AUGMENTED INVERSE PROBABILITY WEIGHTING

The propensity score for binary exposure, denoted $e(X)$,

$$e(x) = f^{\mathcal{O}}_{Z|X}(1|x) = \Pr^{\mathcal{O}}_{Z|X}[Z = 1|X = x]$$

can be used to construct generalized versions of the inverse probability weighted (IPW) estimators based on augmentation via the conditional mean model $\mu(x, z)$

$$\mathbb{E}^{\mathcal{O}}_{Y|X,Z}[Y|X = x, Z = z] = \mu(x, z)$$

We deduce that augmented IPW (AIPW) estimators are

$$\widetilde{\mu}_{\text{AIPW}}(0) = \frac{1}{n}\sum_{i=1}^{n} \frac{(1 - Z_i)(Y_i - \mu(X_i, 0))}{1 - e(X_i)} + \frac{1}{n}\sum_{i=1}^{n} \mu(X_i, 0)$$

$$\widetilde{\mu}_{\text{AIPW}}(1) = \frac{1}{n}\sum_{i=1}^{n} \frac{Z_i(Y_i - \mu(X_i, 1))}{e(X_i)} + \frac{1}{n}\sum_{i=1}^{n} \mu(X_i, 1)$$

or, in standardized weight form

$$\widehat{\mu}_{\text{AIPW}}(0) = \sum_{i=1}^{n} W_{i0}(Y_i - \mu(X_i, 0)) + \frac{1}{n}\sum_{i=1}^{n} \mu(X_i, 0)$$

$$\widehat{\mu}_{\text{AIPW}}(1) = \sum_{i=1}^{n} W_{i1}(Y_i - \mu(X_i, 1)) + \frac{1}{n}\sum_{i=1}^{n} \mu(X_i, 1)$$

with

$$W_{i0} = \frac{\dfrac{(1 - Z_i)}{(1 - e(X_i))}}{\displaystyle\sum_{j=1}^{n} \frac{(1 - Z_j)}{(1 - e(X_j))}} \qquad\qquad W_{i1} = \frac{\dfrac{Z_i}{e(X_i)}}{\displaystyle\sum_{j=1}^{n} \frac{Z_j}{e(X_j)}}$$

These estimators are generalizations of IPW estimators, and are more robust to misspecification: the estimators are consistent provided **at least one of** the two models $e(x)$ or $\mu(x, z)$ is correctly specified. That is, estimators that use instead the models

$$g(x) \qquad \text{and} \qquad m(x, z)$$

for the propensity score and conditional mean model respectively are still consistent provided either $g(x) \equiv e(x)$ or $m(x, z) \equiv \mu(x, z)$. If **both** are correctly specified, the estimators are the optimal IPW estimators.

Simulation study: In the following simulation, we have $k = 10$ predictors, but only the first three are confounders. We compare the results for the following models:

Model 0 : Outcome regression under correct specification of $\mu(x, z)$

Model 1 : IPW with correct specification of $e(x)$

Model 2 : AIPW with correct specification of both $e(x)$ and $\mu(x, z)$

Model 3 : AIPW with correct specification of $e(x)$ only

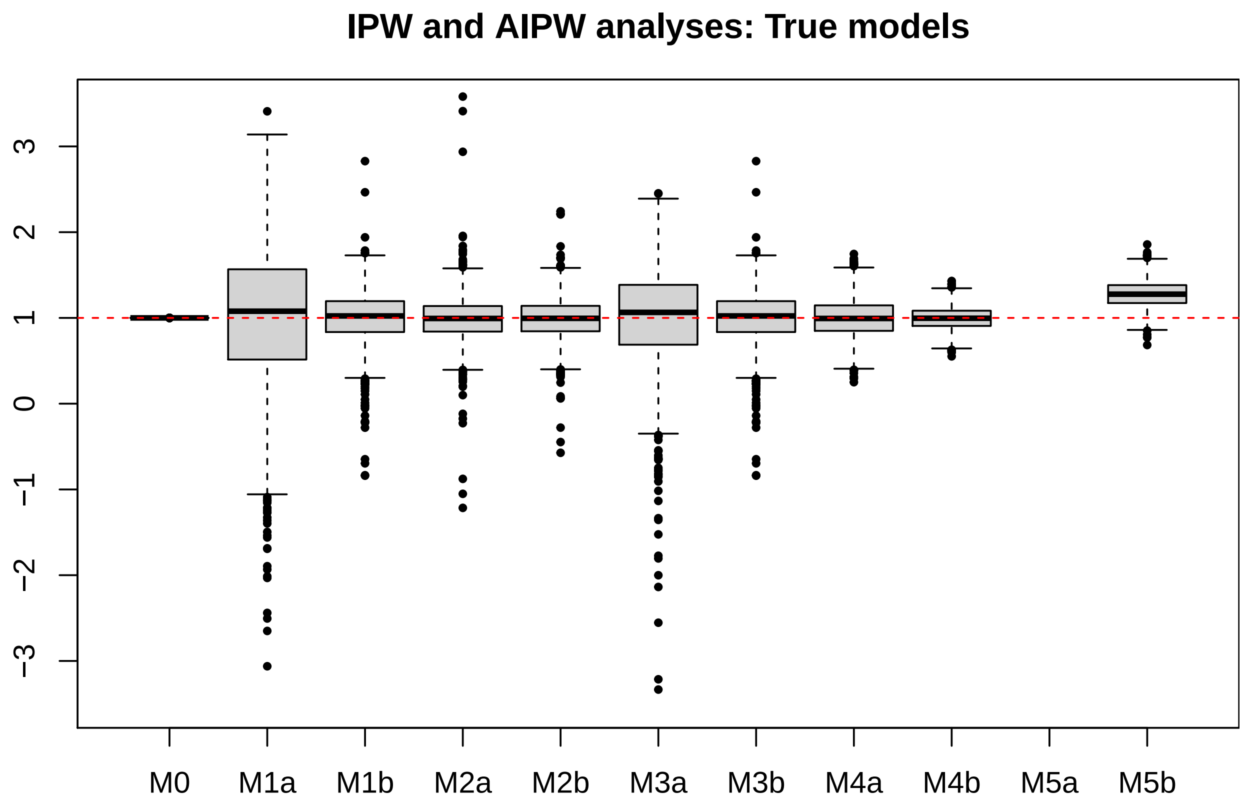
Model 4 : AIPW with correct specification of $\mu(x, z)$ only

Model 5 : AIPW with both models mis-specified

We compute both the original (denoted a) and standardized weight (denoted b) estimators. We perform 2000 replicate analyses with $n = 1000$. In the first analysis, we assume that the propensity score is known.

```
#Monte Carlo study
library(mvnfast)
set.seed(23987)
n<-1000
k<-10
Mu<-sample(-5:5,size=k,rep=T)/2
Sigma<-0.1*0.8^abs(outer(1:k,1:k,'-'))
al<-c(-4,rep(1,6),rep(-2,4))
expit<-function(x){return(1/(1+exp(-x)))};logit<-function(x){return(log(x)-log(1-x))}
be<-rep(0,k+2)
be[1:5]<-c(2,1,-2.0,2.2,3.6)
sig<-2
nreps<-2000
ests<-matrix(0,nrow=nreps,ncol=11)
for(irep in 1:nreps){

    X<-rmvn(n,mu=Mu,Sigma)
    Xal<-cbind(1,X)
    ps.true<-expit(Xal %*% al)
    Z<-rbinom(n,1,ps.true)
    Xm<-cbind(1,Z,X)
    muval<-as.vector(Xm %*% be)

    Y<-rnorm(n,muval,sig)
    eX<-ps.true
    w<-(1-Z)/(1-eX)+Z/eX
    w0<-(1-Z)/(1-eX)
    W0<-w0/sum(w0)
    w1<-Z/eX
    W1<-w1/sum(w1)

    mu0<- cbind(1,0,X) %*% be
    mu1<- cbind(1,1,X) %*% be

    #Intercept only mis-specified model
    m0<- cbind(1,0,0*X) %*% be
    m1<- cbind(1,1,0*X) %*% be

    ests[irep,1]<-mean(mu1)-mean(mu0)
    ests[irep,2]<-mean(w1*Y)-mean(w0*Y)
    ests[irep,3]<-sum(W1*Y)-sum(W0*Y)
    ests[irep,4]<-mean(w1*(Y-mu1))+mean(mu1)-mean(w0*(Y-mu0))-mean(mu0)
    ests[irep,5]<-sum(W1*(Y-mu1))+mean(mu1)-sum(W0*(Y-mu0))-mean(mu0)
    ests[irep,6]<-mean(w1*(Y-m1))+mean(m1)-mean(w0*(Y-m0))-mean(m0)
    ests[irep,7]<-sum(W1*(Y-m1))+mean(m1)-sum(W0*(Y-m0))-mean(m0)

    #Mis-specified PS model using X1 only.
    #Obtain intercept from data
    gX<-expit(logit(mean(Z))+al[2]*X[,1])
    wg0<-(1-Z)/(1-gX)
    Wg0<-wg0/sum(wg0)
    wg1<-Z/gX
    Wg1<-wg1/sum(wg1)

    ests[irep,8]<-mean(wg1*(Y-mu1))+mean(mu1)-mean(wg0*(Y-mu0))-mean(mu0)
    ests[irep,9]<-sum(Wg1*(Y-mu1))+mean(mu1)-sum(Wg0*(Y-mu0))-mean(mu0)

    ests[irep,10]<-mean(wg1*(Y-m1))+mean(m1)-mean(wg0*(Y-m0))-mean(m0)
    ests[irep,11]<-sum(Wg1*(Y-m1))+mean(m1)-sum(Wg0*(Y-m0))-mean(m0)

}
```

```
par(mar=c(4,4,3,0))
nvec<-as.character(c(0,rep(1:5,each=2)))
nvec<-paste('M',nvec,sep='')
nvec[c(2,4,6,8,10)]<-paste(nvec[c(2,4,6,8,10)],'a',sep='')
nvec[c(3,5,7,9,11)]<-paste(nvec[c(3,5,7,9,11)],'b',sep='')
boxplot(ests,ylim=range(-3.5,3.5),pch=19,cex=0.5,names=nvec)
title('IPW and AIPW analyses: True models')
abline(h=1,lty=2,col='red')
```



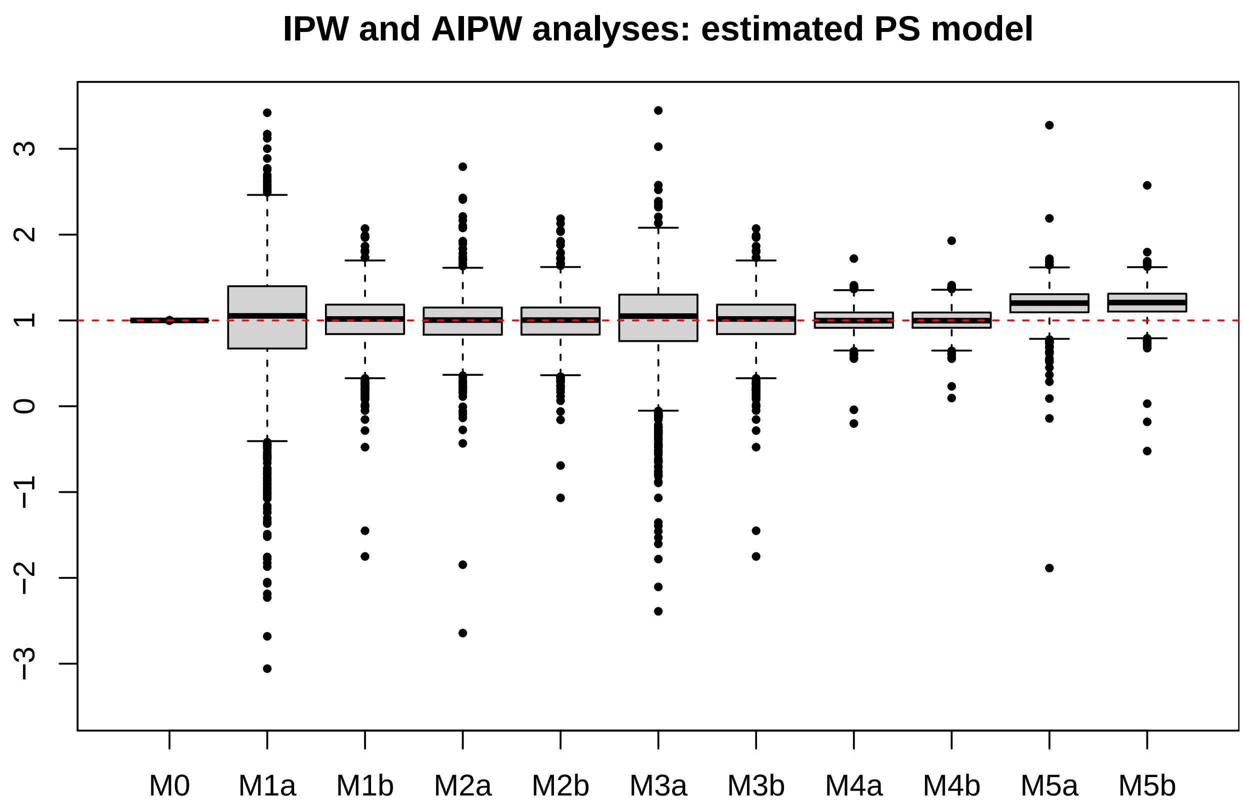

```
#Bias
b<-sqrt(n)*apply(ests-1,2,mean)
#Variance
v<-n*apply(ests,2,var)
#MSE
mse<-b^2+v
res<-rbind(b,v,mse)
colnames(res)<-nvec
rownames(res)<-c('Bias','Var.','MSE')
round(res,2)

+       M0    M1a   M1b   M2a   M2b    M3a   M3b   M4a   M4b      M5a    M5b
+ Bias  0   0.15  0.01 -0.30 -0.30  -0.04  0.01 -0.10 -0.14   231.94   8.80
+ Var.  0 779.64 86.73 70.00 56.71 384.10 86.73 47.20 16.99    84.50  24.18
+ MSE   0 779.66 86.73 70.09 56.80 384.10 86.73 47.21 17.01 53881.25 101.63
```

The results are as expected in light of the theory.

In a second analysis, we now **estimate** the propensity score model.

```
#Monte Carlo study
set.seed(23987)
nreps<-2000
ests2<-matrix(0,nrow=nreps,ncol=11)
for(irep in 1:nreps){

    X<-rmvn(n,mu=Mu,Sigma)
    Xal<-cbind(1,X)
    ps.true<-expit(Xal %*% al)
    Z<-rbinom(n,1,ps.true)
    Xm<-cbind(1,Z,X)
    muval<-as.vector(Xm %*% be)

    Y<-rnorm(n,muval,sig)
    ps.fitted<-fitted(glm(Z~X,family=binomial))
    eX<-ps.fitted
    w<-(1-Z)/(1-eX)+Z/eX
    w0<-(1-Z)/(1-eX)
    W0<-w0/sum(w0)
    w1<-Z/eX
    W1<-w1/sum(w1)

    mu0<- cbind(1,0,X) %*% be
    mu1<- cbind(1,1,X) %*% be

    #Intercept only mis-specified model
    m0<- cbind(1,0,0*X) %*% be
    m1<- cbind(1,1,0*X) %*% be

    ests2[irep,1]<-mean(mu1)-mean(mu0)
    ests2[irep,2]<-mean(w1*Y)-mean(w0*Y)
    ests2[irep,3]<-sum(W1*Y)-sum(W0*Y)
    ests2[irep,4]<-mean(w1*(Y-mu1))+mean(mu1)-mean(w0*(Y-mu0))-mean(mu0)
    ests2[irep,5]<-sum(W1*(Y-mu1))+mean(mu1)-sum(W0*(Y-mu0))-mean(mu0)
    ests2[irep,6]<-mean(w1*(Y-m1))+mean(m1)-mean(w0*(Y-m0))-mean(m0)
    ests2[irep,7]<-sum(W1*(Y-m1))+mean(m1)-sum(W0*(Y-m0))-mean(m0)

    #Mis-specified PS model using X1 only.
    gX<-fitted(glm(Z~X[,1]),family=binomial)
    wg0<-(1-Z)/(1-gX)
    Wg0<-wg0/sum(wg0)
    wg1<-Z/gX
    Wg1<-wg1/sum(wg1)

    ests2[irep,8]<-mean(wg1*(Y-mu1))+mean(mu1)-mean(wg0*(Y-mu0))-mean(mu0)
    ests2[irep,9]<-sum(Wg1*(Y-mu1))+mean(mu1)-sum(Wg0*(Y-mu0))-mean(mu0)

    ests2[irep,10]<-mean(wg1*(Y-m1))+mean(m1)-mean(wg0*(Y-m0))-mean(m0)
    ests2[irep,11]<-sum(Wg1*(Y-m1))+mean(m1)-sum(Wg0*(Y-m0))-mean(m0)

}
```

```
par(mar=c(4,4,3,0))
nvec<-as.character(c(0,rep(1:5,each=2)))
nvec<-paste('M',nvec,sep='')
nvec[c(2,4,6,8,10)]<-paste(nvec[c(2,4,6,8,10)],'a',sep='')
nvec[c(3,5,7,9,11)]<-paste(nvec[c(3,5,7,9,11)],'b',sep='')
boxplot(ests2,ylim=range(-3.5,3.5),pch=19,cex=0.5,names=nvec)
title('IPW and AIPW analyses: estimated PS model')
abline(h=1,lty=2,col='red')
```




```
#Bias
b<-sqrt(n)*apply(ests2-1,2,mean)
#Variance
v<-n*apply(ests2,2,var)
#MSE
mse<-b^2+v
res2<-rbind(b,v,mse)
colnames(res2)<-nvec
rownames(res2)<-c('Bias','Var.','MSE')
round(res2,2)

+        M0    M1a   M1b   M2a   M2b    M3a   M3b   M4a   M4b   M5a   M5b
+ Bias   0  -0.41  0.15 -0.15 -0.14  -0.28  0.15  0.00  0.02  6.19  6.56
+ Var.   0 571.46 84.49 80.33 65.25 303.00 84.49 19.42 18.93 55.76 28.30
+ MSE    0 571.63 84.51 80.35 65.27 303.08 84.51 19.42 18.93 94.01 71.31
```

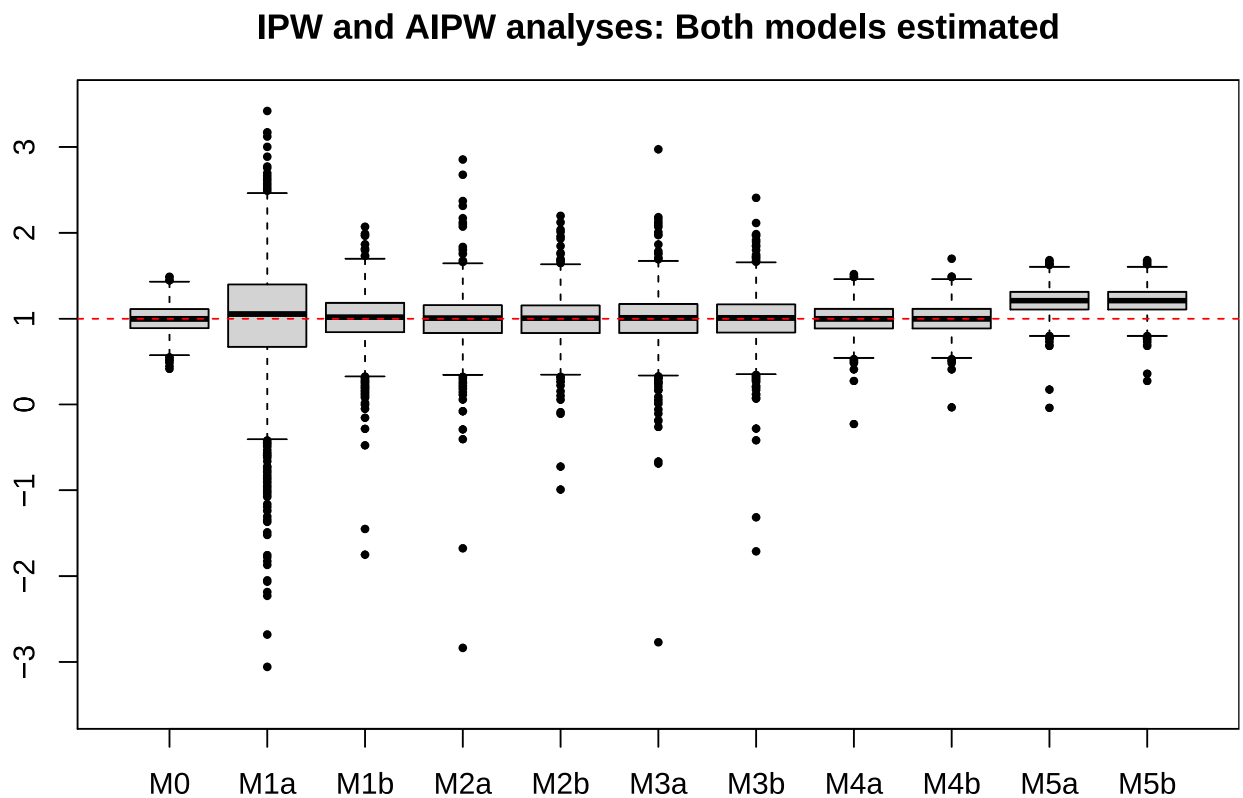
The variances are decreased after estimation

In a third analysis, we now estimate the propensity score **and** the conditional mean models.

```
#Monte Carlo study
set.seed(23987)
nreps<-2000
ests3<-matrix(0,nrow=nreps,ncol=11)
for(irep in 1:nreps){

    X<-rmvn(n,mu=Mu,Sigma)
    Xal<-cbind(1,X)
    ps.true<-expit(Xal %*% al)
    Z<-rbinom(n,1,ps.true)
    Xm<-cbind(1,Z,X)
    muval<-as.vector(Xm %*% be)

    Y<-rnorm(n,muval,sig)
    ps.fitted<-fitted(glm(Z~X,family=binomial))
    eX<-ps.fitted
    w<-(1-Z)/(1-eX)+Z/eX
    w0<-(1-Z)/(1-eX)
    W0<-w0/sum(w0)
    w1<-Z/eX
    W1<-w1/sum(w1)

    fit.mu<-lm(Y~Z+X)
    mu0<- predict(fit.mu,newdata=data.frame(Z=0,X=X))
    mu1<- predict(fit.mu,newdata=data.frame(Z=1,X=X))

    #Intercept only mis-specified model
    X1<-X[,1]
    fit.m<-lm(Y~Z+X1)
    m0<- predict(fit.m,newdata=data.frame(Z=0,X1=X[,1]))
    m1<- predict(fit.m,newdata=data.frame(Z=1,X1=X[,1]))

    ests3[irep,1]<-mean(mu1)-mean(mu0)
    ests3[irep,2]<-mean(w1*Y)-mean(w0*Y)
    ests3[irep,3]<-sum(W1*Y)-sum(W0*Y)
    ests3[irep,4]<-mean(w1*(Y-mu1))+mean(mu1)-mean(w0*(Y-mu0))-mean(mu0)
    ests3[irep,5]<-sum(W1*(Y-mu1))+mean(mu1)-sum(W0*(Y-mu0))-mean(mu0)
    ests3[irep,6]<-mean(w1*(Y-m1))+mean(m1)-mean(w0*(Y-m0))-mean(m0)
    ests3[irep,7]<-sum(W1*(Y-m1))+mean(m1)-sum(W0*(Y-m0))-mean(m0)

    #Mis-specified PS model using X1 only.
    gX<-fitted(glm(Z~X[,1]),family=binomial)
    wg0<-(1-Z)/(1-gX)
    Wg0<-wg0/sum(wg0)
    wg1<-Z/gX
    Wg1<-wg1/sum(wg1)

    ests3[irep,8]<-mean(wg1*(Y-mu1))+mean(mu1)-mean(wg0*(Y-mu0))-mean(mu0)
    ests3[irep,9]<-sum(Wg1*(Y-mu1))+mean(mu1)-sum(Wg0*(Y-mu0))-mean(mu0)

    ests3[irep,10]<-mean(wg1*(Y-m1))+mean(m1)-mean(wg0*(Y-m0))-mean(m0)
    ests3[irep,11]<-sum(Wg1*(Y-m1))+mean(m1)-sum(Wg0*(Y-m0))-mean(m0)

}
```

```
par(mar=c(4,4,3,0))
nvec<-as.character(c(0,rep(1:5,each=2)))
nvec<-paste('M',nvec,sep='')
nvec[c(2,4,6,8,10)]<-paste(nvec[c(2,4,6,8,10)],'a',sep='')
nvec[c(3,5,7,9,11)]<-paste(nvec[c(3,5,7,9,11)],'b',sep='')
```

```
boxplot(ests3,ylim=range(-3.5,3.5),pch=19,cex=0.5,names=nvec)
title('IPW and AIPW analyses: Both models estimated')
abline(h=1,lty=2,col='red')
```




```
#Bias
b<-sqrt(n)*apply(ests3-1,2,mean)
#Variance
v<-n*apply(ests3,2,var)
#MSE
mse<-b^2+v
res3<-rbind(b,v,mse)
colnames(res3)<-nvec
rownames(res3)<-c('Bias','Var.','MSE')
round(res3,2)

+          M0    M1a   M1b   M2a   M2b    M3a   M3b   M4a   M4b   M5a   M5b
+ Bias -0.01   -0.41  0.15 -0.16 -0.14  -0.13 -0.01 -0.03 -0.02  6.70  6.71
+ Var. 25.98 571.46 84.49 80.93 65.24 104.90 79.83 27.30 27.02 25.39 24.81
+ MSE  25.98 571.63 84.51 80.95 65.26 104.92 79.83 27.30 27.02 70.30 69.82
```