

CAUSAL CONTRASTS FROM OUTCOME REGRESSION

In estimating the APO, we refer to the *experimental* distribution where X and Z are *independent*

$$f_{X,Y,Z}^{\mathcal{E}}(x, y, z) = f_X^{\mathcal{E}}(x) f_Z^{\mathcal{E}}(z) f_{Y|X,Z}^{\mathcal{E}}(y|x, z)$$

and compute, for any fixed target z

$$\mathbb{E}[Y(z)] \equiv \mathbb{E}_{Y|Z}^{\mathcal{E}}[Y|Z = z] = \int y f_{Y|Z}^{\mathcal{E}}(y|z) dy = \iint y f_{Y|Z}^{\mathcal{E}}(y|z) f_X^{\mathcal{E}}(x) dy dx$$

However, data may only be available from the more general *observational* distribution where X and Z are *not independent*

$$f_{X,Y,Z}^{\mathcal{O}}(x, y, z) = f_X^{\mathcal{O}}(x) f_{Z|X}^{\mathcal{O}}(z|x) f_{Y|X,Z}^{\mathcal{O}}(y|x, z).$$

In the causal context, where we wish to make statements in a defined population, it follows that we may consider $f_X^{\mathcal{E}}(x) = f_X^{\mathcal{O}}(x)$ for all x . In addition, we will assume

$$f_{Y|X,Z}^{\mathcal{E}}(y|x, z) \equiv f_{Y|X,Z}^{\mathcal{O}}(y|x, z) \quad \text{for all } x, y, z$$

as our hypothetical experimental distribution is identical to the observational distribution except for the fact that we imagine intervening to set Z independent of X . It follows that

$$\mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = z] \equiv \mathbb{E}_{Y|X,Z}^{\mathcal{O}}[Y|X = x, Z = z]$$

even though, in general,

$$\mathbb{E}_{Y|Z}^{\mathcal{E}}[Y|Z = z] \neq \mathbb{E}_{Y|Z}^{\mathcal{O}}[Y|Z = z].$$

However, we may write

$$\begin{aligned} \mathbb{E}_{Y|Z}^{\mathcal{E}}[Y|Z = z] &= \iint y f_{Y|Z}^{\mathcal{E}}(y|z) f_X^{\mathcal{E}}(x) dy dx \\ &= \int \left\{ y f_{Y|Z}^{\mathcal{E}}(y|z) f_X^{\mathcal{E}}(x) dy \right\} f_X^{\mathcal{E}}(x) dx \\ &= \int \mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = z] f_X^{\mathcal{E}}(x) dx \\ &\equiv \int \mathbb{E}_{Y|X,Z}^{\mathcal{O}}[Y|X = x, Z = z] f_X^{\mathcal{O}}(x) dx \\ &= \mathbb{E}_X^{\mathcal{O}} \left[\mathbb{E}_{Y|X,Z}^{\mathcal{O}}[Y|X, Z = z] \right] \quad (\text{note } \neq \mathbb{E}_{Y|Z}^{\mathcal{O}}[Y|Z = z]) \end{aligned}$$

which means that if we *know* the functional form of

$$\mu(x, z) = \mathbb{E}_{Y|X,Z}^{\mathcal{O}}[Y|X = x, Z = z]$$

we may define a *model-based estimator* of the APO as

$$\hat{\mathbb{E}}_{Y|Z}^{\mathcal{E}}[Y|Z = z] = \frac{1}{n} \sum_{i=1}^n \mu(X_i, z)$$

and deduce from standard frequentist arguments that this estimator is unbiased. However, if we need to estimate $\mu(x, z)$ from the observational data (in some parametric model say), this approach is not guaranteed to produce an unbiased estimator.

Experimental Case: In a randomized experimental study, inference concerning the causal effect of binary variable Z on Y can be made by direct comparison of sample averages. Suppose that $Z \sim \text{Bernoulli}(p)$ for $0 < p < 1$. Two estimators of the causal contrast $\delta = \mathbb{E}[Y(1) - Y(0)]$ can be proposed if the conditional mean model

$$\mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = z] = \mathbb{E}_{Y|X,Z}^{\mathcal{O}}[Y|X = x, Z = z] = \mu(x, z)$$

is presumed known. We have for the standard estimator

$$\hat{\delta}_{OR} = \frac{1}{n} \sum_{i=1}^n \mu(X_i, 1) - \frac{1}{n} \sum_{i=1}^n \mu(X_i, 0) \quad (1)$$

but also for a group-specific estimator

$$\hat{\delta}_{OR2} = \frac{1}{n\hat{p}} \sum_{i=1}^n Z_i \mu(X_i, Z_i) - \frac{1}{n(1-\hat{p})} \sum_{i=1}^n (1 - Z_i) \mu(X_i, Z_i) \quad (2)$$

where $\hat{p} = n^{-1} \sum_{i=1}^n Z_i$.

(a) **Correct specification:** In the simulation below, we assume that $X \sim \text{Normal}(1, 1)$ and

$$Y|X = x, Z = z \sim \text{Normal}(0.5x + \delta z, 1)$$

so that $\mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = z] = 0.5x + \delta z$, and

$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_X^{\mathcal{E}} \left[\mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X, Z = 1] - \mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X, Z = 0] \right] = \delta.$$

In this case, $\mu(X, 1) - \mu(X, 0) = \delta$ which does not depend on X , so we can estimate δ without error using δ_{OR}

```
set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
  X<-rnorm(n,1,1)
  Z<-rbinom(n,1,p)
  Y<-rnorm(n,delta*Z+0.5*X,1)
  p.hat<-mean(Z)
  ests.mat[irep,1]<-mean(delta+0.5*X)-mean(0.5*X)
  ests.mat[irep,2]<-sum(Z*(delta*Z+0.5*X))/(n*p.hat)-
    sum((1-Z)*(delta*Z+0.5*X))/(n*(1-p.hat))
}
apply(ests.mat,2,mean) #True value=delta=2.0

+ [1] 2.000000 1.997433

apply(ests.mat,2,var)

+ [1] 7.254914e-33 9.881624e-03
```

(b) **Interaction model:** Suppose now that

$$\mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = z] = 0.5x + \delta z + 2xz$$

so that

$$\mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = 1] - \mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = 0] = \delta + 2x.$$

and thus

$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_X^{\mathcal{E}}[\delta + 2X] = \delta + 2\mathbb{E}_X^{\mathcal{E}}[X] = \delta + 2$$

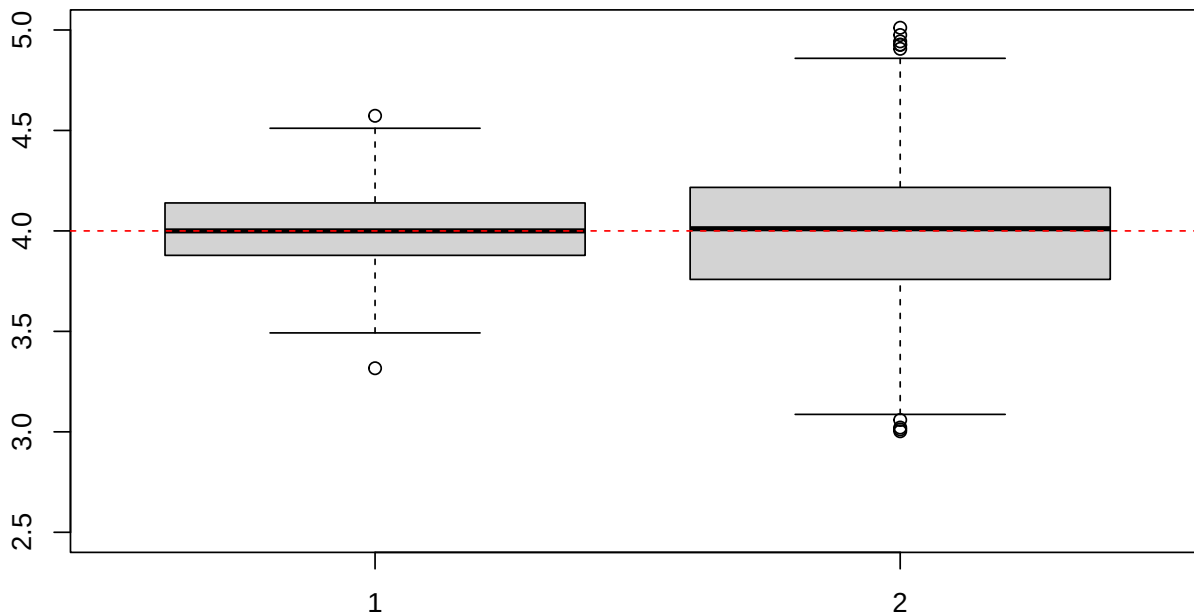
```
set.seed(23); nreps<-1000; n<-100; p<-0.5; delta<-2
ests.mat<-matrix(0, nrow=nreps, ncol=2)
for(irep in 1:nreps){
  X<-rnorm(n, 1, 1)
  Z<-rbinom(n, 1, p)
  Y<-rnorm(n, delta*Z+0.5*X + 2*Z*X, 1)
  p.hat<-mean(Z)
  ests.mat[irep,1]<-mean(delta+0.5*X+2*X)-mean(0.5*X)
  ests.mat[irep,2]<-sum(Z*(delta*Z+0.5*X+ 2*X*Z))/(n*p.hat)-
    sum((1-Z)*(delta*Z+0.5*X+2*X*Z))/(n*(1-p.hat))
}
apply(ests.mat, 2, mean) #True value=delta+2 = 4.0

+ [1] 4.005106 3.997452

apply(ests.mat, 2, var)

+ [1] 0.03626926 0.12555203

par(mar=c(4, 2, 1, 1))
boxplot(ests.mat, ylim=range(2.5, 5)); abline(h=delta+2, lty=2, col='red')
```



(c) **Mis-specification:** Suppose we **mis-specify** the modelled mean using

$$m(x, z) = x - 2x^2 + \delta z + 2xz$$

that is, where the dependence on z is **correct**, but it is not the true conditional mean. We can still unbiasedly estimate δ using the same estimator:

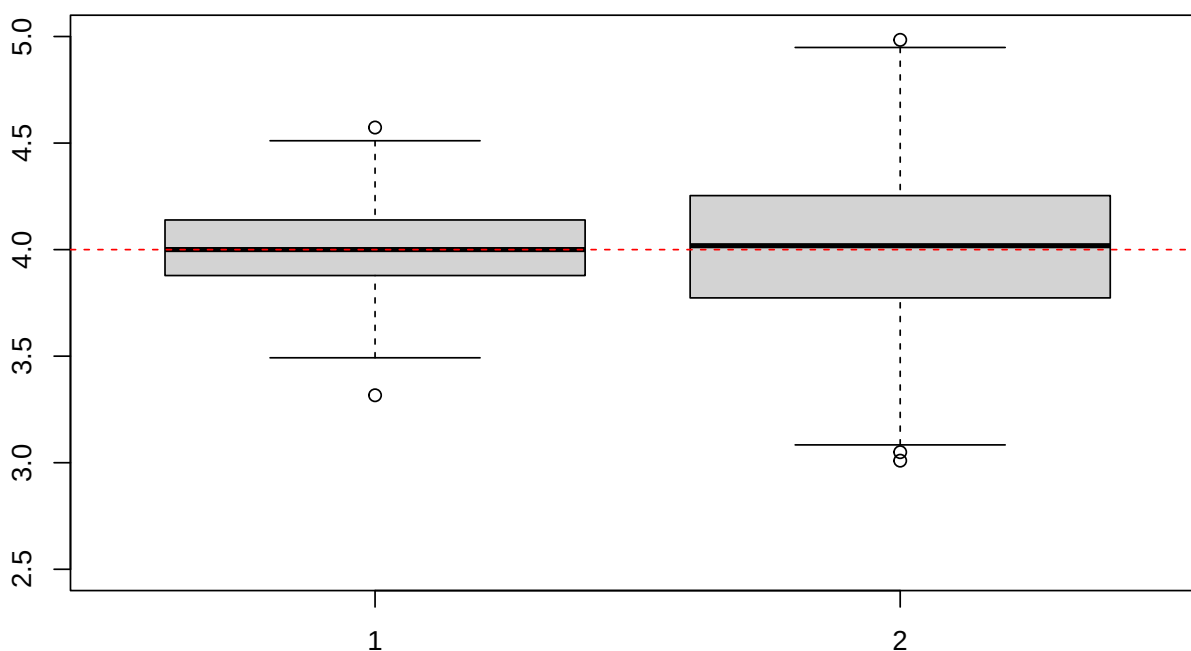
```
set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
  X<-rnorm(n,1,1)
  Z<-rbinom(n,1,p)
  Y<-rnorm(n,delta*Z+0.5*X + 2*Z*X,1)
  p.hat<-mean(Z)
  ests.mat[irep,1]<-mean(delta*X-X^2+2*X)-mean(X-X^2)
  ests.mat[irep,2]<-sum(Z*(delta*Z+X-X^2+2*Z*X))/(n*p.hat)-
    sum((1-Z)*(delta*Z+X-X^2+2*Z*X))/(n*(1-p.hat))
}
apply(ests.mat,2,mean) #True value=delta+2=4.0

+ [1] 4.005106 4.009763

apply(ests.mat,2,var)

+ [1] 0.03626926 0.11866337

par(mar=c(4,2,1,1))
boxplot(ests.mat,ylim=range(2.5,5));abline(h=delta+2,lty=2,col='red')
```



However, if we **mis-specify** the modelled mean using

$$m(x, z) = x - 2x^2 + \delta z + xz$$

that is, where the dependence on z is **incorrect**, and it is not the true conditional mean, we can no longer unbiasedly estimate δ using the same estimator:

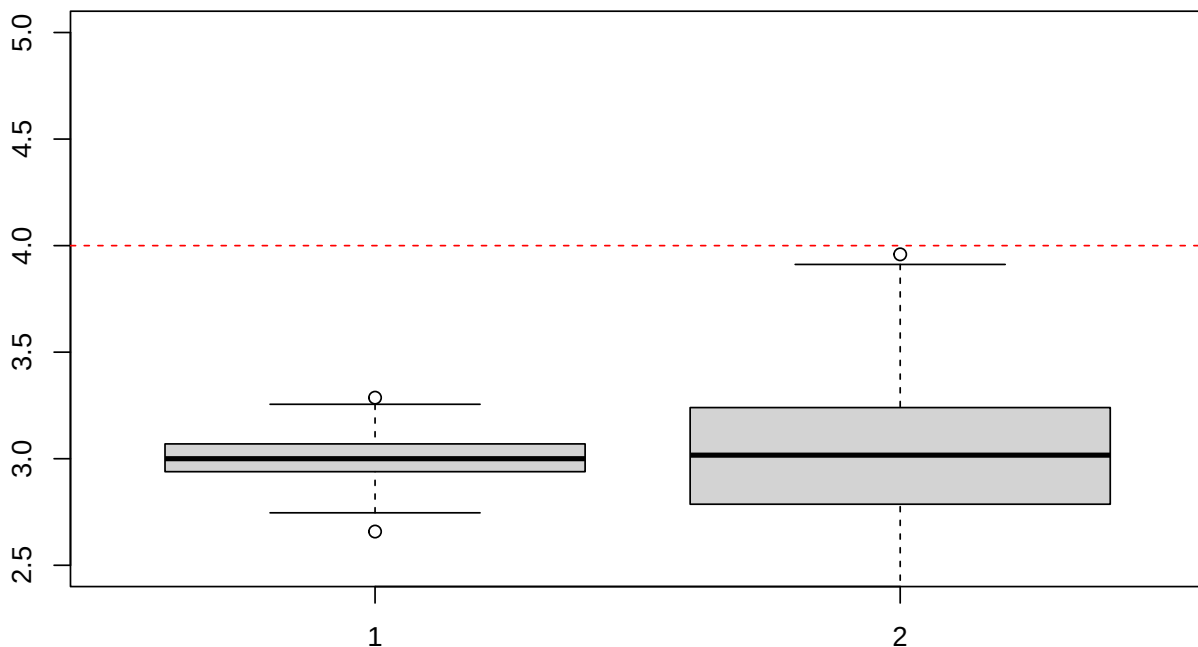
```
set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
  X<-rnorm(n,1,1)
  Z<-rbinom(n,1,p)
  Y<-rnorm(n,delta*Z+0.5*X + 2*Z*X,1)
  p.hat<-mean(Z)
  ests.mat[irep,1]<-mean(delta*X-X^2+X)-mean(X-X^2)
  ests.mat[irep,2]<-sum(Z*(delta*Z+X-X^2+Z*X))/(n*p.hat)-
    sum((1-Z)*(delta*Z+X-X^2+Z*X))/(n*(1-p.hat))
}
apply(ests.mat,2,mean) #True value=delta+2=4.0

+ [1] 3.002553 3.009753

apply(ests.mat,2,var)

+ [1] 0.009067314 0.101345327

par(mar=c(4,2,1,1))
boxplot(ests.mat,ylim=range(2.5,5));abline(h=delta+2,lty=2,col='red')
```



Observational Case: Suppose that $Z|X = x \sim \text{Bernoulli}(p(x))$ where

$$p(x) = \frac{e^x}{1 + e^x}$$

so that Z and X are not independent. We examine the estimator of the causal contrast from (1) again assuming different forms for $\mu(x, z)$, and compare with the direct contrast estimator that compares the mean in the two groups.

(a) **Correct specification:** In the simulation below, we assume that $X \sim \text{Normal}(1, 1)$ and

$$Y|X = x, Z = z \sim \text{Normal}(0.5x + \delta z, 1)$$

so that $\mu(x, z) = 0.5x + \delta z$, so $\mu(X, 1) - \mu(X, 0) = \delta$.

```
set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat.obs<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
  X<-rnorm(n,1,1)
  px<-exp(X)/(1+exp(X))
  Z<-rbinom(n,1,px)
  Y<-rnorm(n,delta*Z+0.5*X,1)
  ests.mat.obs[irep,1]<-mean(delta+0.5*X)-mean(0.5*X)
  ests.mat.obs[irep,2]<-mean(Y[Z==1])-mean(Y[Z==0]) #Comparison of means
}
apply(ests.mat.obs,2,mean) #True value=delta=2.0

+ [1] 2.000000 2.415635

apply(ests.mat.obs,2,var)

+ [1] 7.254914e-33 5.792122e-02
```

(b) **Interaction model:** If

$$\mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = z] = 0.5x + \delta z + 2xz$$

so that

$$\mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = 1] - \mathbb{E}_{Y|X,Z}^{\mathcal{E}}[Y|X = x, Z = 0] = \delta + 2x.$$

and thus

$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}_X^{\mathcal{E}}[\delta + 2X] = \delta + 2\mathbb{E}_X^{\mathcal{E}}[X] = \delta + 2$$

```
set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat.obs<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
  X<-rnorm(n,1,1)
  px<-exp(X)/(1+exp(X))
  Z<-rbinom(n,1,px)
  Y<-rnorm(n,delta*Z+0.5*X + 2*Z*X,1)
  p.hat<-mean(Z)
```

```

ests.mat.obs[irep,1]<-mean(delta+0.5*X+2*X)-mean(0.5*X)
ests.mat.obs[irep,2]<-mean(Y[Z==1])-mean(Y[Z==0])
}
apply(ests.mat.obs,2,mean) #True value=delta+2=4.0

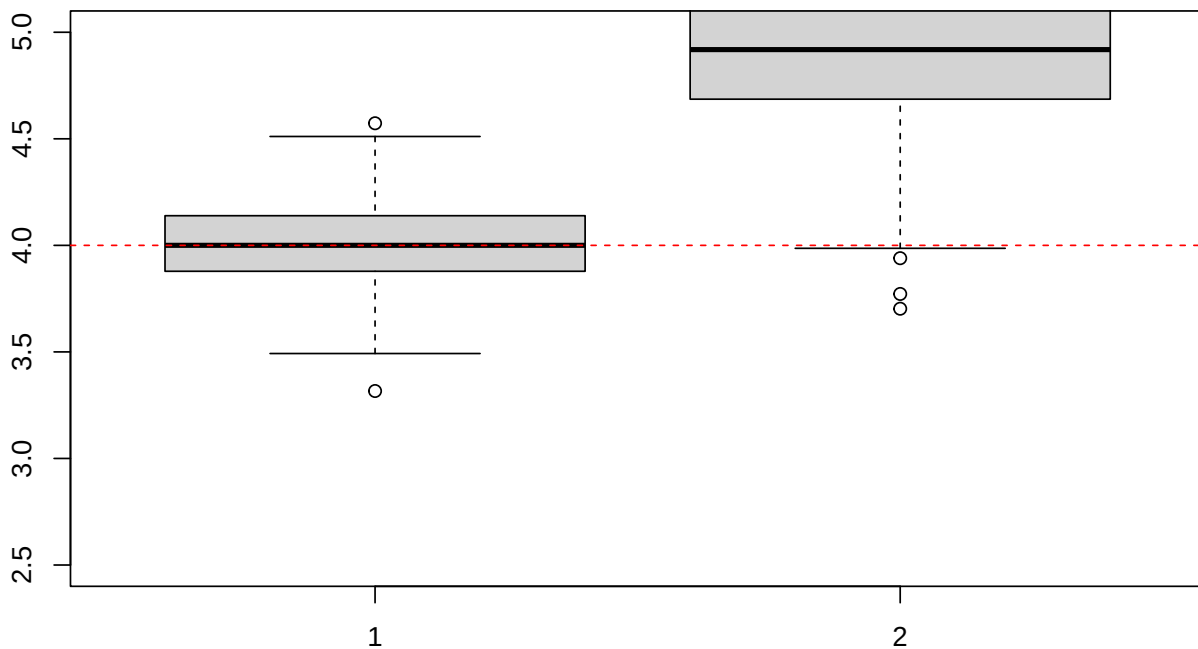
+ [1] 4.005106 4.929942

apply(ests.mat.obs,2,var)

+ [1] 0.03626926 0.13022673

par(mar=c(4,2,1,1))
boxplot(ests.mat.obs,ylim=range(2.5,5));abline(h=delta+2,lty=2,col='red')

```



(c) **Mis-specification:** Suppose we **mis-specify** the modelled mean using

$$m(x, z) = x - 2x^2 + \delta z + 2xz$$

that is, where the dependence on z is **correct**, but it is not the true conditional mean. We can still unbiasedly estimate δ using the same estimator:

```

set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat.obs<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
  X<-rnorm(n,1,1)
  px<-exp(X)/(1+exp(X))
  Z<-rbinom(n,1,px)
  Y<-rnorm(n,delta*Z+0.5*X + 2*Z*X,1)
}

```

```

ests.mat.obs[irep,1]<-mean(delta+X-X^2+2*X)-mean(X-X^2)
ests.mat.obs[irep,2]<-mean(Y[Z==1])-mean(Y[Z==0])
}
apply(ests.mat.obs,2,mean) #True value=delta+2=4.0

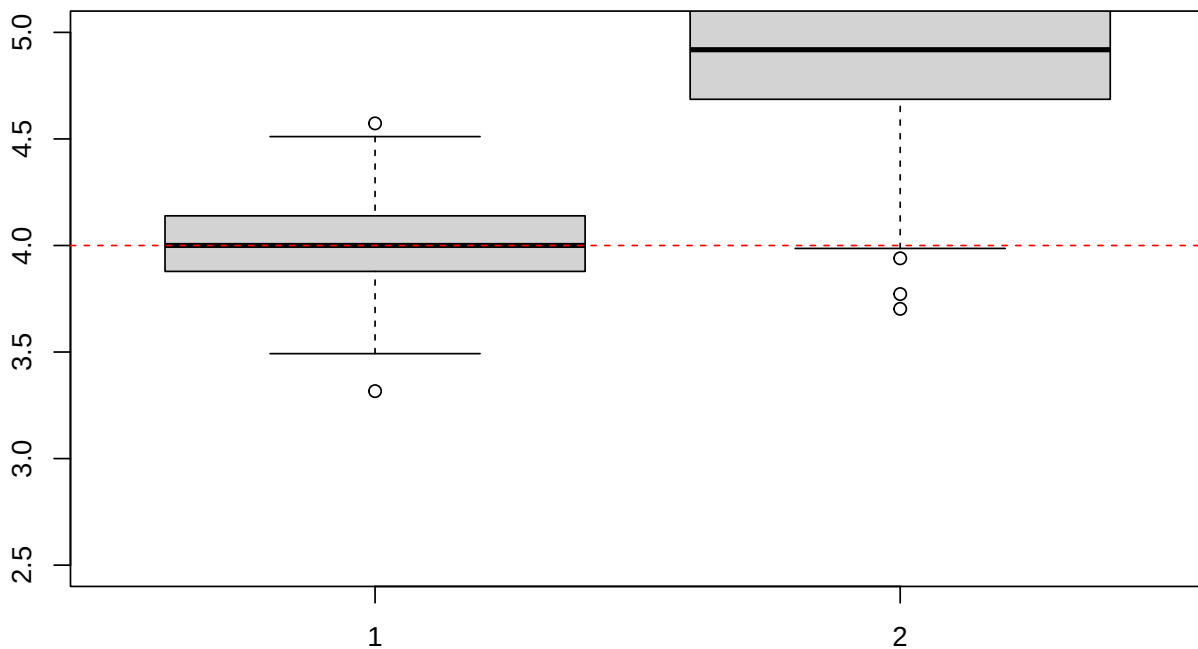
+ [1] 4.005106 4.929942

apply(ests.mat.obs,2,var)

+ [1] 0.03626926 0.13022673

par(mar=c(4,2,1,1))
boxplot(ests.mat.obs,ylim=range(2.5,5));abline(h=delta+2,lty=2,col='red')

```



However, if we **mis-specify** the modelled mean using

$$m(x, z) = x - 2x^2 + \delta z + xz$$

that is, where the dependence on z is **incorrect**, and it is not the true conditional mean, we can no longer unbiasedly estimate δ using the same estimator:

```

set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat.obs<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
  X<-rnorm(n,1,1)
  px<-exp(X)/(1+exp(X))
  Z<-rbinom(n,1,px)
  Y<-rnorm(n,delta*Z+0.5*X + 2*Z*X,1)
}

```

```

p.hat<-mean(Z)
ests.mat.obs[irep,1]<-mean(delta+X-X^2+X)-mean(X-X^2)
ests.mat.obs[irep,2]<-mean(Y[Z==1])-mean(Y[Z==0])
}
apply(ests.mat.obs,2,mean) #True value=delta+2=4.0

+ [1] 3.002553 4.929942

apply(ests.mat.obs,2,var)

+ [1] 0.009067314 0.130226727

par(mar=c(4,2,1,1))
boxplot(ests.mat.obs,ylim=range(2.5,5));abline(h=delta+2,lty=2,col='red')

```

