z}^*)}{\sum_{j=1}^n w_j(\mathsf{z},\mathsf{z}^*)}$$

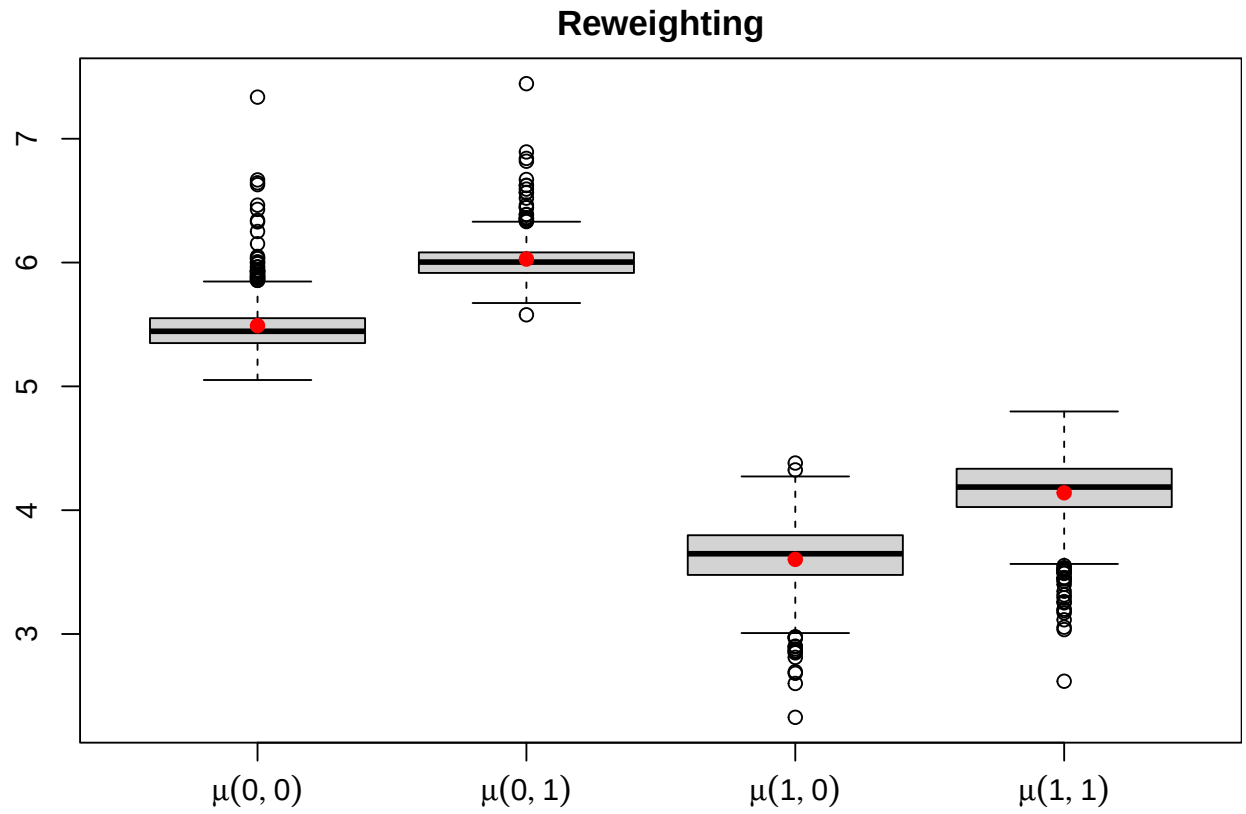
where

$$w_i(\mathsf{z},\mathsf{z}^*) = \mathbb{1}_{\mathsf{z}}(Z_i) \frac{f_{M|X,Z}(M_i|X_i,\mathsf{z}^*)}{f_{M|X,Z}(M_i|X_i,\mathsf{z})} \frac{1}{f_{Z|X}(Z_i|X_i)}$$

```
set.seed(1234)
n<-500
nreps<-1000
pred.mat.w<-matrix(0,nrow=nreps,ncol=4)
for(irep in 1:nreps){
    X<-rmvn(n,mu=Mu,Sigma)
```

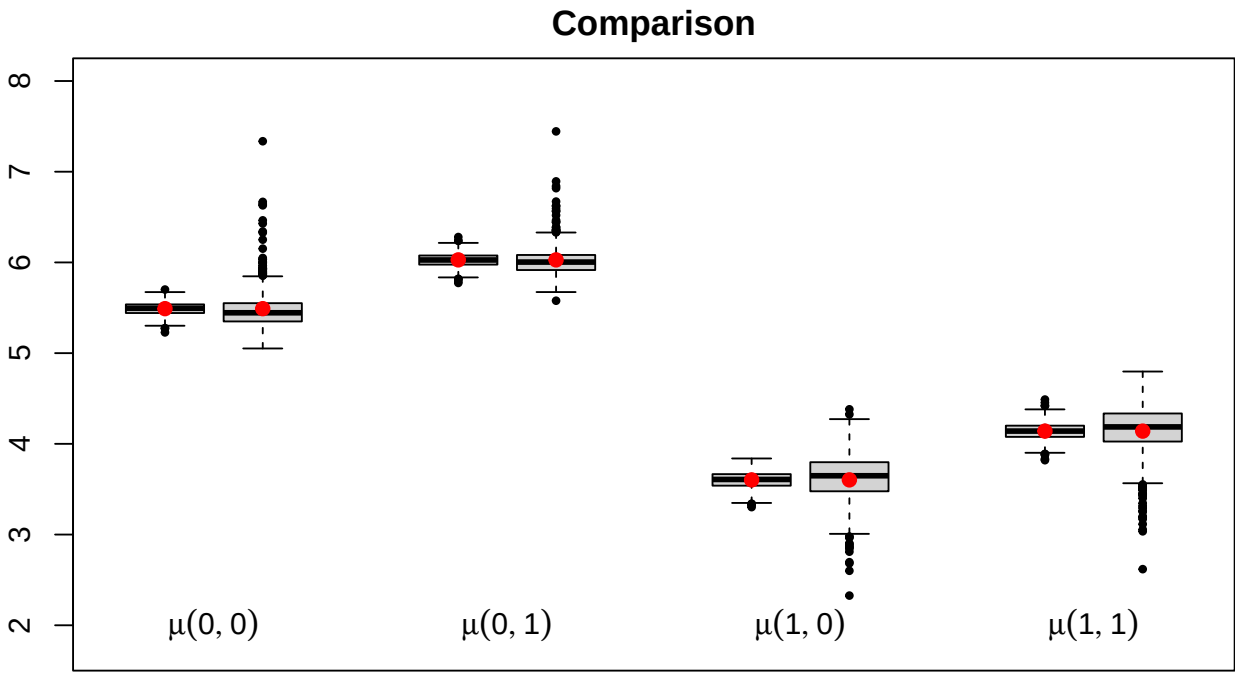
```
    Xal<-cbind(1,X[,1],X[,2],X[,3],X[,1]*X[,2])
    ps.true<-expit(Xal %*% al)
    Z<-rbinom(n,1,ps.true)
    Xgam<-cbind(1,X[,1],X[,2],X[,3],X[,1]*X[,2],Z)
    ps.med.true<-expit(Xgam %*% gam)
    M<-rbinom(n,1,ps.med.true)
    Xb<-cbind(1,X[,1],X[,2],X[,1]*X[,2])
    mu0<- Xb %*% be
    Xm<-M
    mu1<-Xm*phi
    Xp<-cbind(1,X[,1],X[,2],X[,1]*X[,2])
    mu2<- (Xp %*% psi)
    eta<-mu0 + mu1 + Z * mu2 ; sig<-0.1
    Y<-rnorm(n,eta,sig)

    X1<-X[,1]
    X2<-X[,2]
    X3<-X[,3]
    X12<-X1*X2

    fit0<-glm(M~X1+X2+X3+X12+Z,family=binomial)
    iptw<-Z/ps.true+(1-Z)/(1-ps.true)

    d0<-data.frame(X1,X2,X3,X12,Z=0)
    d1<-data.frame(X1,X2,X3,X12,Z=1)

    p0<-predict(fit0,newdata=d0,type='response')
    p1<-predict(fit0,newdata=d1,type='response')

    ipmw00<-1
    ipmw01<-(1-M)*(1-p1)/(1-p0)+M*p1/p0
    ipmw10<-(1-M)*(1-p0)/(1-p1)+M*p0/p1
    ipmw11<-1

    p00<-sum((Z==0)*ipmw00*iptw*Y)/sum((Z==0)*ipmw00*iptw)
    p10<-sum((Z==1)*ipmw10*iptw*Y)/sum((Z==1)*ipmw10*iptw)
    p01<-sum((Z==0)*ipmw01*iptw*Y)/sum((Z==0)*ipmw01*iptw)
    p11<-sum((Z==1)*ipmw11*iptw*Y)/sum((Z==1)*ipmw11*iptw)
    pred.mat.w[irep,]<-c(p00,p01,p10,p11)
}
```

```
par(mar=c(4,4,2,0))
mnames<-c(expression(mu(0,0)),expression(mu(0,1)),expression(mu(1,0)),expression(mu(1,1)))
boxplot(pred.mat.w,names=mnames,main='Reweighting')
points(1:4,c(mu00,mu01,mu10,mu11),col='red',pch=19)
```




```
apply(pred.mat.reg,2,var)

## [1] 0.004821077 0.005268265 0.008289915 0.008121462

apply(pred.mat.w,2,var)

## [1] 0.03840914 0.02183572 0.06091249 0.06439552

apply(pred.mat.reg,2,var)/apply(pred.mat.w,2,var)

## [1] 0.1255190 0.2412682 0.1360955 0.1261184
```

```
res<-cbind(pred.mat.reg,pred.mat.w)[,c(1,5,2,6,3,7,4,8)]
par(mar=c(6,4,2,0),xaxt='n')
mnames<-c(expression(mu(0,0)),expression(mu(0,1)),expression(mu(1,0)),expression(mu(1,1)))
boxplot(res,main='Comparison',at=c(1,2,4,5,7,8,10,11),ylim=range(1.75,8),pch=19,cex=0.5)
points(c(1,2,4,5,7,8,10,11),rep(c(mu00,mu01,mu10,mu11),each=2),col='red',pch=19)
text(c(1.5,4.5,7.5,10.5),rep(2,4),mnames)
```


Comparison
2
3
4
5
6
7
8
μ(0, 0)
μ(0, 1)
μ(1, 0)
μ(1, 1)