Project 4: Solutions

Denote the model in equation (1) from the paper

$$Y_i = \psi S_i + \mathbf{x}_i \beta + \epsilon_i \tag{1}$$

where ψ is the parameter of interest (replacing β in the notation of the paper) and $\beta = (\beta_1, \ldots, \beta_K)$. As per equation (11), consider the second model

$$S_i = \mathbf{x}_i \alpha + \varepsilon_i$$

Let $\mathbf{S} = (S_1, \ldots, S_n)^\top$ and $\mathbf{X}$ be the stacked row vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. We have from standard OLS theory that for the fitted value

$$\widehat{S}_i = \mathbf{x}_i \widehat{\alpha} = \mathbf{x}_i (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{S} \qquad i = 1, \ldots, n$$

or in vector form

$$\widehat{\mathbf{S}} = \mathbf{X} (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{S} = \mathbf{H}\mathbf{S} \tag{2}$$

say.

In vector form, equation (1) becomes

$$\mathbf{Y} = \mathbf{S}\psi + \mathbf{X}\beta + \epsilon.$$

Now, using the profiling method from lectures, we have that the reduced-form (and efficient) estimating equation for ψ is given by

$$\mathbf{S}^\top (\mathbf{I} - \mathbf{H})(\mathbf{Y} - \mathbf{S}\psi) = \mathbf{0}$$

or equivalently

$$(\mathbf{S} - \mathbf{H}\mathbf{S})^\top (\mathbf{Y} - \mathbf{S}\psi) = \mathbf{0}.$$

But from (2) $\mathbf{H}\mathbf{S} = \widehat{\mathbf{S}}$, which yields the equation

$$(\mathbf{S} - \widehat{\mathbf{S}})^\top (\mathbf{Y} - \mathbf{S}\psi) = \mathbf{0}.$$

Solving this using OLS methods yields

$$\widehat{\psi} = \frac{(\mathbf{S} - \widehat{\mathbf{S}})^\top \mathbf{Y}}{(\mathbf{S} - \widehat{\mathbf{S}})^\top \mathbf{S}} = \frac{\sum_{i=1}^n Y_i (S_i - \widehat{S}_i)}{\sum_{i=1}^n S_i (S_i - \widehat{S}_i)} = \frac{\sum_{i=1}^n Y_i (S_i - \widehat{S}_i)}{\sum_{i=1}^n (S_i - \widehat{S}_i)^2}$$

where the final equality follows from the fact that

$$\sum_{i=1}^n \widehat{S}_i (S_i - \widehat{S}_i) = \widehat{\mathbf{S}}^\top (\mathbf{S} - \widehat{\mathbf{S}}) = 0$$

as $\widehat{\mathbf{S}}$ is orthogonal to $(\mathbf{S} - \widehat{\mathbf{S}})$ by construction.