# MATH 559 Fall 2023
## Formula Sheet

- **Complete-the-square** formulae for quadratic forms:
  - *Univariate case:*

$$A(x-a)^2 + B(x-b)^2 = (A+B)\left(x - \frac{Aa+Bb}{A+B}\right)^2 + \frac{AB}{A+B}(a-b)^2$$

  - *Multivariate case:*

$$(\mathbf{x}-\mathbf{a})^\top \mathbf{A}(\mathbf{x}-\mathbf{a}) + (\mathbf{x}-\mathbf{b})^\top \mathbf{B}(\mathbf{x}-\mathbf{b}) = (\mathbf{x}-\mathbf{m})^\top \mathbf{M}(\mathbf{x}-\mathbf{m}) + \mathbf{c}$$

    where

$$\mathbf{m} = (\mathbf{A}+\mathbf{B})^{-1}(\mathbf{A}\mathbf{a}+\mathbf{B}\mathbf{b}) \qquad \mathbf{M} = (\mathbf{A}+\mathbf{B})$$

    and

$$\mathbf{c} = \mathbf{a}^\top \mathbf{A}\mathbf{a} + \mathbf{b}^\top \mathbf{B}\mathbf{b} - \mathbf{m}^\top \mathbf{M}\mathbf{m}$$

- **Useful frequentist theory:** In the theory of maximum likelihood, we have under regularity conditions that the log-likelihood

$$\ell_n(\theta) = \sum_{i=1}^n \log f_Y(y_i;\theta) \qquad \theta \in \mathbb{R}^d$$

  admits a quadratic expansion around the maximizing value $\widehat{\theta}_n$

$$\ell_n(\theta) \simeq \ell_n(\widehat{\theta}_n) + \dot{\ell}_n(\widehat{\theta}_n)^\top(\widehat{\theta}_n - \theta) + \frac{1}{2}(\widehat{\theta}_n - \theta)^\top \ddot{\ell}_n(\widehat{\theta}_n)(\widehat{\theta}_n - \theta)$$

  where

$$\ddot{\ell}(\theta) = \sum_{i=1}^n \ddot{\ell}(y_i;\theta)$$

  with

$$\ddot{\ell}(y;\theta) = \frac{\partial^2 \ell(y;\theta)}{\partial\theta\partial\theta^\top} \qquad (d \times d).$$

  Noting that $\dot{\ell}_n(\widehat{\theta}_n) = 0$, we have that

$$\begin{aligned}\exp\{\ell_n(\theta)\} &\simeq \exp\{\ell_n(\widehat{\theta}_n)\}\exp\left\{\frac{1}{2}(\widehat{\theta}_n - \theta)^\top \ddot{\ell}_n(\widehat{\theta}_n)(\widehat{\theta}_n - \theta)\right\} \\ &\propto \exp\left\{-\frac{1}{2}(\theta - \widehat{\theta}_n)^\top\{-\ddot{\ell}_n(\widehat{\theta}_n)\}(\theta - \widehat{\theta}_n)\right\}.\end{aligned}$$

  Thus, for large $n$,

$$\pi_n(\theta) \simeq Normal_d\left(\widehat{\theta}_n, \left\{-\ddot{\ell}_n(\widehat{\theta}_n)\right\}^{-1}\right).$$

  and we have a Normal approximation to the posterior.

- **1-1 Transformations:** For continuous variables $(X_1, \ldots, X_d)$ with joint pdf $f_{X_1,\ldots,X_d}$ we can construct the pdf of a transformed set of variables $(Y_1, \ldots, Y_d)$ where $\mathbf{Y} = g(\mathbf{X})$ is a $d$-dimensional transformation. In the 1-1 case, the computation proceeds using the following steps:

  1. Write down the set of *component transformation functions* $g_1, \ldots, g_d$

  $$\begin{aligned} Y_1 &= g_1(X_1, \ldots, X_d) \\ &\vdots \\ Y_d &= g_d(X_1, \ldots, X_d) \end{aligned} .$$

  2. Write down the set of component *inverse transformation functions* $g_1^{-1}, \ldots, g_d^{-1}$

  $$\begin{aligned} X_1 &= g_1^{-1}(Y_1, \ldots, Y_d) \\ &\vdots \\ X_d &= g_d^{-1}(Y_1, \ldots, Y_d) \end{aligned}$$

  3. Consider the joint support of the new variables, $\mathbb{Y}^{(k)}$.
  4. Compute the Jacobian of the transformation: first form the matrix of partial derivatives

  $$D_y = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} & \cdots & \dfrac{\partial x_1}{\partial y_d} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} & \cdots & \dfrac{\partial x_2}{\partial y_d} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial x_d}{\partial y_1} & \dfrac{\partial x_d}{\partial y_2} & \cdots & \dfrac{\partial x_d}{\partial y_d} \end{bmatrix}$$

  where, for each $(i, j)$

  $$\frac{\partial x_i}{\partial y_j} = \frac{\partial}{\partial y_j}\left\{g_i^{-1}(y_1, \ldots, y_d)\right\}$$

  and then set $|J(y_1, \ldots, y_d)| = |\det D_y|$. Note that $\det D_y = \det D_y^\top$ so that an alternative but equivalent Jacobian calculation can be carried out by forming $D_y^\top$. Note also that

  $$|J(y_1, \ldots, y_d)| = \frac{1}{|J(x_1, \ldots, x_d)|}$$

  where $J(x_1, \ldots, x_d)$ is the Jacobian of the transformation regarded in the reverse direction (that is, if we start with $(Y_1, \ldots, Y_d)$ and transform to $(X_1, \ldots, X_d)$.

  5. Write down the joint pdf of $(Y_1, \ldots, Y_d)$ as

  $$f_{Y_1,\ldots,Y_d}(y_1, \ldots, y_d) = f_{X_1,\ldots,X_d}\left(g_1^{-1}(y_1, \ldots, y_d), \ldots, g_d^{-1}(y_1, \ldots, y_d)\right) \times |J(y_1, \ldots, y_d)|$$

  for $(y_1, \ldots, y_d) \in \mathbb{R}^d$.

- **Multivariate distributions**

1. **The Multinomial Distribution**: Let $n \geq 1$ be a positive integer. The joint pmf of vector $\mathbf{X} = (X_1, \ldots, X_d)^\top$ where $0 \leq X_i \leq n$ are integers, and

$$0 \leq \sum_{i=1}^{d} X_i \leq n$$

is given by

$$p_{X_1,\ldots,X_d}(x_1, \ldots, x_d) = \frac{n!}{x_1! \ldots, x_d! x_{d+1}!} \theta_1^{x_1} \ldots . \theta_d^{x_d} \theta_{d+1}^{x_{d+1}} = \frac{n!}{x_1! \ldots, x_d! x_{d+1}!} \prod_{i=1}^{d+1} \theta_i^{x_i}$$

where $0 \leq \theta_i \leq 1$ for all $i$, and $\theta_1 + \cdots + \theta_d + \theta_{d+1} = 1$, and where $x_{d+1}$ is defined by

$$x_{d+1} = n - (x_1 + \ldots + x_d).$$

This is the joint pmf for the **multinomial distribution**. We write

$$\mathbf{X} \sim Multinomial(n; \theta_1, \ldots, \theta_d).$$

The pmf reduces to the Binomial if $d = 1$. The marginal and conditional distributions derived from the joint pmf are also multinomials.

2. **The Dirichlet Distribution** The joint pdf of vector $\mathbf{X} = (X_1, \ldots, X_d)^\top$ where $0 \leq X_i \leq 1$ for $i = 1, \ldots, d$ are real-valued, and

$$0 \leq \sum_{i=1}^{d} X_i \leq 1.$$

is given by

$$f_{X_1,\ldots,X_d}(x_1, \ldots, x_d) = \frac{\Gamma(\alpha)}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_d) \Gamma(\alpha_{d+1})} x_1^{\alpha_1 - 1} \ldots, x_d^{\alpha_d - 1} {x_{d+1}}^{\alpha_{d+1} - 1}$$

for $0 \leq x_i \leq 1$ for all $i$ such that $x_1 + \cdots + x_d + x_{d+1} = 1$, where $\alpha = \alpha_1 + \cdots + \alpha_{d+1}$ and where

$$x_{d+1} = 1 - (x_1 + \cdots + x_d).$$

This is the density function which reduces to the Beta distribution if $d = 1$. It can also be shown that the marginal distribution of $X_i$ is $Beta(\alpha_i, \alpha - \alpha_i)$. We write

$$\mathbf{X} \sim Dirichlet(d; \alpha_1, \ldots, \alpha_{d+1}).$$

that is, there are $d + 1$ $X$s, but this is in fact a $d$-dimensional distribution.

The Dirichlet distribution can be generated by considering independent random variables $Z_1, \ldots, Z_{d+1}$, with $Z_j \sim Gamma(\alpha_j, 1)$, and then defining

$$X_j = \frac{Z_j}{\sum_{k=1}^{d+1} Z_k} \qquad j = 1, \ldots, d + 1.$$

3. **The Multivariate Normal Distribution**: The joint pdf of $\mathbf{X} = (X_1, \ldots, X_d)^\top$ takes the form

$$f_{X_1,\ldots,X_d}(x_1, \ldots, x_d) = \left(\frac{1}{2\pi}\right)^{d/2} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

where $\mathbf{x} = (x_1, \ldots, x_d)^\top$, $\boldsymbol{\mu}$ is a $d \times 1$ vector, and $\Sigma$ is a symmetric, positive-definite $d \times d$ matrix. The distribution is obtained by taking a vector $\mathbf{Z} = (Z_1, \ldots, Z_d)^\top$ of independent standard Normal random variables with joint pdf

$$f_{Z_1,\ldots,Z_d}(z_1, \ldots, z_d) = \left(\frac{1}{2\pi}\right)^{d/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{d} z_i^2\right\} = \left(\frac{1}{2\pi}\right)^{d/2} \exp\left\{-\frac{1}{2}\mathbf{z}^\top\mathbf{z}\right\}$$

and taking the linear transformation

$$\mathbf{X} = \mathbf{L}\mathbf{Z} + \boldsymbol{\mu}$$

where $\mathbf{L}$ is the Cholesky factor of $\Sigma$, that is,

$$\Sigma = \mathbf{L}\mathbf{L}^\top.$$

It can be shown that for any linear combination

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$$

for constant matrix $\mathbf{A}$ and vector $\mathbf{b}$ (compatible in dimension) also has a multivariate normal distribution. The distribution of $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ can be deduced to be

$$\mathbf{Y} \sim Normal_d(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\Sigma\mathbf{A}^\top).$$

All marginal and all conditional distributions derived from the multivariate normal are also multivariate normal. Let

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}.$$

with subvectors of dimensions $d_1$ and $d_2 = d - d_1$ respectively. Let

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

where $\Sigma_{11}$ is $d_1 \times d_1$, $\Sigma_{22}$ is $d_2 \times d_2$, $\Sigma_{21} = \Sigma_{12}^\top$.

(i) **Marginal Distributions:** We have

$$\mathbf{X}_1 \sim Normal_{d_1}(\boldsymbol{\mu}_1, \Sigma_{11})$$

(ii) **Conditional Distributions:** We have

$$\mathbf{X}_2 | \mathbf{X}_1 = \mathbf{x}_1 \sim Normal_{d_2}\left(\boldsymbol{\mu}_2 + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{x}_1 - \boldsymbol{\mu}_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)$$

- **Kullback-Leibler Divergence**

  The *Kullback-Leibler* (KL) divergence is used when measuring the discrepancy between distributions. For two distributions with cdfs $F_0, F_1$

$$KL(F_0, F_1) = \int \log\left\{\frac{dF_0(y)}{dF_1(y)}\right\} dF_0(y)$$

  which is defined when $F_1$ is absolutely continuous with respect to $F_0$, that is for the corresponding probability measures

$$P_0(B) = 0 \quad \Longrightarrow \quad P_1(B) = 0$$

  for any set $B$.

  - Discrete case:

$$KL(p_0, p_1) = \sum_y \log\left\{\frac{p_0(y)}{p_1(y)}\right\} p_0(y) = \mathbb{E}_{p_0}\left[\log\left\{\frac{p_0(Y)}{p_1(Y)}\right\}\right].$$

  - Continuous case:

$$KL(f_0, f_1) = \int \log\left\{\frac{f_0(y)}{f_1(y)}\right\} f_0(y)\,dy = \mathbb{E}_{f_0}\left[\log\left\{\frac{f_0(Y)}{f_1(Y)}\right\}\right].$$

  1. $KL(F_0, F_1) \geq 0$;
  2. $KL(F_0, F_1) \neq KL(F_1, F_0)$ in general;
  3. $KL(F_0, F_1) = 0$ if and only if the two distributions are identical.

## DISCRETE DISTRIBUTIONS

| | SUPPORT $\mathbb{X}$ | PARAMETERS | MASS FUNCTION $f_X$ | CDF $F_X$ | $\mathbb{E}_X[X]$ | $\text{Var}_X[X]$ | MGF $M_X$ |
|---|---|---|---|---|---|---|---|
| $Bernoulli(\theta)$ | $\{0,1\}$ | $\theta \in (0,1)$ | $\theta^x(1-\theta)^{1-x}$ | | $\theta$ | $\theta(1-\theta)$ | $1-\theta+\theta e^t$ |
| $Binomial(n,\theta)$ | $\{0,1,...,n\}$ | $n \in \mathbb{Z}^+, \theta \in (0,1)$ | $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ | | $n\theta$ | $n\theta(1-\theta)$ | $(1-\theta+\theta e^t)^n$ |
| $Poisson(\lambda)$ | $\{0,1,2,...\}$ | $\lambda \in \mathbb{R}^+$ | $\dfrac{e^{-\lambda}\lambda^x}{x!}$ | | $\lambda$ | $\lambda$ | $\exp\{\lambda(e^t-1)\}$ |
| $Geometric(\theta)$ | $\{1,2,...\}$ | $\theta \in (0,1)$ | $(1-\theta)^{x-1}\theta$ | $1-(1-\theta)^x$ | $\dfrac{1}{\theta}$ | $\dfrac{(1-\theta)}{\theta^2}$ | $\dfrac{\theta e^t}{1-e^t(1-\theta)}$ |
| $NegBinomial(n,\theta)$ | $\{n,n+1,...\}$ | $n \in \mathbb{Z}^+, \theta \in (0,1)$ | $\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$ | | $\dfrac{n}{\theta}$ | $\dfrac{n(1-\theta)}{\theta^2}$ | $\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$ |
| or | $\{0,1,2,...\}$ | $n \in \mathbb{Z}^+, \theta \in (0,1)$ | $\binom{n+x-1}{x}\theta^n(1-\theta)^x$ | | $\dfrac{n(1-\theta)}{\theta}$ | $\dfrac{n(1-\theta)}{\theta^2}$ | $\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$ |

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION** for $\alpha > 0$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathbb{E}_Y[Y] = \mu + \sigma\mathbb{E}_X[X] \qquad \text{Var}_Y[Y] = \sigma^2\text{Var}_X[X]$$

## CONTINUOUS DISTRIBUTIONS

| | SUPPORT<br>$\mathbb{X}$ | PARAMETERS | PDF<br>$f_X$ | CDF<br>$F_X$ | $\mathbb{E}_X[X]$ | $\mathrm{Var}_X[X]$ | MGF<br>$M_X$ |
|---|---|---|---|---|---|---|---|
| $Uniform(\alpha,\beta)$<br>(standard model $\alpha=0,\beta=1$) | $(\alpha,\beta)$ | $\alpha<\beta\in\mathbb{R}$ | $\frac{1}{\beta-\alpha}$ | $\frac{x-\alpha}{\beta-\alpha}$ | $\frac{(\alpha+\beta)}{2}$ | $\frac{(\beta-\alpha)^2}{12}$ | $\frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$ |
| $Exponential(\lambda)$<br>(standard model $\lambda=1$) | $\mathbb{R}^+$ | $\lambda\in\mathbb{R}^+$ | $\lambda e^{-\lambda x}$ | $1-e^{-\lambda x}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | $\left(\frac{\lambda}{\lambda-t}\right)$ |
| $Gamma(\alpha,\beta)$<br>(standard model $\beta=1$) | $\mathbb{R}^+$ | $\alpha,\beta\in\mathbb{R}^+$ | $\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ | | $\frac{\alpha}{\beta}$ | $\frac{\alpha}{\beta^2}$ | $\left(\frac{\beta}{\beta-t}\right)^\alpha$ |
| $Weibull(\alpha,\beta)$<br>(standard model $\beta=1$) | $\mathbb{R}^+$ | $\alpha,\beta\in\mathbb{R}^+$ | $\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$ | $1-e^{-\beta x^\alpha}$ | $\frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{\beta^{1/\alpha}}$ | $\frac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$ | |
| $Normal(\mu,\sigma^2)$<br>(standard model $\mu=0,\sigma=1$) | $\mathbb{R}$ | $\mu\in\mathbb{R}$<br>$\sigma\in\mathbb{R}^+$ | $\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$ | | $\mu$ | $\sigma^2$ | $\exp\left\{\mu t+\frac{\sigma^2t^2}{2}\right\}$ |
| $Student(\nu)$ | $\mathbb{R}$ | $\nu\in\mathbb{R}^+$ | $\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$ | | $0$<br>(if $\nu>1$) | $\frac{\nu}{\nu-2}$<br>(if $\nu>2$) | |
| $Pareto(\theta,\alpha)$ | $\mathbb{R}^+$ | $\theta,\alpha\in\mathbb{R}^+$ | $\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$ | $1-\left(\frac{\theta}{\theta+x}\right)^\alpha$ | $\frac{\theta}{\alpha-1}$<br>(if $\alpha>1$) | $\frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$<br>(if $\alpha>2$) | |
| $Beta(\alpha,\beta)$ | $(0,1)$ | $\alpha,\beta\in\mathbb{R}^+$ | $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ | | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | |

# Facts About Discrete Distributions

- Bernoulli/Binomial

$$Y_1, \ldots, Y_n \sim Bernoulli(p) \qquad \Longrightarrow \qquad Y = \sum_{i=1}^{n} Y_i \sim Binomial(n, p)$$

- Geometric/Negative Binomial

$$Y_1, \ldots, Y_n \sim Geometric(p) \qquad \Longrightarrow \qquad Y = \sum_{i=1}^{n} Y_i \sim NegBinomial(n, p)$$

- Binomial/Poisson

$$Y_n \sim Binomial(n, p) \longrightarrow Y \sim Poisson(\lambda)$$

where $\lambda = np$ is held fixed and $n \longrightarrow \infty$.

- Negative Binomial/Poisson

$$Y_n \sim NegBinomial(n, p) \qquad X_n = Y_n - n \longrightarrow X \sim Poisson(\lambda)$$

where $\lambda = n(1-p)$ is held fixed and $n \longrightarrow \infty$.

**Sums of Independent Random Variables:**

- Binomial

$$\left.\begin{array}{l} Y_1 \sim Binomial(m, p) \\ Y_2 \sim Binomial(n, p) \end{array}\right\} \qquad \Longrightarrow \qquad Y = Y_1 + Y_2 \sim Binomial(m+n, p)$$

- Negative Binomial

$$\left.\begin{array}{l} Y_1 \sim NegBinomial(m, p) \\ Y_2 \sim NegBinomial(n, p) \end{array}\right\} \qquad \Longrightarrow \qquad Y = Y_1 + Y_2 \sim NegBinomial(m+n, p)$$

- Poisson

$$\left.\begin{array}{l} Y_1 \sim Poisson(\lambda_1) \\ Y_2 \sim Poisson(\lambda_2) \end{array}\right\} \qquad \Longrightarrow \qquad Y = Y_1 + Y_2 \sim Poisson(\lambda_1 + \lambda_2)$$

# Facts About Continuous Distributions

- Distributions on $\mathbb{R}^+$: Begin with $Y \sim Uniform(0,1)$:
  - $U = -\dfrac{1}{\beta} \log Y \sim Exponential(\beta)$, for $\beta > 0$.
  - $X = (\beta U/\lambda)^{1/\alpha} \sim Weibull(\alpha, \lambda)$, for $\alpha, \lambda > 0$.
  - If $X_1, \ldots, X_n \sim Exponential(\beta)$, independent, then $Z = \sum_{i=1}^{n} X_i \sim Gamma(n, \beta)$.
  - If $Y_1 \sim Gamma(\alpha_1, \beta)$ and $Y_2 \sim Gamma(\alpha_2, \beta)$ are independent, then
  $$S = Y_1 + Y_2 \sim Gamma(\alpha_1 + \alpha_2, \beta)$$
  - If $X \sim Gamma(\alpha, \beta)$ then
  $$Z = \frac{1}{X} \sim InverseGamma(\alpha, \beta)$$
  with
  $$f_Z(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{z}\right)^{\alpha+1} \exp\left\{-\frac{\beta}{z}\right\} \qquad z > 0.$$

- Distributions on $\mathbb{R}$: The Normal distribution and connections
  - Suppose $Y \sim Normal(0,1)$. Then $X = \mu + \sigma Y \sim Normal(\mu, \sigma^2)$.
  - Suppose $Y \sim Normal(0,1)$. Then $U = Y^2 \sim Gamma(1/2, 1/2) \equiv Chisquared(1)$.
  - If $Y_i \sim Gamma(\alpha_i/2, 1/2) \equiv Chisquared(\alpha_i)$ for $i = 1, \ldots, n$ are independent, then
  $$V = \sum_{i=1}^{n} Y_i \sim Gamma\left(\nu/2, 1/2\right) \equiv Chisquared(\nu)$$
  where
  $$\nu = \sum_{i=1}^{n} \alpha_i.$$
  - If $Y_1 \sim Normal(\mu_1, \sigma_1^2)$ and $Y_2 \sim Normal(\mu_2, \sigma_2^2)$ are independent, then
  $$Y = Y_1 + Y_2 \sim Normal(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

- Distribution on $(0,1)$: The Beta distribution
  - If $Y_1 \sim Gamma(\alpha_1, \beta)$ and $Y_2 \sim Gamma(\alpha_2, \beta)$ are independent, then
  $$Y = \frac{Y_1}{Y_1 + Y_2} \sim Beta(\alpha_1, \alpha_2)$$
  This result follows by multivariate transformations.