MATH 559: BAYESIAN THEORY AND METHODS

BAYESIAN NONPARAMETRIC METHODS

The *Dirichlet Process* is a probability distribution on the space of discrete distributions on $\mathbb{R}$ characterized by

- a *base distribution*, G_X, which is some probability measure on $\mathbb{R}$, from which are drawn a countable collection of random variables $X_1, X_2, \ldots$.
- a countable collection of random variables $\pi_1, \pi_2, \ldots$ with $\pi_i > 0$ such that

$$\sum_{i=1}^{\infty} \pi_i = 1$$

that is constructed in the following way: $\pi_1 = V_1 \sim Beta(1, \alpha)$ and for $i = 2, 3, \ldots$,

$$\pi_i = V_i \prod_{j=1}^{i-1} (1 - V_j)$$

that is

$$\begin{aligned} \pi_1 &= V_1 \\ \pi_2 &= (1 - V_1)V_2 \\ \pi_3 &= (1 - V_1)(1 - V_2)V_3 \end{aligned}$$

etc., where for each j, $V_j \sim Beta(1, \alpha)$ for some parameter $\alpha > 0$. Suppose that Z is a random variable on $\{1, 2, \ldots, \}$ with mass function defined by $\{\pi_1, \pi_2, \ldots\}$, that is $\Pr[Z = i] = \pi_i$. Then

$$\Pr[Z > 2] = 1 - \pi_1 - \pi_2 = 1 - V_1 - (1 - V_1)V_2 = (1 - V_1)(1 - V_2)$$

$$\Pr[Z > 3] = 1 - \pi_1 - \pi_2 - \pi_3 = (1 - V_1)(1 - V_2) - (1 - V_2)(1 - V_2)V_3 = (1 - V_1)(1 - V_2)(1 - V_3)$$

etc. and in general, for $n \geq 1$,

$$P_n = \Pr[Z > n] = \prod_{j=1}^{n} (1 - V_j).$$

The sequence of **random variables** $\{P_n\}$ is **strictly stochastically decreasing**

$$P_1 > P_2 > \cdots > P_n > \cdots$$

and is bounded below by zero. Let $\epsilon > 0$. Then

$$\Pr[P_n < \epsilon] = \Pr\left[\prod_{j=1}^{n} (1 - V_j) < \epsilon\right] \leq \Pr[T_n^n < \epsilon] \qquad T_n = \max\{1 - V_1, \ldots, 1 - V_n\}.$$

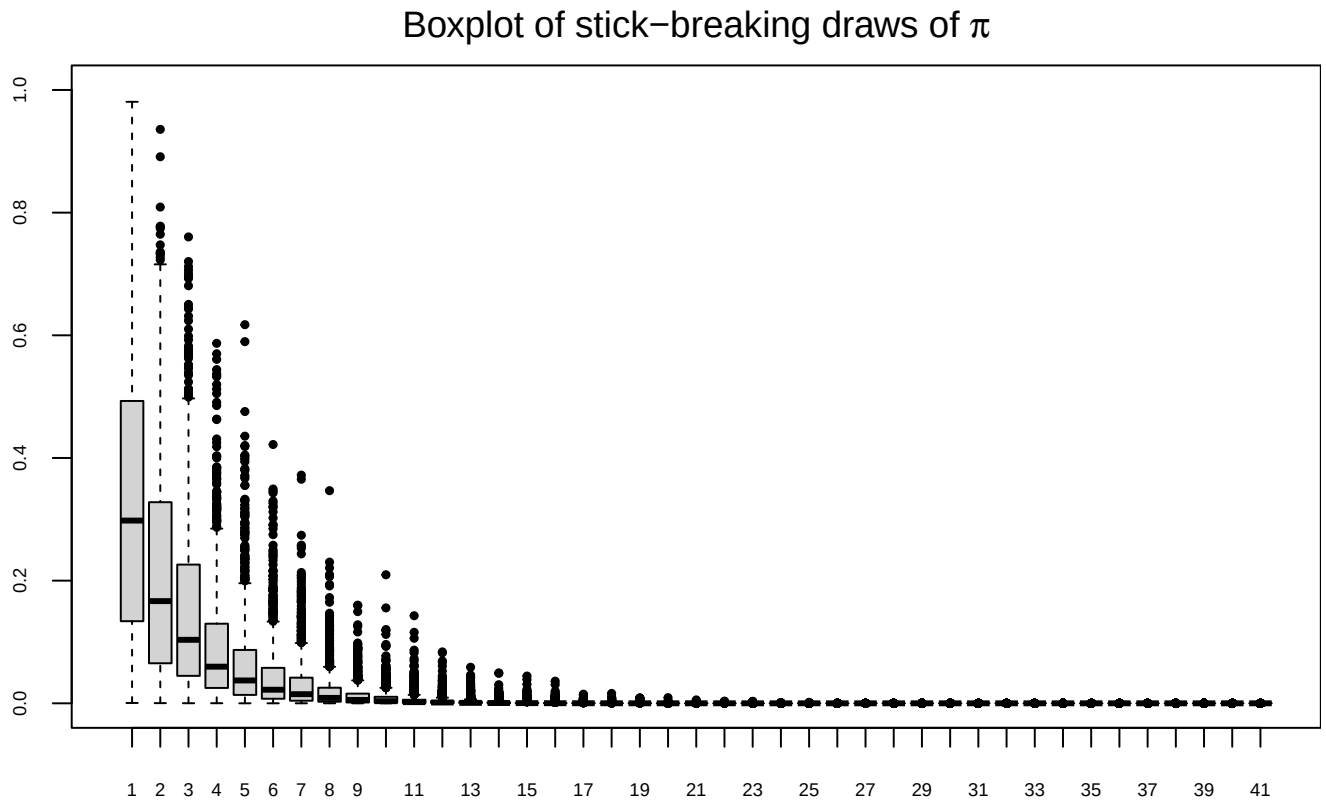
As the random variables V_j are independent, we have that if F is the cdf of the $1 - V_j \sim Beta(\alpha, 1)$ random variables, then

$$\Pr[P_n < \epsilon] \leq \{F(\epsilon^{1/n})\}^n \longrightarrow 0 \qquad n \longrightarrow \infty.$$

and the sequence converges (almost surely) to 0 as $n \longrightarrow \infty$. Thus the generated πs constitute a random discrete distribution. Also by construction, in expectation $\pi_1 > \pi_2 > \cdots$, which facilitates truncation at some finite value M. Note that, up to machine accuracy, the value of π_i will become zero eventually in any case.

Simulation: 1000 replicated runs of the generated probabilities, with $\alpha = 2$.

```
M<-41
alpha<-2
pi.mat<-matrix(0,nrow=1000,ncol=M)
for(i in 1:1000){
    V<-rbeta(M,1,alpha)
    pi.mat[i,]<-c(V[1],cumprod(1-V[-M])*V[-1])
}
par(mar=c(2,2,3,0))
boxplot(pi.mat,pch=19,cex=0.5,ylim=range(0,1),cex.axis=0.6)
title(expression(paste('Boxplot of stick-breaking draws of ', pi)),line=1)
```



The collections $\{X_1, X_2, \ldots\}$ and $\{\pi_1, \pi_2, \ldots, \}$ define a random discrete distribution with mass function $\widetilde{f}$ where

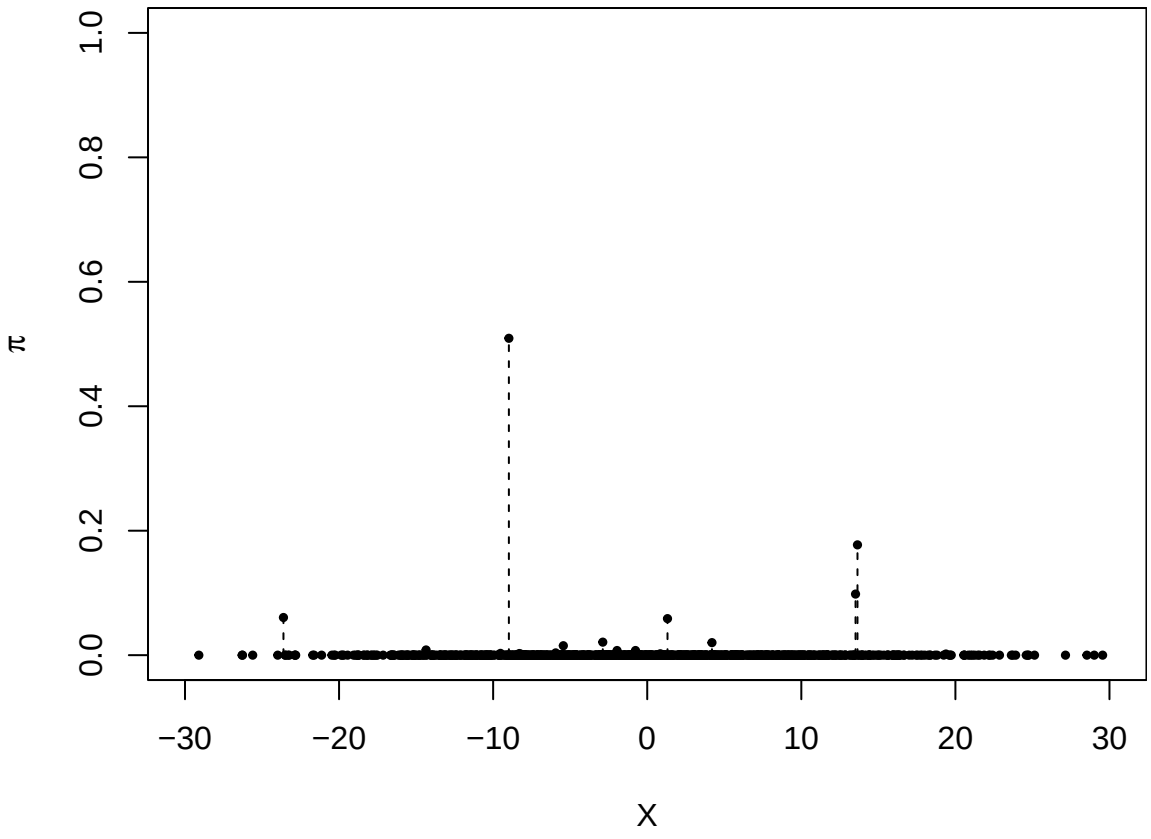
$$\widetilde{f}(x) = \sum_{i=1}^{\infty} \pi_i \delta_{X_i}(x)$$

where $\delta_{x_i}(x) = 1 \iff x = x_i$. We write $\widetilde{f} \sim DP(\alpha, G_X)$ to denote that $\widetilde{f}$ is drawn from the Dirichlet Process.

EXAMPLE: Here we choose $\alpha = 2$, and set $G_X \equiv Normal(0, 10^2)$.

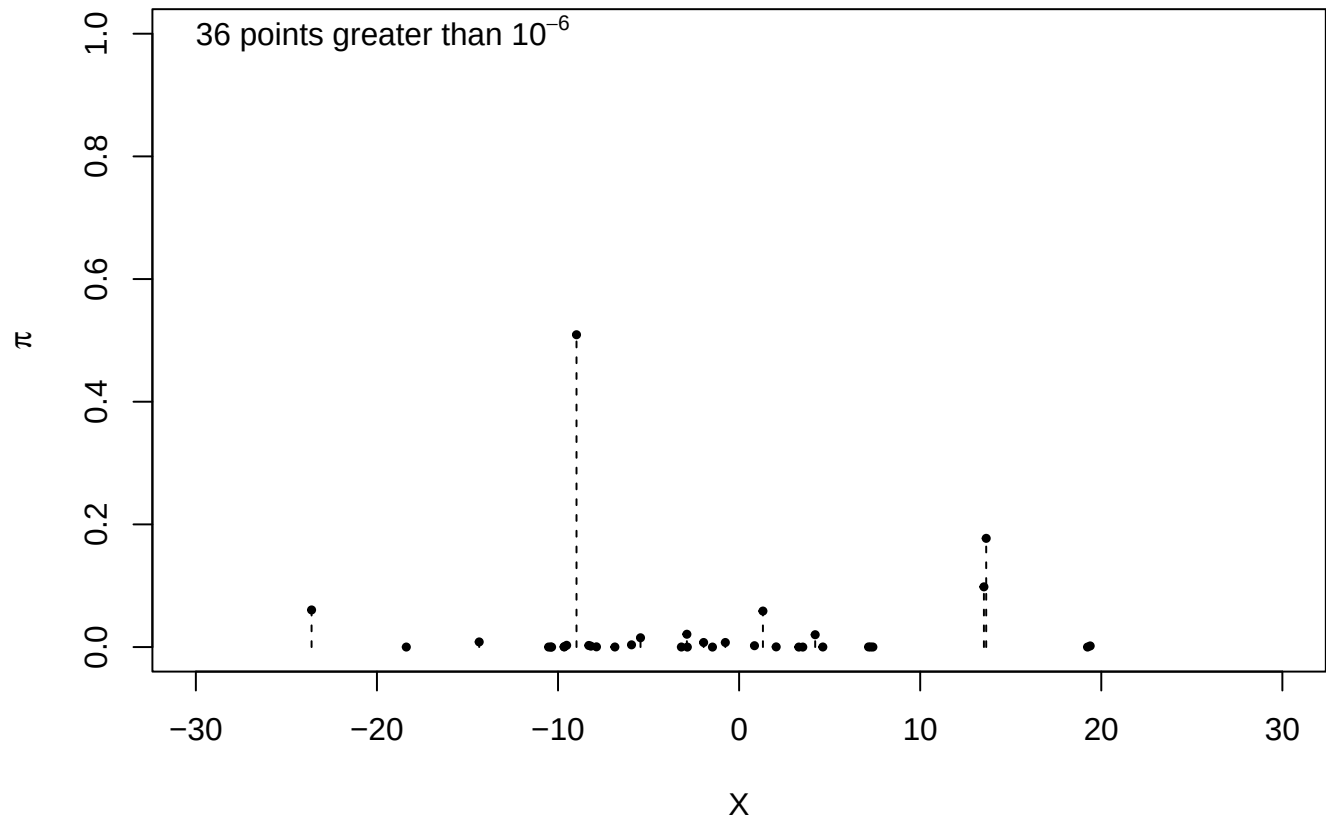
```
M<-1001;alpha<-2
set.seed(1010)
#Stick-breaking
Xi<-rnorm(M,0,10)    #Base measure
V<-rbeta(M,1,alpha)
```

```
pi.vec<-c(V[1],cumprod(1-V[-M])*V[-1])
Xi1<-Xi
pi1<-pi.vec

Xi<-rnorm(M,0,10); V<-rbeta(M,1,alpha)
pi.vec<-c(V[1],cumprod(1-V[-M])*V[-1])
Xi2<-Xi
pi2<-pi.vec

Xi<-rnorm(M,0,10);V<-rbeta(M,1,alpha)
pi.vec<-c(V[1],cumprod(1-V[-M])*V[-1])
Xi3<-Xi
pi3<-pi.vec

par(mar=c(4,4,1,0))
plot(Xi1,pi1,type="n",ylim=range(0,1),xlab='X',ylab=expression(pi),xlim=range(-30,30))
points(Xi1,pi1,pch=19,xlim=range(-30,30),cex=0.5)
for(i in 1:M){lines(c(Xi1[i],Xi1[i]),c(0,pi1[i]),lty=2)}
```

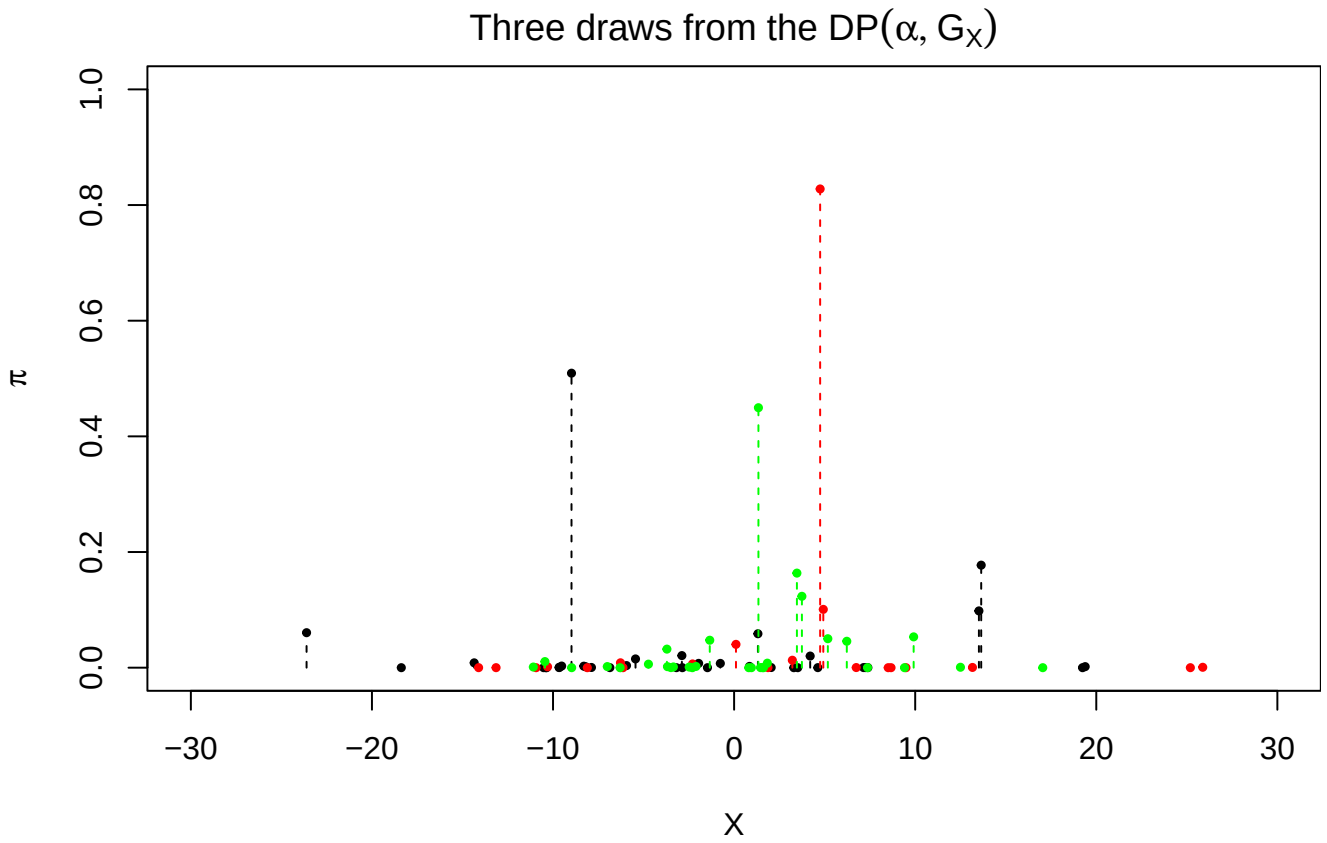



```
par(mar=c(4,4,1,0))
plot(Xi1,pi1,type="n",ylim=range(0,1),xlab='X',ylab=expression(pi),xlim=range(-30,30))
points(Xi1[pi1>1e-6],pi1[pi1>1e-6],pch=19,xlim=range(-30,30),cex=0.5)
for(i in 1:M){
        if(pi1[i] < 1e-6) next
        lines(c(Xi1[i],Xi1[i]),c(0,pi1[i]),lty=2)
}
```

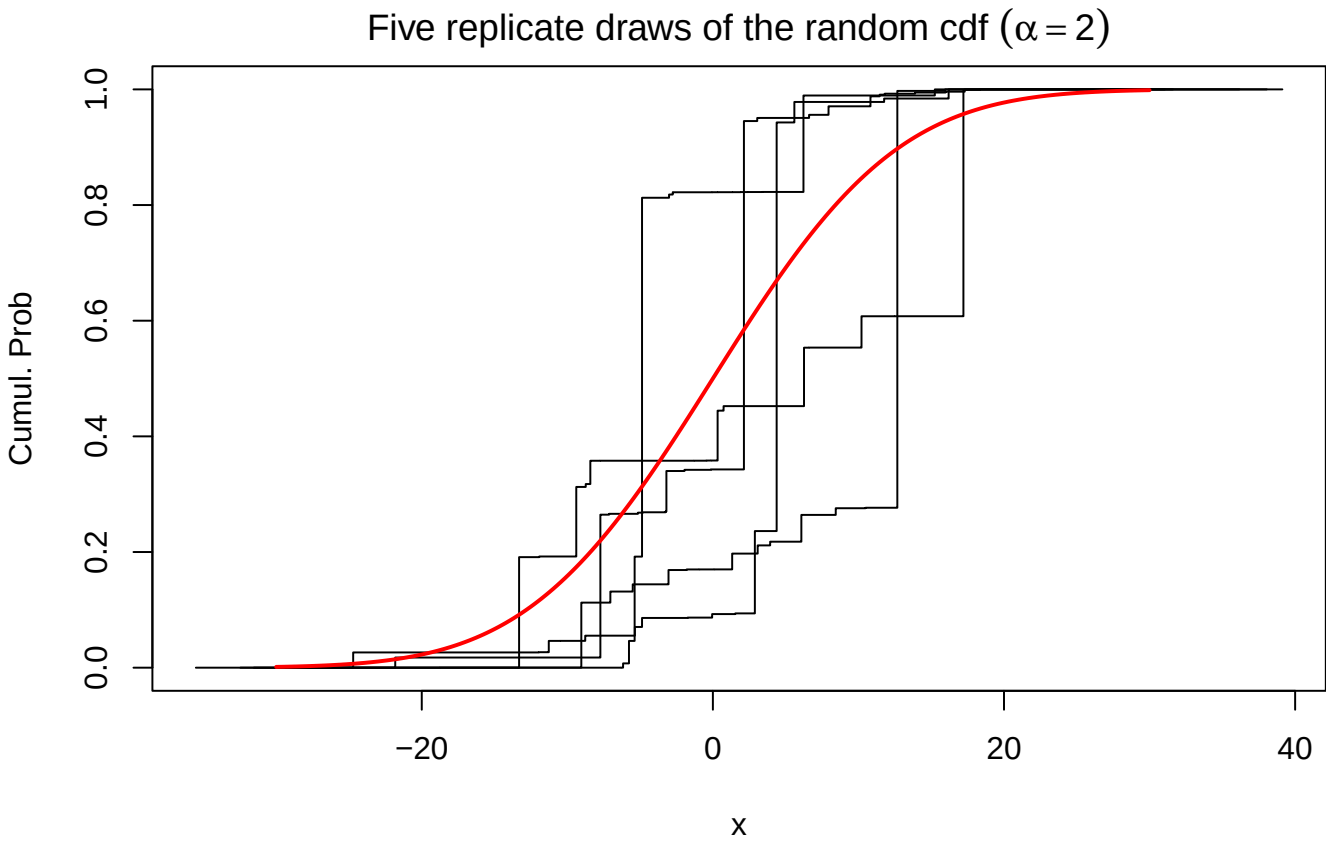
```
nval<-sum(pi1>1e-6)
text(-30,1,substitute(paste(nv,' points greater than ',10^{-6}),list(nv=nval)),adj=0)
```



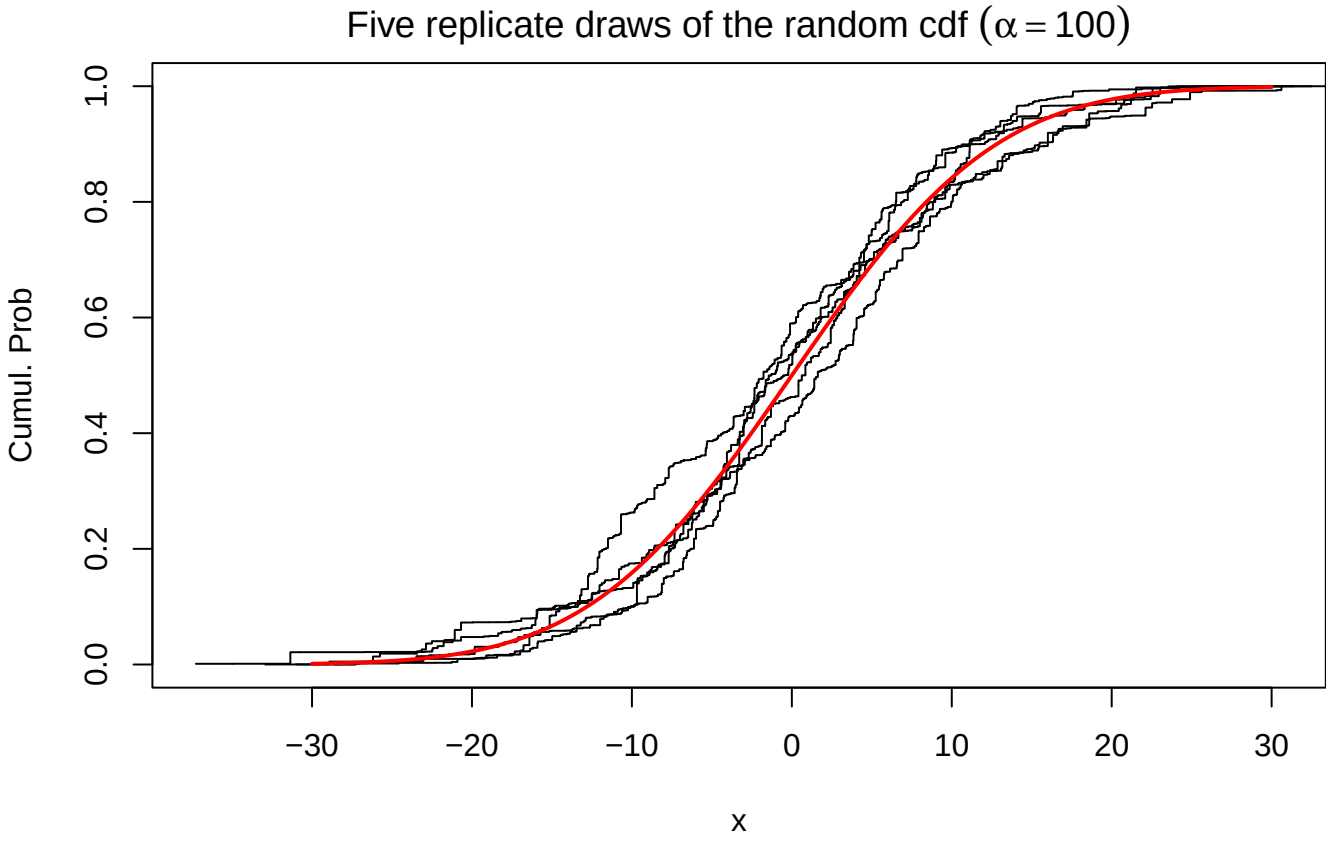

```
par(mar=c(4,4,2,0))
plot(Xi1,pi1,type="n",ylim=range(0,1),xlab='X',ylab=expression(pi),xlim=range(-30,30))
title(expression(paste('Three draws from the ',DP(alpha,G[X]))))
points(Xi1[pi1>1e-6],pi1[pi1>1e-6],pch=19,xlim=range(-30,30),cex=0.5)
for(i in 1:M){
        if(pi1[i] < 1e-6) next
        lines(c(Xi1[i],Xi1[i]),c(0,pi1[i]),lty=2)
}
points(Xi2[pi2>1e-6],pi2[pi2>1e-6],pch=19,xlim=range(-30,30),col="red",cex=0.5)
for(i in 1:M){
        if(pi2[i] < 1e-6) next
        lines(c(Xi2[i],Xi2[i]),c(0,pi2[i]),lty=2,col="red")
}
points(Xi3[pi3>1e-6],pi3[pi3>1e-6],pch=19,xlim=range(-30,30),col="green",cex=0.5)
for(i in 1:M){
        if(pi3[i] < 1e-6) next
        lines(c(Xi3[i],Xi3[i]),c(0,pi3[i]),lty=2,col="green")
}
```



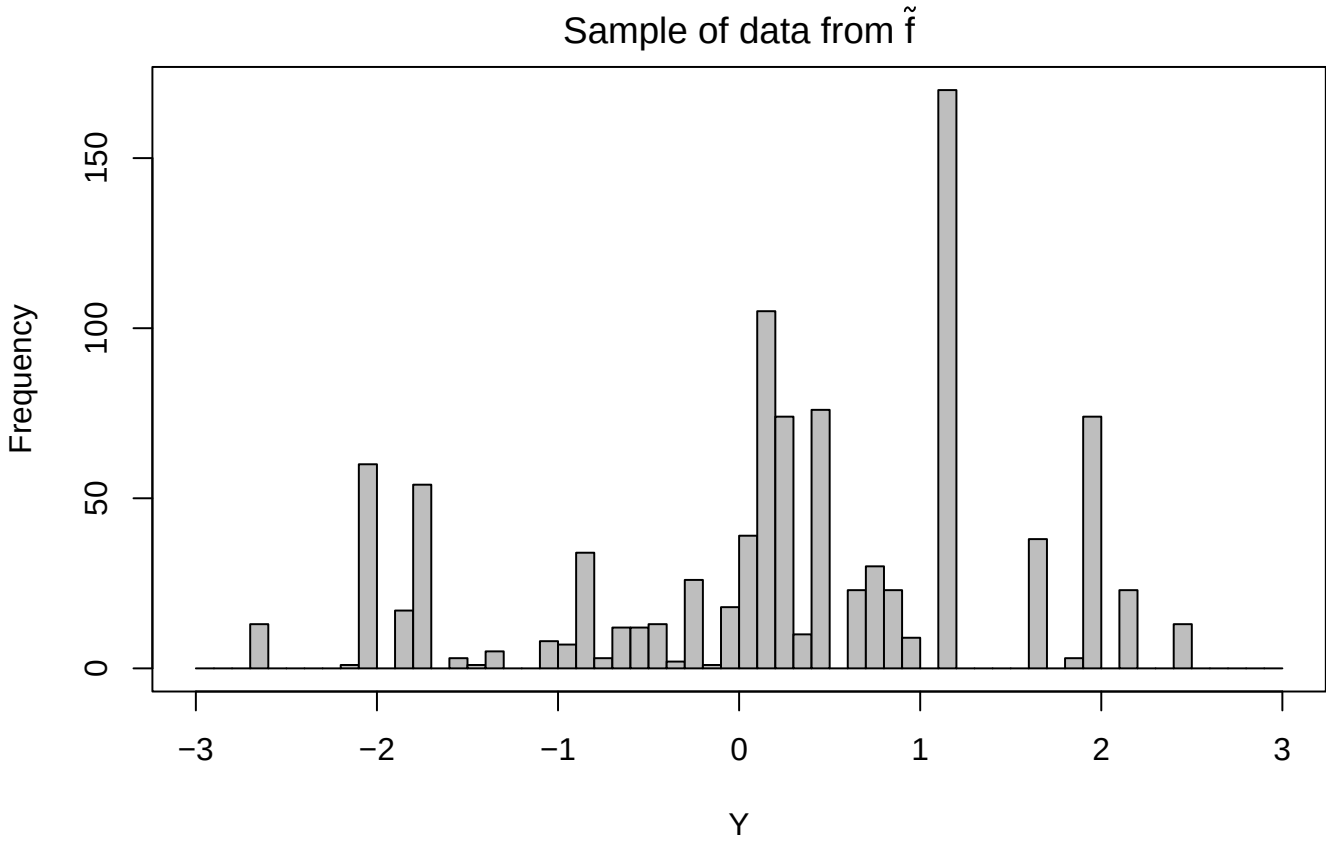

```
Xi<-rnorm(M,0,10)
V<-rbeta(M,1,alpha)
pi.vec<-c(V[1],cumprod(1-V[-M])*V[-1])
XiLarge<-Xi
piLarge<-pi.vec

xi.sort<-XiLarge[order(XiLarge)]
pi.sort<-pi.vec[order(XiLarge)]
Pi<-cumsum(pi.sort)
par(mar=c(4,4,2,0))
plot(xi.sort,Pi,type="s",ylim=range(0,1),xlab="x",ylab="Cumul. Prob")
title(expression(paste('Five replicate draws of the random cdf ',(alpha==2))))
for(i in 1:4){
        Xi<-rnorm(M,0,10)
        V<-rbeta(M,1,alpha)
    pi.vec<-c(V[1],cumprod(1-V[-M])*V[-1])
        XiLarge<-Xi
        piLarge<-pi.vec
        xi.sort<-XiLarge[order(XiLarge)]
        pi.sort<-pi.vec[order(XiLarge)]
        Pi<-cumsum(pi.sort)
        lines(xi.sort,Pi,type="s")
}
xv<-seq(-30,30,length=1001)
yv<-pnorm(xv,0,10)
lines(xv,yv,col="red",lwd=2)
```



The parameter α is a precision parameter; as α increases, the random distribution becomes more concentrated on the base measure.



The random distribution $\widetilde{f}$ can be sampled to produce iid data $Y_1, \ldots, Y_n$ in a straightforward fashion. In the simulation below, we simulate $\widetilde{f}$ with $\alpha = 10$ and $G_X \equiv Normal(0, 1)$



Note that in the sample of size $n = 1000$, there are only 64 distinct Y values

```
tabY<-table(Y)
names(tabY)<-as.character(round(as.numeric(names(tabY)),5))
tabY

+ -2.63151 -2.16276 -2.08518 -1.81156 -1.75768 -1.70421 -1.54031 -1.40852
+       13        1       60       17       45        9        3        1
+ -1.32673 -1.09304 -0.90964 -0.86634 -0.83135 -0.78091 -0.73734 -0.69322
+        5        8        7        1       33        2        1        1
+ -0.66846 -0.62998 -0.53459 -0.50782 -0.47949 -0.41675 -0.33048 -0.32752
+        8        3       11        1       12        1        1        1
+ -0.29525 -0.28987 -0.26925  -0.1323 -0.09247 -0.08415  -0.0704 -0.05806
+        1        4       21        1        1        6        5        5
+  -0.0233  0.01556  0.10959  0.18201  0.21553   0.2375  0.25179  0.25219
+        1       39      104        1        4       13        1        8
+  0.25484  0.28758  0.31203   0.3336   0.3526  0.38624  0.41278  0.49196
+        1       47        5        1        3        1       69        7
+  0.60717  0.61285  0.65872  0.70102  0.71598  0.72779  0.81569  0.94265
+       11        9        3        2        2       26       23        9
+  1.11499  1.13418  1.60025  1.68618  1.85047  1.99737  2.13939  2.43685
+      132       38        3       35        3       74       23       13
```

and thus we observe *clustering*: this is due to the fact that $\widetilde{f}$ is discrete.

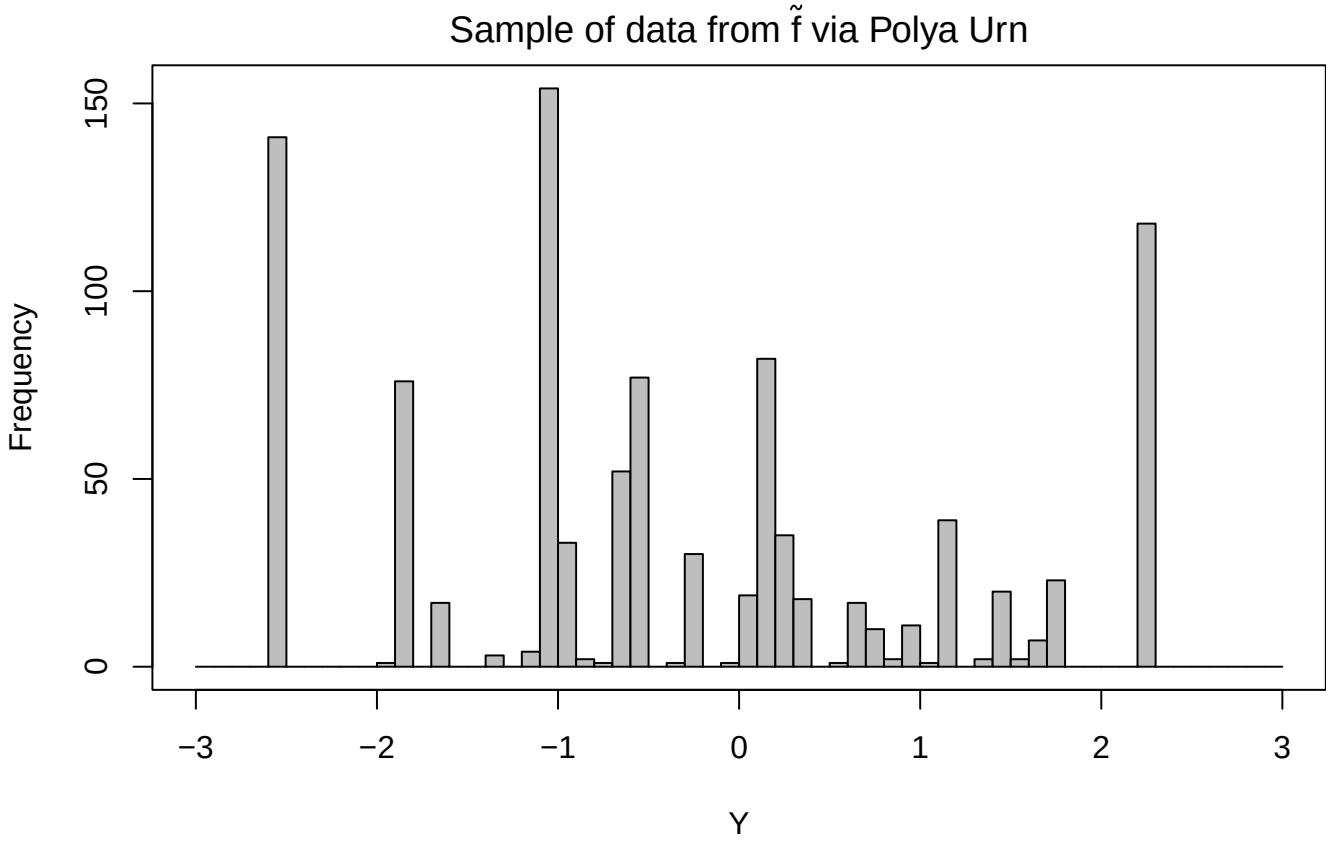
Polya Urn: We can also use a *Polya Urn* scheme to sample these data:

1. simulate $Y_1 \sim G_X$;
2. for $i = 2, 3, \ldots, n$ generate Y_i from the mixture distribution

$$\frac{\alpha}{\alpha + i - 1} G_X(.) + \frac{1}{\alpha + i - 1} \sum_{j=1}^{i-1} \delta_{Y_j}(.)$$

that is

- with probability $\alpha/(\alpha + i - 1)$, simulate $Y_i \sim G_X(.)$;
- with probability $1/(\alpha + i - 1)$, simulate Y_i uniformly on $\{Y_1, \ldots, Y_{i-1}\}$.



Here there are 52 distinct Y values. The Polya Urn method reconstructs a sample of exchangeable observables according to the de Finetti representation

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \int \prod_{i=1}^{n} \widetilde{f}(y_i)\, \pi_0(d\widetilde{f})$$

by sampling directly from the left hand side using the factorization

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = f_{Y_1}(y_1) \prod_{i=2}^{n} f_{Y_i|Y_1,\ldots,Y_{i-1}}(y_i|y_1, \ldots, y_{i-1})$$

where, denoting $g_X(.)$ is the density corresponding to $G_X(.)$, we have

$$f_{Y_i|Y_1,\ldots,Y_{i-1}}(y_i|y_1, \ldots, y_{i-1}) = \frac{\alpha}{\alpha + i - 1} g_X(y_i) + \frac{1}{\alpha + i - 1} \sum_{j=1}^{i-1} \delta_{Y_j}(y_i)$$

Dirichlet Process Mixtures: In a *Dirichlet Process Mixture* model we add another stage that brings in a continuous distribution. For example, could treat each x_i as the location of a normal density, and consider generating a y for each

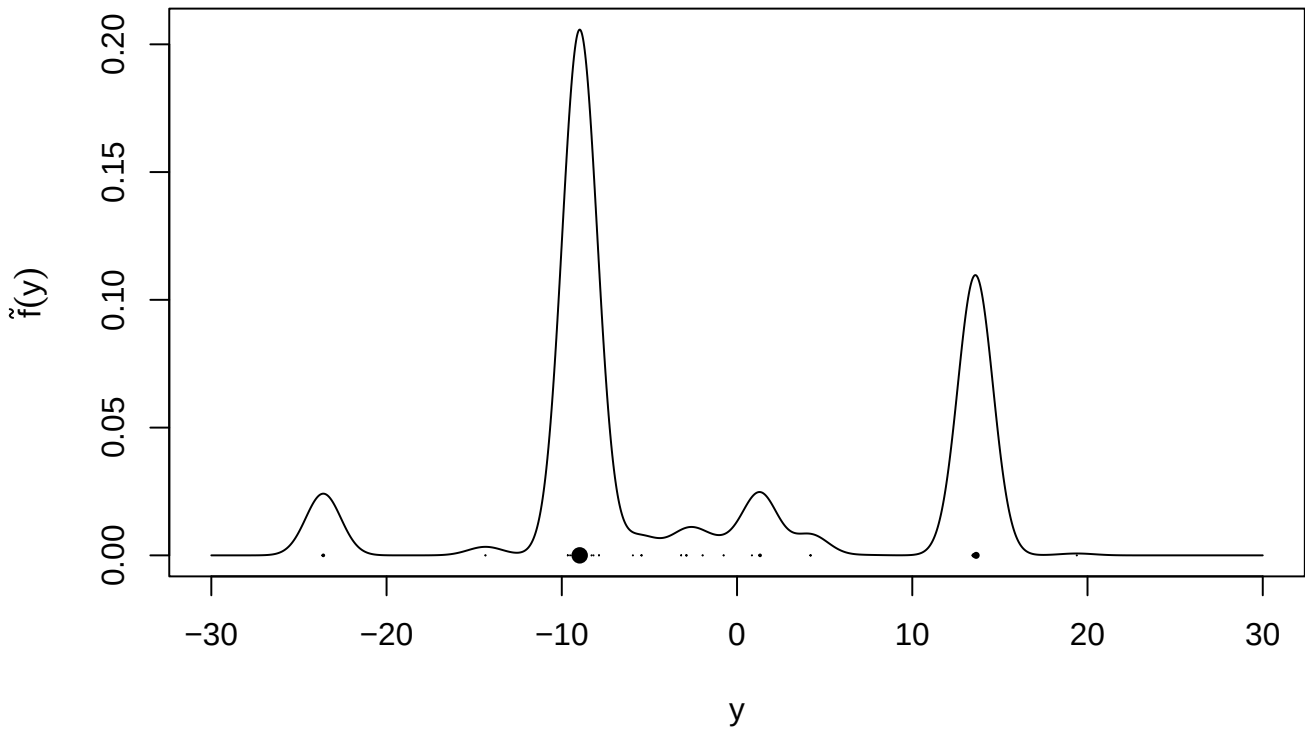
$$\begin{aligned} x_1, x_2, \ldots &\sim G_X \\ \pi_1, \pi_2, \ldots &\quad \text{generated by stick-breaking.} \\ y_i &\sim \phi((y_i - x_i)/\sigma) \qquad i = 1, 2, \ldots \end{aligned}$$

Then the random density function generated takes the form

$$\widetilde{f}(y) = \sum_{i=1}^{\infty} \pi_i \phi((y - x_i)/\sigma)/\sigma$$

that is, an **infinite mixture model**. Below we have $\alpha = 2$, with $G_X(x) \equiv Normal(0, \lambda^2)$ with $\lambda = 10$.

```
set.seed(1010)
M<-1001
alpha<-2; sigma<-1;lambda<-10
Xi<-rnorm(M,0,lambda);V<-rbeta(M,1,alpha)
pi.vec<-c(V[1],cumprod(1-V[-M])*V[-1])
xv<-seq(-30,30,by=0.01);yv<-xv*0
for(i in 1:M){
    yv<-yv+pi.vec[i]*dnorm(xv,Xi[i],sigma)
}
par(mar=c(4,5,2,0))
plot(xv,yv,type='l',xlab='y',ylab=expression(widetilde(f)(y)))
points(Xi,Xi*0,cex=2*pi.vec,pch=19)
```



Gibbs sampler: For fully Bayesian inference consider the de Finetti construction

- a prior model for f that is $DPM(\alpha, G_X, g_Y; \theta)$ where θ represents the parameters that appear in G_X and g_Y.
- conditional on f, data $y_1, y_2, \ldots, y_n \sim f$ independently

We wish to compute the posterior for f. We use the hierarchical formulation

$$\begin{aligned} y_j|x_j &\overset{\text{ind.}}{\sim} g_Y(y_j|x_j;\theta) \qquad j = 1, \ldots, n \\ x_1, \ldots, x_n &\sim DP(\alpha, G_X; \theta) \\ \theta &\sim \pi_0(\theta). \end{aligned}$$

The latent variables $x_1, \ldots, x_n$ are also treated as parameters. They can be sampled using an MCMC **Gibbs sampler** scheme. For $j = 1, \ldots, n$, we sample

$$x_j \mid \boldsymbol{x}_{(j)}, \boldsymbol{y} \sim w_0 p(x_j|y_j) + \sum_{l \neq j} w_l \delta_{x_l}$$

where

- $\boldsymbol{y} = (y_1, \ldots, y_n)^\mathsf{T}$
- $\boldsymbol{x}_{(j)} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)^\mathsf{T}$.
- w_0 is proportional to α times the **prior predictive** of y_j
- w_l is proportional to the **likelihood** of $y_j|x_l$
- $p(x_j|y_j)$ is the **posterior** for x_j given y_j.

Marginalized sampler: To be more efficient, we can use the **clustering** property: suppose that at a given iteration of the MCMC, there are K clusters labelled 1 to K, where $K \leq n$. Label the K distinct x values $z_1, \ldots, z_K$ and for each j, define the corresponding cluster label c_j where

$$c_j = k \qquad \Longleftrightarrow \qquad x_j = z_k$$

We can update the c_js instead of the x_js which will be more computationally efficient; we are clustering xs to the **cluster centres** at the z values. For $i = 1, \ldots, n$, let

- $n_1(i), \ldots, n_K(i)$ denote the number of items in clusters $1, \ldots, K$
- $\boldsymbol{y}_1(i), \ldots, \boldsymbol{y}_K(i)$ denote the vectors of y values currently allocated to the K clusters

if the ith data point is removed. For $i = 1, \ldots, n$, we sample the cluster labels in a Gibbs sampler with conditional probabilities

$$\Pr[c_i = k \mid c_{(i)}] \propto \frac{n_k(i)}{\alpha + n - 1} p(y_i|\boldsymbol{y}_k(i)) \qquad k = 1, \ldots, K$$

and

$$\Pr[c_i = K + 1 \mid c_{(i)}] \propto \frac{\alpha}{\alpha + n - 1} p(y_i)$$

In this expression

- $p(y_i|\boldsymbol{y}_k(i))$ is the **posterior predictive** density for y_i, assuming that y_i comes from cluster k.
- $p(y_i)$ is the **prior predictive** density for y_i, assuming that y_i comes from a **new cluster** not currently represented in the data.

In this formulation, we have **integrated out** the Dirichlet Process analytically. Thus we can simply sample the cluster labels in turn, and then sample the $z_1, \ldots, z_k$ values; this will allow us to do density estimation. By the usual calculation

$$p(y_i|\boldsymbol{y}_k(i)) = \int g_Y(y_i \mid x) p(x \mid \boldsymbol{y}_k(i))\, dx$$

where

$$p(x \mid \boldsymbol{y}_k(i)) \quad \propto \quad p(\boldsymbol{y}_k(i) \mid x) p(x) = \left\{ \prod_{l \neq i} g_Y(y_l \mid x) \right\} p(x)$$

gives the posterior distribution for the kth cluster centre. Similarly

$$p(y_i) = \int g_Y(y_i \mid x) p(x)\, dx$$

In the earlier Normal model, suppose for simplicity that G_X is the $Normal(0, \lambda^2)$ density:

$$x_i \sim Normal(0, \lambda^2) \qquad\qquad y_i \mid x_i \sim Normal(x_i, \sigma^2)$$

Then by standard calculations

$$p(y_i|\boldsymbol{y}_k(i)) \equiv Normal\left(\frac{n_k(i)\overline{y}_k(i)/\sigma^2}{n_k(i)/\sigma^2 + 1/\lambda^2}, \frac{(n_k(i)+1)/\sigma^2 + 1/\lambda^2}{n_k(i)/\sigma^2 + 1/\lambda^2} \sigma^2 \right)$$

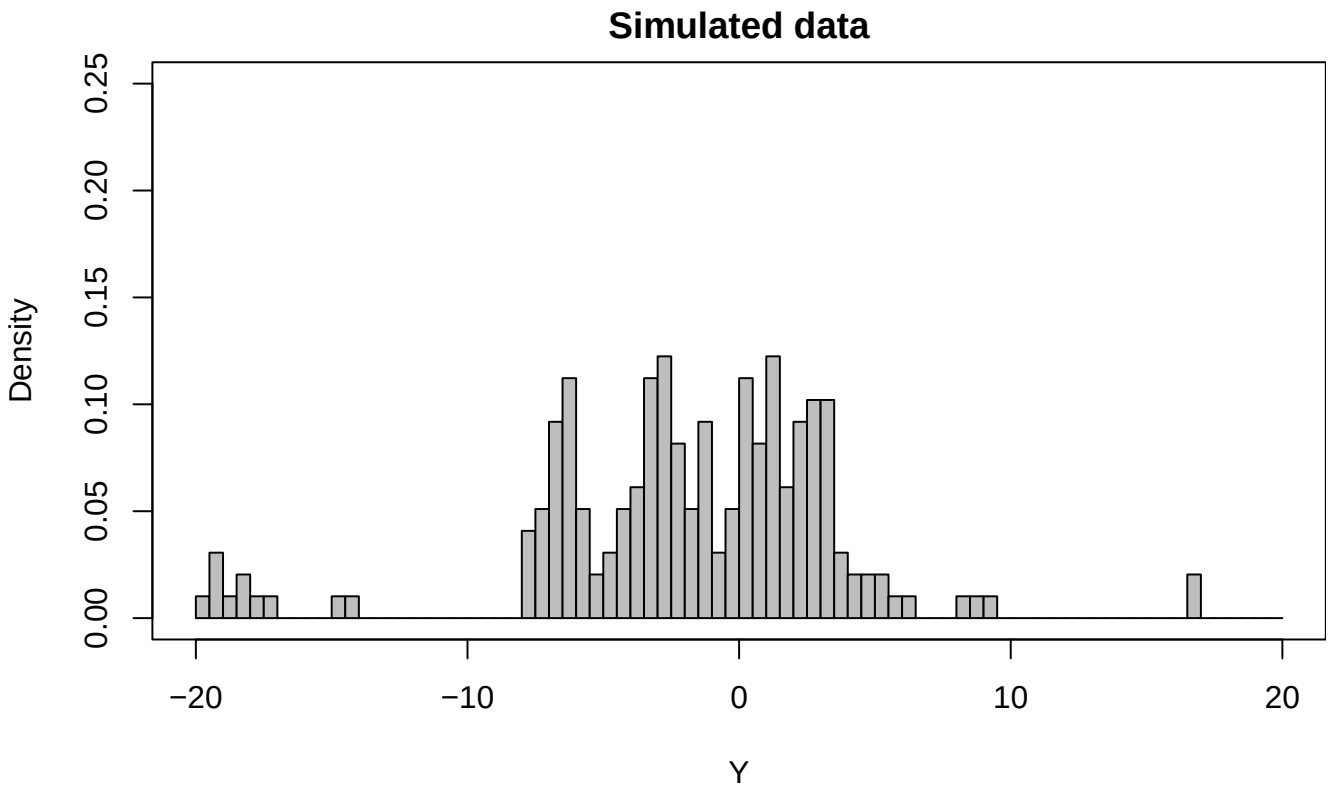
and

$$p(y_i) \equiv Normal\left(0, \sigma^2 + \lambda^2\right).$$

The marginalized version of the algorithm often mixes more quickly than the Gibbs sampler on the cluster centres.

```
set.seed(1091)
n<-200
alpha<-2;sigma<-1; lambda<-10
X<-numeric(n);X[1]<-rnorm(1,0,lambda)
for(j in 2:n){
        u<-runif(1)
        if(u < alpha/(alpha+j-1)){
                X[j]<-rnorm(1,0,lambda)
        }else{
                X[j]<-sample(X[1:(j-1)],size=1)
        }
}
True.K<-length(table(X))
Y<-rnorm(n,X,sigma)
```

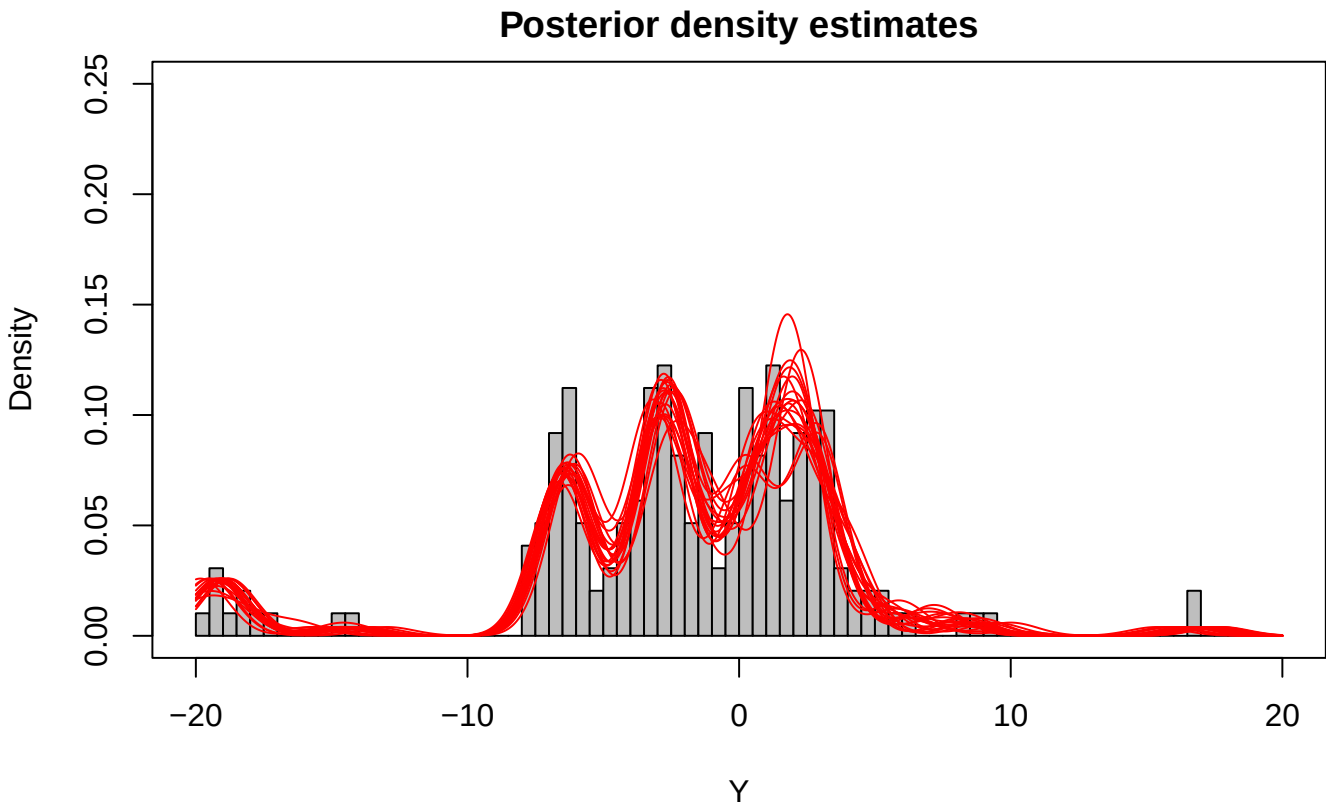
```
par(mar=c(4,4,2,0))
inc<-abs(Y) < 20
hist(Y[inc],breaks=seq(-20,20,by=0.50),freq=FALSE,xlab='Y',
    ylim=range(0,0.25),col='gray',main='Simulated data');box()
```



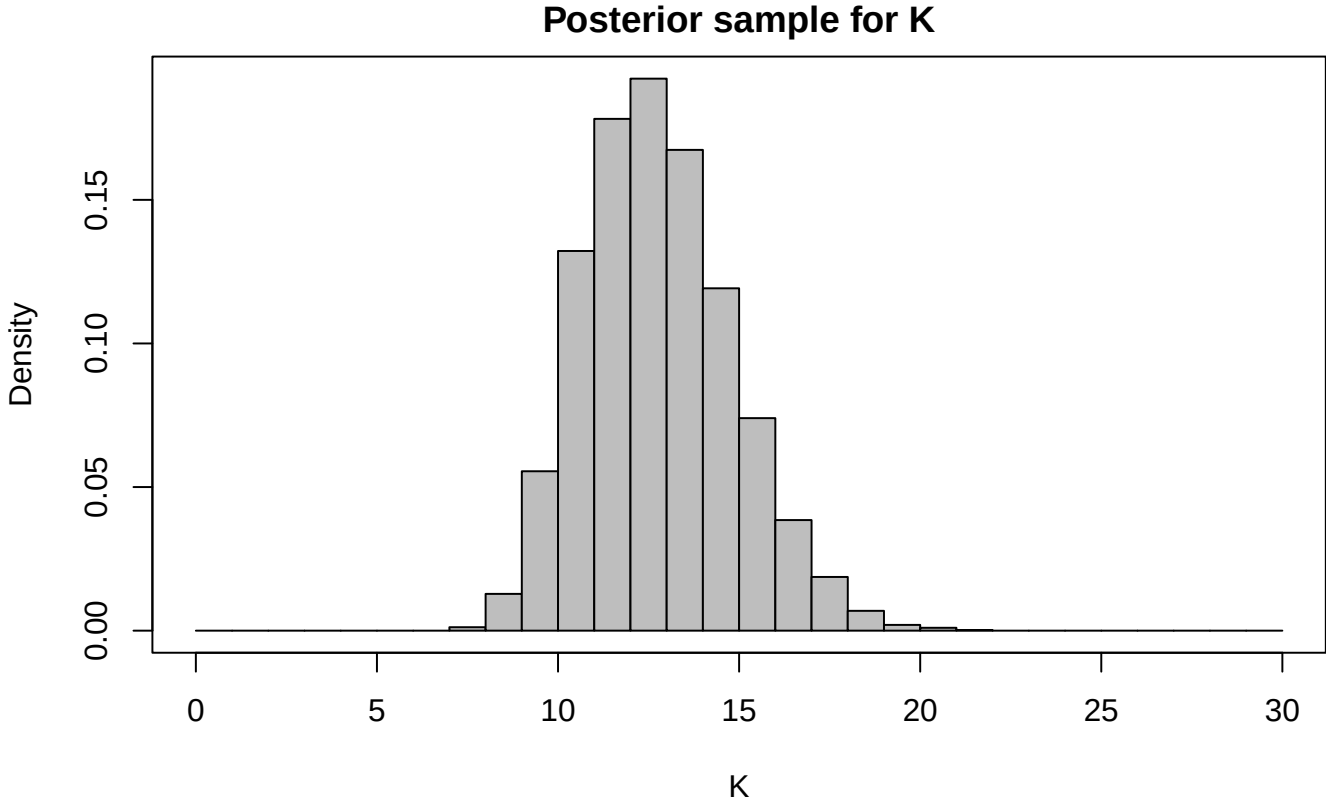

```
set.seed(1010)
index<-c(1:n)
Kval<-5
Yq<-quantile(Y,prob=c(0:Kval)/Kval)
Cvec<-as.numeric(cut(Y,Yq,include.lowest=T))
Ysum.Vec<-rep(0,Kval)
for(k in 1:Kval){Ysum.Vec[k]<-sum(Y[Cvec==k])}
Nvec<-as.numeric(table(Cvec))

nreps<-10000;Nsamp<-rep(0,nreps)
yv<-seq(-20,20,by=0.01);fyv.samp<-numeric()
for(irep in 1:nreps){
        for(i in 1:n){
                sum.Vec<-Ysum.Vec
                sum.Vec[Cvec[i]]<-sum.Vec[Cvec[i]]-Y[i]
                nvec<-Nvec
                nvec[Cvec[i]]<-nvec[Cvec[i]]-1
                if(nvec[Cvec[i]] > 0){
                        mvec<-sum.Vec/nvec
                        Mvec<-(sum.Vec/sigma^2)/(nvec/sigma^2+1/lambda^2)
                        Svec<-sigma^2*((nvec+1)/sigma^2+1/lambda^2)/(nvec/sigma^2+1/lambda^2)
                        pvec<-nvec*dnorm(rep(Y[i],Kval),Mvec,sqrt(Svec))
                        pvec<-c(pvec,alpha*dnorm(Y[i],0,sqrt(sigma^2 + lambda^2)))/(n-1+alpha)
                        pvec<-pvec/sum(pvec)
                        Ksamp<-sample(c(1:(Kval+1)),size=1,prob=pvec)
                        Cvec[i]<-Ksamp
                        if(Ksamp == Kval+1){
                                Kval<-Kval+1
                                Ysum.Vec<-c(sum.Vec,Y[i])
```

```
                        Nvec<-c(nvec,1)
                    }else{
                        Ysum.Vec<-sum.Vec
                        Ysum.Vec[Ksamp]<-sum.Vec[Ksamp]+Y[i]
                        Nvec<-nvec
                        Nvec[Ksamp]<-nvec[Ksamp]+1
                    }
                }else{
                    Ktmp<-Cvec[i]
                    sum.Vec<-sum.Vec[-Ktmp]
                    nvec<-nvec[-Ktmp]
                    Kval<-Kval-1
                    Cvec[Cvec > Ktmp]<-Cvec[Cvec > Ktmp]-1

                    mvec<-sum.Vec/nvec
                    Mvec<-(sum.Vec/sigma^2)/(nvec/sigma^2+1/lambda^2)
                    Svec<-sigma^2*((nvec+1)/sigma^2+1/lambda^2)/(nvec/sigma^2+1/lambda^2)
                    pvec<-nvec*dnorm(rep(Y[i],Kval),Mvec,sqrt(Svec))
                    pvec<-c(pvec,alpha*dnorm(Y[i],0,sqrt(sigma^2 + lambda^2)))/(n-1+alpha)

                    pvec<-pvec/sum(pvec)
                    Ksamp<-sample(c(1:(Kval+1)),size=1,prob=pvec)
                    Cvec[i]<-Ksamp
                    if(Ksamp == Kval+1){
                        Kval<-Kval+1
                        Ysum.Vec<-c(sum.Vec,Y[i])
                        Nvec<-c(nvec,1)
                    }else{
                        Ysum.Vec<-sum.Vec
                        Ysum.Vec[Ksamp]<-sum.Vec[Ksamp]+Y[i]
                        Nvec<-nvec
                        Nvec[Ksamp]<-nvec[Ksamp]+1
                    }
                }
        }
        Nsamp[irep]<-Kval
        if(irep %% 500 == 0){
                MuVec<-(Ysum.Vec/sigma^2)/(Nvec/sigma^2+1/lambda^2)
                SigVec<-1/(Nvec/sigma^2+1/lambda^2)
                Xvec<-rnorm(Kval)*sqrt(SigVec)+MuVec
                fyv<-yv*0
                for(k in 1:Kval){
                        fyv<-fyv+Nvec[k]*dnorm(yv,Xvec[k],sigma)/n
                }
                fyv.samp<-cbind(fyv.samp,fyv)
        }
}
```

```
par(mar=c(4,4,2,0))
inc<-abs(Y) < 20
hist(Y[inc],breaks=seq(-20,20,by=0.5),freq=FALSE,col='gray',xlab='Y',
    ylim=range(0,0.25),main='Posterior density estimates');box()
for(i in 1:ncol(fyv.samp)){lines(yv,fyv.samp[,i],col='red')}
```




```
hist(Nsamp,col='gray',breaks=seq(0,30,by=1),freq=FALSE,
    xlab='K',main='Posterior sample for K');box()
```

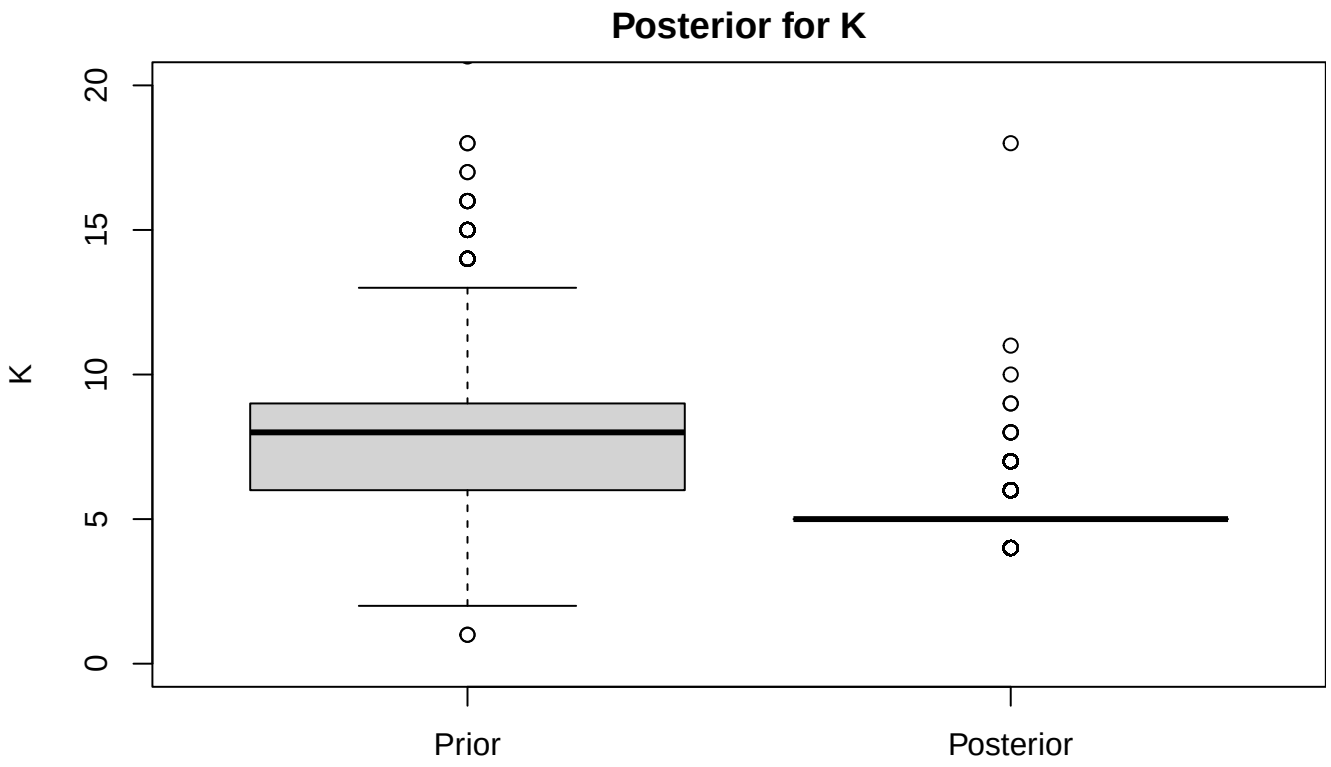


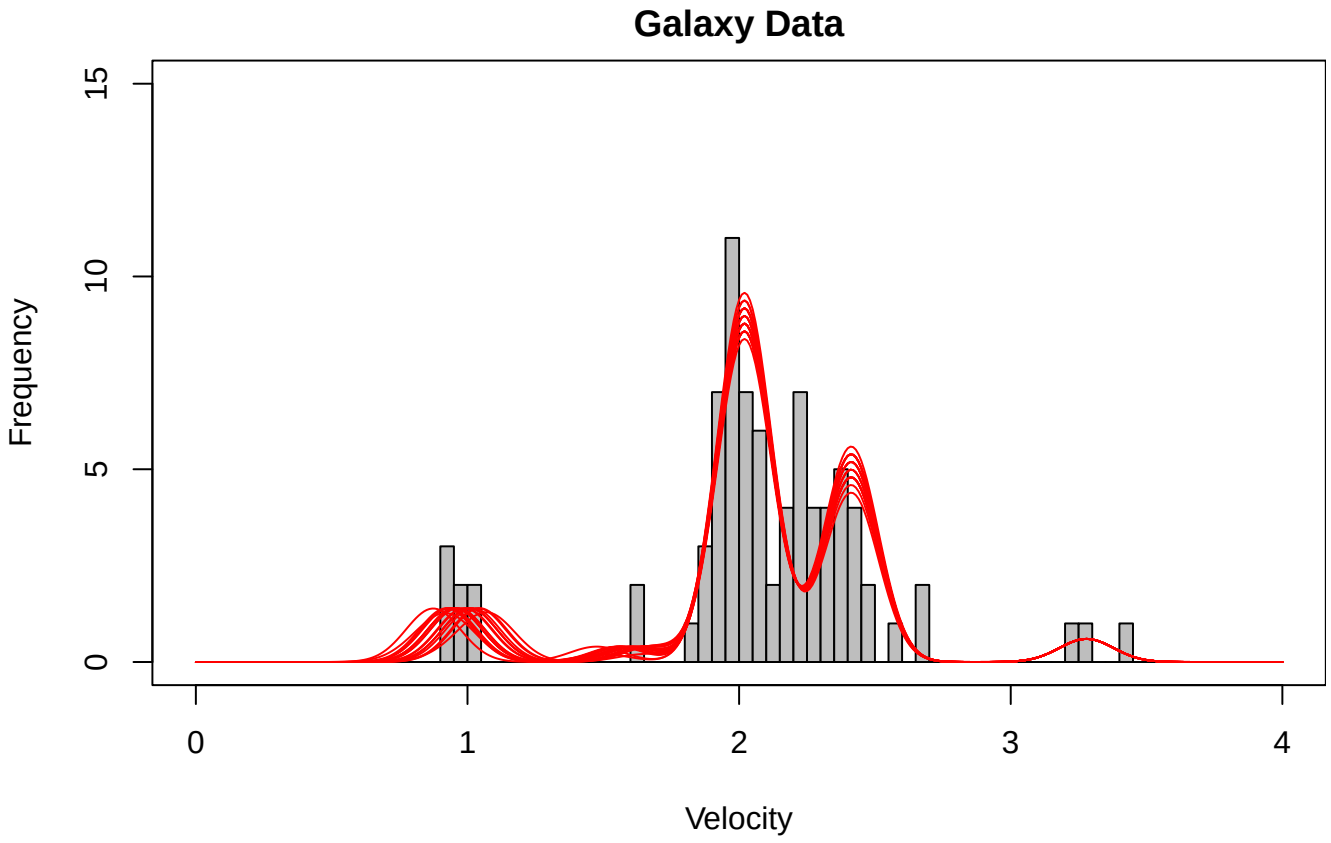
Analysis of the Galaxy data: From the `MASS::galaxies` help: the galaxies data are the velocities in km/sec of 82 galaxies from six conic sections of an unfilled survey of the Corona Borealis region. Multimodality in such surveys is evidence for voids and superclusters in the far universe.

```
#Galaxy data using Gibbs sampler
Y.galaxy<-MASS::galaxies/10000
n.galaxy<-length(Y.galaxy)

set.seed(1010)
index<-c(1:n.galaxy)
nreps<-10000

alpha<-2.0

Pi<-Y.galaxy
Prior.K.galaxy<-numeric()

for(irep in 1:nreps){
        for(i in 1:n.galaxy){
                pvec<-rep(1/(alpha+n.galaxy-1),n.galaxy)
                pvec[i]<-alpha/(alpha+n.galaxy-1)
                ival<-sample(index,size=1,prob=pvec)
                if(ival == i){
                        Pi[i]<-rnorm(1,0,lambda)
                }else{
                        Pi[i]<-Pi[ival]
                }
        }

        Kval<-length(table(Pi))
        Prior.K.galaxy<-c(Prior.K.galaxy,Kval)
}

set.seed(1010)
index<-c(1:n.galaxy)
nreps<-10000
sigma<-0.1
lambda<-0.5

Pi<-Y.galaxy
Pvec<-rep(0,n.galaxy)
Posterior.K.galaxy<-numeric()
yv<-seq(0,4,by=0.01)
fyv.samp.GAL<-numeric()
for(irep in 1:nreps){

        for(i in 1:n.galaxy){
                pvec<-dnorm(Y.galaxy[i],Pi,rep(sigma,n.galaxy))
                pvec[i]<-alpha*dnorm(Y.galaxy[i],0,sqrt(sigma^2+lambda^2))
                ival<-sample(index,size=1,prob=pvec)
                if(ival == i){
                        muval<-(Y.galaxy[i]/sigma^2)/(1/sigma^2+1/lambda^2)
                        sval<-sqrt(1/(1/sigma^2+1/lambda^2))
                        Pi[i]<-rnorm(1,muval,sval)
                }else{
```

```
                        Pi[i]<-Pi[ival]
                }
        }

        Kval<-length(table(Pi))
    Posterior.K.galaxy<-c(Posterior.K.galaxy,Kval)

        if(irep %% 100 == 0){
                fyv<-yv*0
                for(k in 1:n.galaxy){
                        fyv<-fyv+dnorm(yv,Pi[k],sigma)/n.galaxy
                }
                fyv.samp.GAL<-cbind(fyv.samp.GAL,fyv)
        }

}
```

```
table(Posterior.K.galaxy)/length(Posterior.K.galaxy)

+ Posterior.K.galaxy
+      4      5      6      7      8      9     10     11     18     28
+ 0.0512 0.8120 0.1260 0.0096 0.0006 0.0002 0.0001 0.0001 0.0001 0.0001

table(Prior.K.galaxy)/length(Prior.K.galaxy)

+ Prior.K.galaxy
+      1      2      3      4      5      6      7      8      9     10     11
+ 0.0002 0.0019 0.0135 0.0382 0.0862 0.1399 0.1638 0.1686 0.1483 0.1072 0.0651
+     12     13     14     15     16     17     18     21     28
+ 0.0383 0.0179 0.0062 0.0032 0.0009 0.0002 0.0002 0.0001 0.0001
```

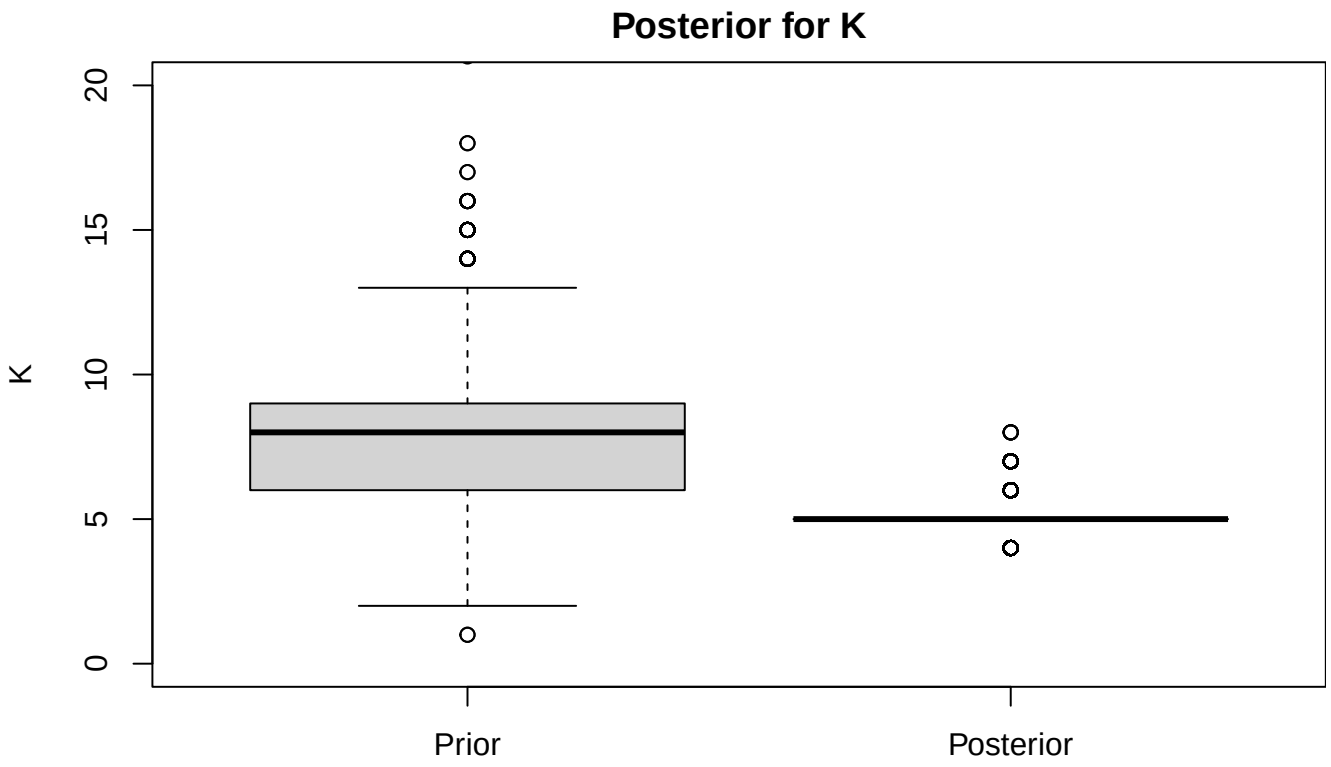

Posterior for K
K
0
5
10
15
20
Prior
Posterior

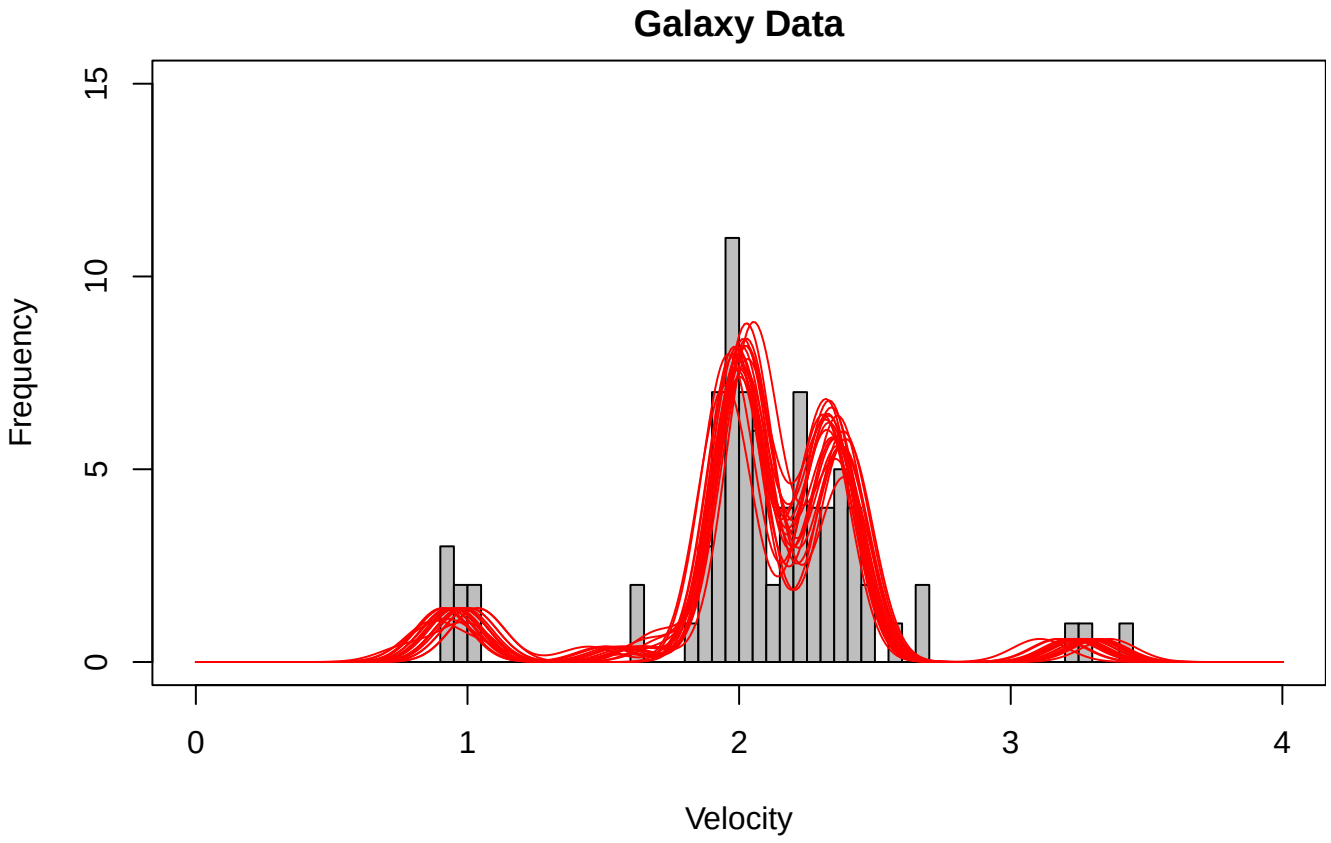

Galaxy Data
Frequency
0
5
10
15
0
1
2
3
4
Velocity

```r
#Galaxy data using marginalized algorithm
Y.galaxy<-MASS::galaxies/10000
n.galaxy<-length(Y.galaxy)

set.seed(1010)
index<-c(1:n.galaxy)
nreps<-10000
alpha<-2.0
sigma<-0.1
lambda<-0.5

index<-c(1:n.galaxy)
Kval<-5
Yq<-quantile(Y.galaxy,prob=c(0:Kval)/Kval)
Cvec<-as.numeric(cut(Y.galaxy,Yq,include.lowest=T))
Ysum.Vec<-rep(0,Kval)
for(k in 1:Kval){Ysum.Vec[k]<-sum(Y.galaxy[Cvec==k])}
Nvec<-as.numeric(table(Cvec))

Posterior.K.marg<-rep(0,nreps)
yv<-seq(0,4,by=0.01)
marg.fyv.samp.GAL<-numeric()

for(irep in 1:nreps){

        for(i in 1:n.galaxy){
                sum.Vec<-Ysum.Vec
                sum.Vec[Cvec[i]]<-sum.Vec[Cvec[i]]-Y.galaxy[i]
                nvec<-Nvec
                nvec[Cvec[i]]<-nvec[Cvec[i]]-1
                if(nvec[Cvec[i]] > 0){
                        mvec<-sum.Vec/nvec
                        Mvec<-(sum.Vec/sigma^2)/(nvec/sigma^2+1/lambda^2)
                        Svec<-sigma^2*((nvec+1)/sigma^2+1/lambda^2)/(nvec/sigma^2+1/lambda^2)
                        pvec<-nvec*dnorm(rep(Y.galaxy[i],Kval),Mvec,sqrt(Svec))
                        pvec<-c(pvec,alpha*dnorm(Y.galaxy[i],0,sqrt(sigma^2 + lambda^2)))
                        pvec<-pvec/sum(pvec)
                        Ksamp<-sample(c(1:(Kval+1)),size=1,prob=pvec)
                        Cvec[i]<-Ksamp
                        if(Ksamp == Kval+1){
                                Kval<-Kval+1
                                Ysum.Vec<-c(sum.Vec,Y.galaxy[i])
                                Nvec<-c(nvec,1)
                        }else{
                                Ysum.Vec<-sum.Vec
                                Ysum.Vec[Ksamp]<-sum.Vec[Ksamp]+Y.galaxy[i]
                                Nvec<-nvec
                                Nvec[Ksamp]<-nvec[Ksamp]+1
                        }
                }else{
                        Ktmp<-Cvec[i]
                        sum.Vec<-sum.Vec[-Ktmp]
                        nvec<-nvec[-Ktmp]
                        Kval<-Kval-1
```

```
                        Cvec[Cvec > Ktmp]<-Cvec[Cvec > Ktmp]-1

                        mvec<-sum.Vec/nvec
                        Mvec<-(sum.Vec/sigma^2)/(nvec/sigma^2+1/lambda^2)
                        Svec<-sigma^2*((nvec+1)/sigma^2+1/lambda^2)/(nvec/sigma^2+1/lambda^2)
                        pvec<-nvec*dnorm(rep(Y.galaxy[i],Kval),Mvec,sqrt(Svec))
                        pvec<-c(pvec,alpha*
                dnorm(Y.galaxy[i],0,sqrt(sigma^2 + lambda^2)))/(n.galaxy-1+alpha)

                        pvec<-pvec/sum(pvec)
                        Ksamp<-sample(c(1:(Kval+1)),size=1,prob=pvec)
                        Cvec[i]<-Ksamp
                        if(Ksamp == Kval+1){
                                Kval<-Kval+1
                                Ysum.Vec<-c(sum.Vec,Y.galaxy[i])
                                Nvec<-c(nvec,1)
                        }else{
                                Ysum.Vec<-sum.Vec
                                Ysum.Vec[Ksamp]<-sum.Vec[Ksamp]+Y.galaxy[i]
                                Nvec<-nvec
                                Nvec[Ksamp]<-nvec[Ksamp]+1
                        }
                }
        }


        Posterior.K.marg[irep]<-Kval

        if(irep %% 100 == 0){

                MuVec<-(Ysum.Vec/sigma^2)/(Nvec/sigma^2+1/lambda^2)
                SigVec<-1/(Nvec/sigma^2+1/lambda^2)
                Xvec<-rnorm(Kval)*sqrt(SigVec)+MuVec
                fyv<-yv*0
                for(k in 1:Kval){
                        fyv<-fyv+Nvec[k]*dnorm(yv,Xvec[k],sigma)/n.galaxy
                }
                marg.fyv.samp.GAL<-cbind(marg.fyv.samp.GAL,fyv)
        }
}
```

```
table(Posterior.K.marg)/nreps

+ Posterior.K.marg
+      4      5      6      7      8
+ 0.2340 0.6559 0.1021 0.0075 0.0005

table(Prior.K.galaxy)/length(Prior.K.galaxy)

+ Prior.K.galaxy
+      1      2      3      4      5      6      7      8      9     10     11
+ 0.0002 0.0019 0.0135 0.0382 0.0862 0.1399 0.1638 0.1686 0.1483 0.1072 0.0651
+     12     13     14     15     16     17     18     21     28
+ 0.0383 0.0179 0.0062 0.0032 0.0009 0.0002 0.0002 0.0001 0.0001
```


Posterior for K
K
0
5
10
15
20
Prior
Posterior


Galaxy Data
Frequency
0
5
10
15
0
1
2
3
4
Velocity