Causal Contrasts from Outcome Regression

In a randomized experimental study, inference concerning the causal effect of binary variable Z on Y can be made by direct comparison of sample averages. Suppose that $Z \sim Bernoulli(p)$ for $0 < p < 1$. Two estimators of the causal contrast $\delta = \mathbb{E}^{\mathcal{E}}[Y(1) - Y(0)]$ can be proposed if the conditional mean model

$$\mathbb{E}^{\mathcal{E}}_{Y|X,Z}[Y|X = x, Z = z] = \mathbb{E}^{\mathcal{O}}_{Y|X,Z}[Y|X = x, Z = z] = \mu(x, z)$$

is presumed known. We have

$$\widehat{\delta}_{OR} = \frac{1}{n}\sum_{i=1}^{n} \mu(X_i, 1) - \frac{1}{n}\sum_{i=1}^{n} \mu(X_i, 0) \tag{1}$$

and

$$\widehat{\delta}_{OR2} = \frac{1}{n\widehat{p}}\sum_{i=1}^{n} Z_i \mu(X_i, Z_i) - \frac{1}{n(1-\widehat{p})}\sum_{i=1}^{n}(1 - Z_i)\mu(X_i, Z_i) \tag{2}$$

where $\widehat{p} = n^{-1}\sum_{i=1}^{n} Z_i$. In the simulation below, we assume that $X \sim Normal(1, 1)$ and

$$Y|X = x, Z = z \sim Normal(0.5x + \delta z, 1)$$

so that $\mathbb{E}^{\mathcal{E}}_{Y|X,Z}[Y|X = x, Z = z] = 0.5x + \delta z$, and

$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}^{\mathcal{E}}_{X}\left[\mathbb{E}^{\mathcal{E}}_{Y|X,Z}[Y|X, Z = 1] - \mathbb{E}^{\mathcal{E}}_{Y|X,Z}[Y|X, Z = 0]\right] = \delta.$$

In this case, $\mu(X, 1) - \mu(X, 0) = \delta$ which does not depend on X, so we can estimate δ without error using δ_{OR}

```
set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
    X<-rnorm(n,1,1)
    Z<-rbinom(n,1,p)
    Y<-rnorm(n,delta*Z+0.5*X,1)
    p.hat<-mean(Z)
    ests.mat[irep,1]<-mean(delta+0.5*X)-mean(0.5*X)
    ests.mat[irep,2]<-sum(Z*(delta*Z+0.5*X))/(n*p.hat)-sum((1-Z)*(delta*Z+0.5*X))/(n*(1-p.hat))
}
apply(ests.mat,2,var)

+ [1] 7.254914e-33 9.881624e-03
```

Suppose now that

$$\mathbb{E}^{\mathcal{E}}_{Y|X,Z}[Y|X=x,Z=z] = 0.5x + \delta z + 2xz$$

so that

$$\mathbb{E}^{\mathcal{E}}_{Y|X,Z}[Y|X=x,Z=1] - \mathbb{E}^{\mathcal{E}}_{Y|X,Z}[Y|X=x,Z=0] = \delta + 2x.$$

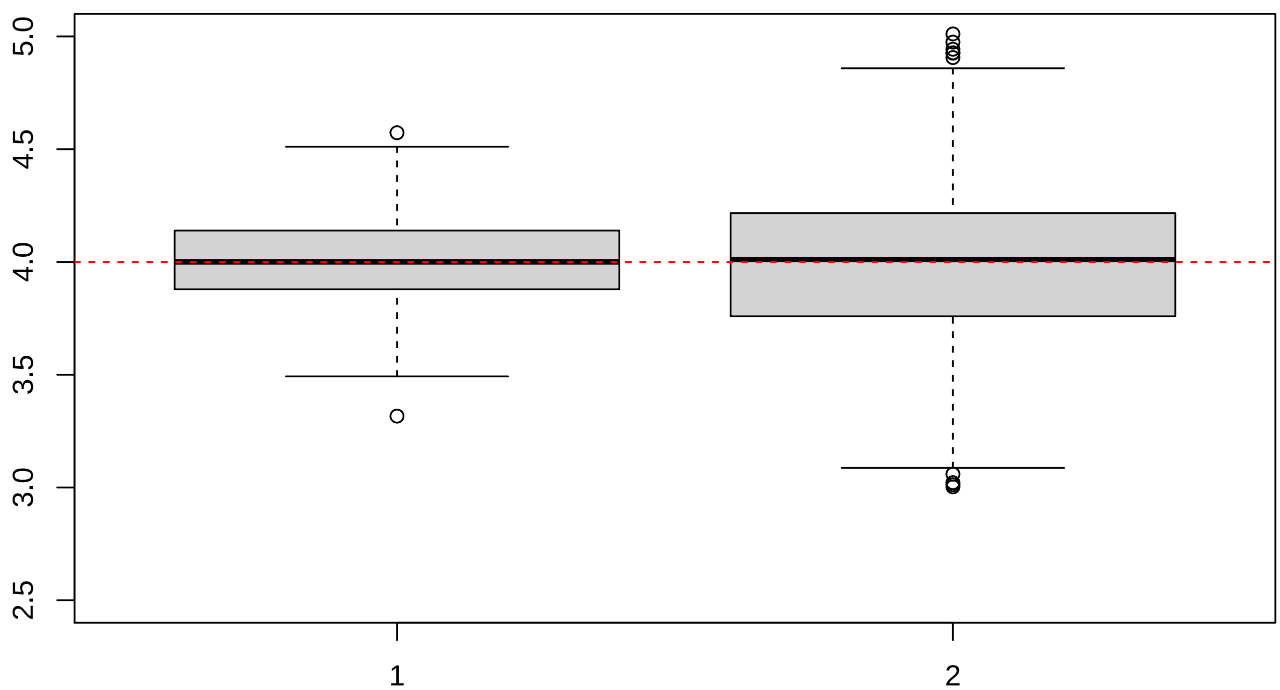
and thus

$$\mathbb{E}^{\mathcal{E}}[Y(1) - Y(0)] = \mathbb{E}^{\mathcal{E}}[\delta + 2X] = \delta + 2\mathbb{E}^{\mathcal{E}}[X] = \delta + 2$$

```
set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
    X<-rnorm(n,1,1)
    Z<-rbinom(n,1,p)
    Y<-rnorm(n,delta*Z+0.5*X + 2*Z*X,1)
    p.hat<-mean(Z)
    ests.mat[irep,1]<-mean(delta+0.5*X+2*X)-mean(0.5*X)
    ests.mat[irep,2]<-sum(Z*(delta*Z+0.5*X+ 2*X*Z))/(n*p.hat)-
                     sum((1-Z)*(delta*Z+0.5*X+2*X*Z))/(n*(1-p.hat))
}
apply(ests.mat,2,var)

+ [1] 0.03626926 0.12555203

par(mar=c(4,2,1,1))
boxplot(ests.mat,ylim=range(2.5,5));abline(h=delta+2,lty=2,col='red')
```



Suppose we **mis-specify** the modelled mean using

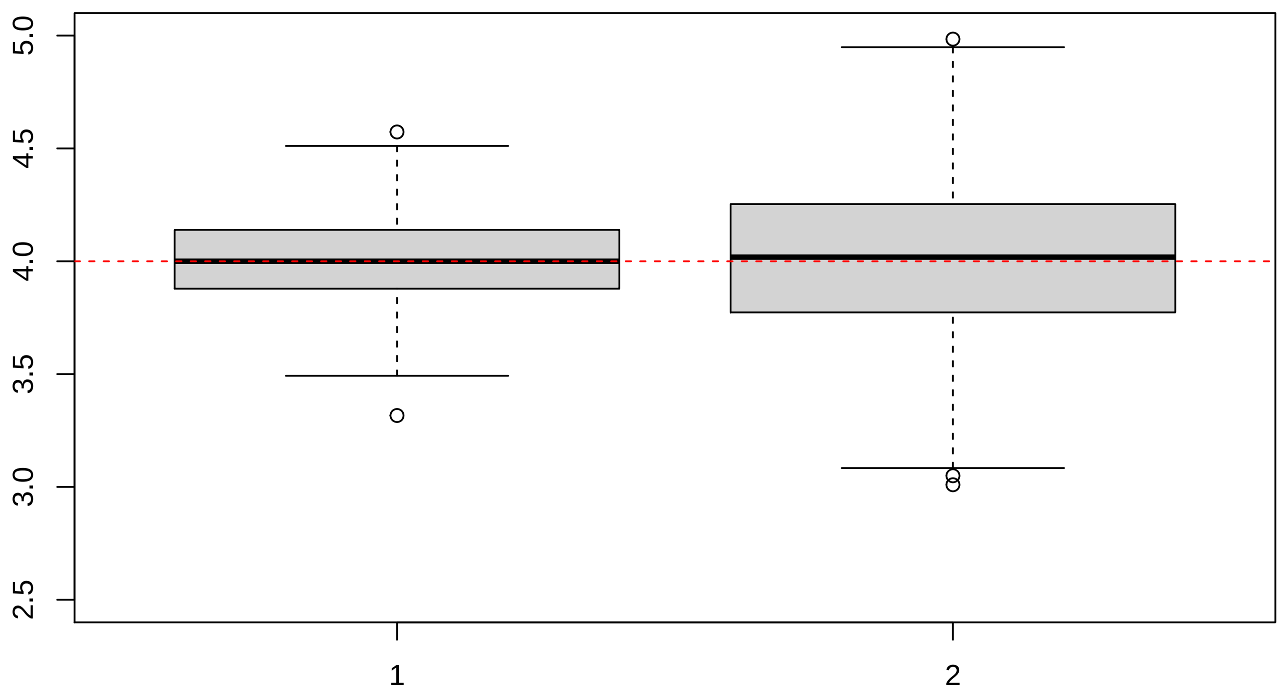
$$m(x, z) = x - 2x^2 + \delta z + 2xz$$

that is, where the dependence on z is correct, but it is not the true conditional mean. We can still unbiasedly estimate δ using the same estimator:

```
set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
    X<-rnorm(n,1,1)
    Z<-rbinom(n,1,p)
    Y<-rnorm(n,delta*Z+0.5*X + 2*Z*X,1)
    p.hat<-mean(Z)
    ests.mat[irep,1]<-mean(delta+X-X^2+2*X)-mean(X-X^2)
    ests.mat[irep,2]<-sum(Z*(delta*Z+X-X^2+2*Z*X))/(n*p.hat)-
                     sum((1-Z)*(delta*Z+X-X^2+2*Z*X))/(n*(1-p.hat))
}
apply(ests.mat,2,var)

+ [1] 0.03626926 0.11866337

par(mar=c(4,2,1,1))
boxplot(ests.mat,ylim=range(2.5,5));abline(h=delta+2,lty=2,col='red')
```



However, if we **mis-specify** the modelled mean using

$$m(x,z) = x - 2x^2 + \delta z + xz$$

that is, where the dependence on z is incorrect, and it is not the true conditional mean, we can no longer unbiasedly estimate δ using the same estimator:

```
set.seed(23)
nreps<-1000;n<-100;p<-0.5;delta<-2
ests.mat<-matrix(0,nrow=nreps,ncol=2)
for(irep in 1:nreps){
    X<-rnorm(n,1,1)
    Z<-rbinom(n,1,p)
    Y<-rnorm(n,delta*Z+0.5*X + 2*Z*X,1)
    p.hat<-mean(Z)
    ests.mat[irep,1]<-mean(delta+X-X^2+X)-mean(X-X^2)
    ests.mat[irep,2]<-sum(Z*(delta*Z+X-X^2+Z*X))/(n*p.hat)-
                      sum((1-Z)*(delta*Z+X-X^2+Z*X))/(n*(1-p.hat))
}
apply(ests.mat,2,var)

+ [1] 0.009067314 0.101345327

par(mar=c(4,2,1,1))
boxplot(ests.mat,ylim=range(2.5,5));abline(h=delta+2,lty=2,col='red')
```