The Metropolis-Hastings Algorithm

The Metropolis-Hastings algorithm is a Markov chain method for exploring a target distribution π.

The algorithm implements a sequence of moves that

(i) propose a new value z from a proposal distribution q that can depend on the current point of the chain x; usually this takes the form of a conditional density

$$q(x \to z) = q(z|x)$$

then

(ii) accept z with probability

$$\alpha(x, z) = \min\left\{1, \frac{\pi(z)}{\pi(x)} \frac{q(z \to x)}{q(x \to z)}\right\}$$

If the proposal is not accepted, the Markov chain stays at x.

The Metropolis-Hastings Algorithm

Suppose that

$$\pi(x_1, x_2) = \frac{1}{\pi} \qquad x_1^2 + x_2^2 < 1$$

and zero otherwise. This distribution is Uniform on the unit disk.

Suppose that

$$q((x_1, x_2) \to (z_1, z_2)) = \frac{1}{\delta^2}$$

for

$$x_1 - \delta/2 < z_1 < x_1 + \delta/2, x_2 - \delta/2 < z_2 < x_2 + \delta/2$$

that is, the proposal is Uniform on the square of edge length δ centered at (x_1, x_2).

The Metropolis-Hastings Algorithm

Then we have that

$$\alpha((x_1, x_2), (z_1, z_2)) = 1$$

provided

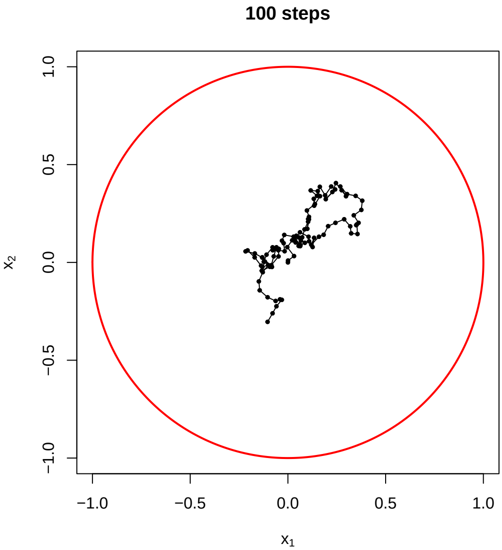
$$z_1^2 + z_2^2 < 1$$

otherwise $\alpha((x_1, x_2), (z_1, z_2)) = 0$.

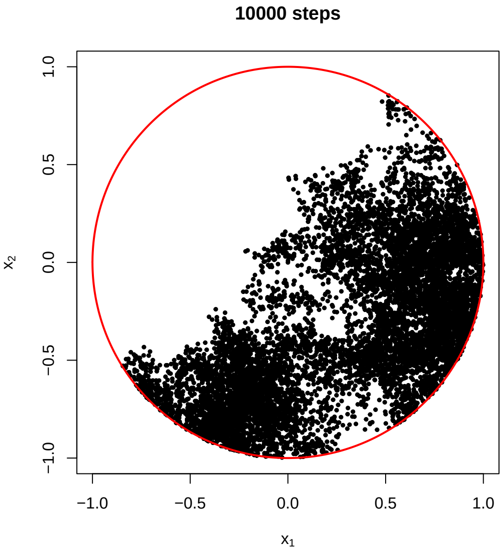
The Metropolis-Hastings Algorithm

```r
set.seed(32)
nits<-100000
xmat<-matrix(0,nrow=nits,ncol=2)
delta<-0.1
for(iter in 2:nits){
        z<-xmat[iter-1,]+runif(2,-delta/2,delta/2)
        if(sum(z^2) < 1){
            xmat[iter,]<-z
        }else{
            xmat[iter,]<-xmat[iter-1,]
        }
}
th<-seq(-pi,pi,by=0.01)
xv<-cos(th)
yv<-sin(th)
par(pty='s',mar=c(4,4,4,2))
plot(xmat[1:100,],pch=19,cex=0.5,main=paste('100 steps'),
        xlim=range(-1,1),ylim=range(-1,1),
        xlab=expression(x[1]),ylab=expression(x[2]))
lines(xmat[1:100,])
lines(xv,yv,col='red',lwd=2)
```

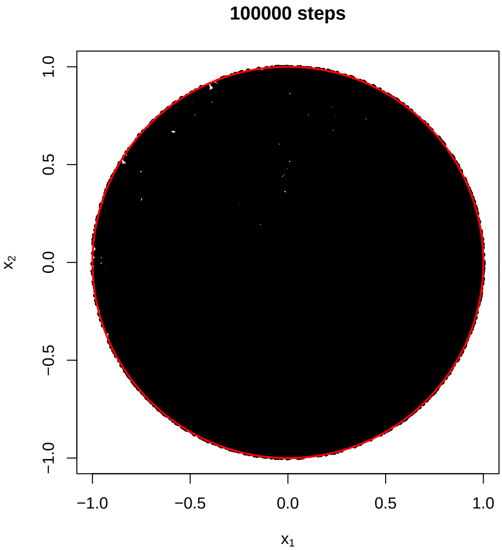
The Metropolis-Hastings Algorithm



The Metropolis-Hastings Algorithm



The Metropolis-Hastings Algorithm



The Metropolis-Hastings Algorithm

Suppose that

$$\pi(x_1, x_2) = c \exp\{-\beta h(x_1, x_2)\} \qquad -5 < x_1, x_2 < 5$$

and zero otherwise, where

$$h(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2$$

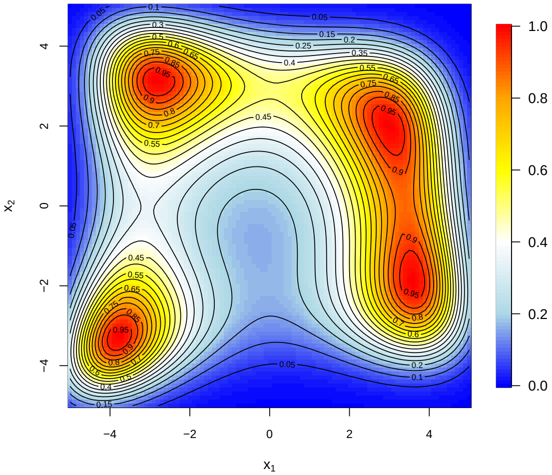
and $\beta > 0$ is a fixed constant, say $\beta = 0.01$.

The Metropolis-Hastings Algorithm

```
test.func<-function(x1,x2,bv=1){
    hval<-(x1^2+x2-11)^2 + (x1+x2^2-7)^2
    fval<-exp(-bv*hval)
    return(fval)
}
f <- Vectorize(test.func,vectorize.args=c("x1","x2"))
x1v<-x2v<-seq(-5,5,by=0.1)
be<-0.01;dmat<-outer(x1v,x2v,f,bv=be)
```

```
library(fields,quietly=TRUE)
par(pty='s',mar=c(4,3,2,2))
cols<-c("blue","lightblue","white","yellow","orange","red")
colfunc <- colorRampPalette(cols)
image.plot(x1v,x2v,dmat,col=colfunc(100),
           xlab=expression(x[1]),ylab=expression(x[2]),cex.axis=0.8)
contour(x1v,x2v,dmat,add=T,nlevels=20)
```

The Metropolis-Hastings Algorithm



The Metropolis-Hastings Algorithm

Suppose that as before

$$q((x_1, x_2) \to (z_1, z_2)) = \frac{1}{\delta^2}$$

for

$$x_1 - \delta/2 < z_1 < x_1 + \delta/2, x_2 - \delta/2 < z_2 < x_2 + \delta/2$$

that is, the proposal is Uniform on the square of edge length δ centered at (x_1, x_2).

The Metropolis-Hastings Algorithm

Then we have that

$$\alpha((x_1, x_2), (z_1, z_2)) = \min\left\{1, \frac{\exp\{-\beta h(z_1, z_2)\}}{\exp\{-\beta h(x_1, x_2)\}}\right\}$$

provided

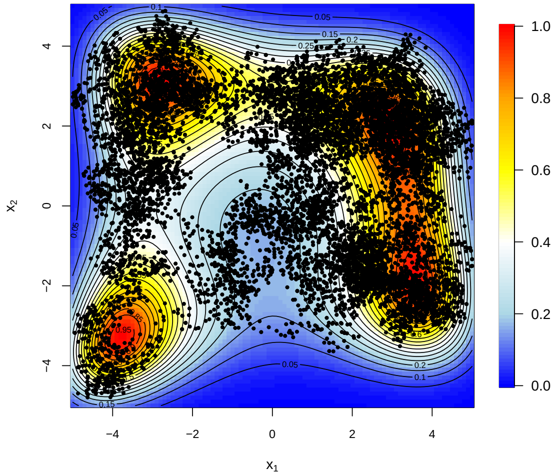
$$-5 < z_1, z_2 < 5$$

otherwise $\alpha((x_1, x_2), (z_1, z_2)) = 0$.

The Metropolis-Hastings Algorithm

```
set.seed(312)
nits<-10000
xmat<-matrix(0,nrow=nits,ncol=2)
delta<-0.5
be<-0.01
for(iter in 2:nits){
        z<-xmat[iter-1,]+runif(2,-delta/2,delta/2)
        if(abs(z[1])>5 | abs(z[2]) > 5){
            xmat[iter,]<-xmat[iter-1,]
        }else{
            u<-runif(1)
            num<-log(test.func(z[1],z[2],be))
            den<-log(test.func(xmat[iter-1,1],xmat[iter-1,2],be))
            if(log(u) < num-den){
                xmat[iter,]<-z
            }else{
                xmat[iter,]<-xmat[iter-1,]
            }
        }
}
```

The Metropolis-Hastings Algorithm



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