

IMPORTANCE REWEIGHTING

Consider two joint distributions on variables X, Y, Z with the following joint mass/density structure:

$$\begin{aligned} f_{X,Y,Z}(x, y, z) &= f_{Y|X,Z}(y|x, z) f_{Z|X}(z|x) f_X(x) \\ f_{X,Y,Z}^*(x, y, z) &= f_{Y|X,Z}(y|x, z) f_Z^*(z) f_X(x) \end{aligned}$$

for all $(x, y, z) \in \{0, 1\}^3$, that is, for $f_{X,Y,Z}^*$ we have independence of X and Z . Suppose further that $f_Z^*(z)$ is degenerate at $z = z_0$, that is $P_Z^*[Z = z_0] = 1$. To compute the expectation $\mathbb{E}_{Y|Z}^*[Y|Z = z_0]$ (derived from the $f_{X,Y,Z}^*$ distribution), we have

$$\begin{aligned} \mathbb{E}_{Y|Z}^*[Y|Z = z_0] &= P_{Y|X,Z}^*[Y = 1|Z = z_0] \\ &= P_{Y|X,Z}^*[Y = 1|X = 0, Z = z_0] P_{X|Z}^*[X = 0|Z = z_0] \\ &\quad + P_{Y|X,Z}^*[Y = 1|X = 1, Z = z_0] P_{X|Z}^*[X = 1|Z = z_0] \\ &\equiv P_{Y|X,Z}[Y = 1|X = 0, Z = z_0] P_X[X = 0] \\ &\quad + P_{Y|X,Z}[Y = 1|X = 1, Z = z_0] P_X[X = 1] \\ &= \sum_{x=0}^1 P_{Y|X,Z}[Y = 1|X = x, Z = z_0] P_X[X = x] \end{aligned}$$

as by assumption, and by independence

$$f_{Y|X,Z}(y|x, z) \equiv f_{Y|X,Z}^*(y|x, z) \quad \text{and} \quad f_{X|Z}^*(x|z) \equiv f_X^*(x) \equiv f_X(x).$$

Now, in this degenerate case with $P_Z^*[Z = z_0] = 1$, we may write

$$P_X[X = x] = P_X[X = x] \times P_Z^*[Z = z_0] \times \frac{P_{Z|X}[Z = z_0|X = x]}{P_{Z|X}[Z = z_0|X = x]}.$$

Therefore we have

$$\begin{aligned} \mathbb{E}_{Y|Z}^*[Y|Z = z_0] &= \sum_{x=0}^1 P_{Y|X,Z}[Y = 1|X = x, Z = z_0] P_X[X = x] P_Z^*[Z = z_0] \frac{P_{Z|X}[Z = z_0|X = x]}{P_{Z|X}[Z = z_0|X = x]} \\ &= \sum_{z=0}^1 \sum_{x=0}^1 \left\{ P_{Y|X,Z}[Y = 1|X = x, Z = z] \frac{P_Z^*[Z = z]}{P_{Z|X}[Z = z|X = x]} \right\} P_{Z|X}[Z = z|X = x] P_X[X = x] \\ &\equiv \mathbb{E}_X \left[\mathbb{E}_{Z|X} \left[\mathbb{E}_{Y|X,Z} [Y|X, Z] \frac{f_Z^*(Z)}{f_{Z|X}(Z|X)} \middle| X \right] \right] \\ &\equiv \mathbb{E}_{X,Z} \left[\mathbb{E}_{Y|X,Z} [Y|X, Z] \frac{f_Z^*(Z)}{f_{Z|X}(Z|X)} \right] \end{aligned}$$

provided $f_{Z|X}(z|x) \equiv P_Z[Z = z|X = x] > 0, \forall (x, z)$ (this is 'positivity' condition). Note that the last line is equal to

$$\mathbb{E}_Z \left[\mathbb{E}_{X|Z} \left[\mathbb{E}_{Y|X,Z} [Y|X, Z] \frac{f_Z^*(Z)}{f_{Z|X}(Z|X)} \right] \middle| Z \right] = \mathbb{E}_{X|Z} \left[\mathbb{E}_{Y|X,Z} [Y|X, Z] \frac{f_Z^*(z_0)}{f_{Z|X}(z_0|X)} \middle| Z = z_0 \right]$$

in this degenerate case.