MATH 559 - EXERCISES 3

Not for Assessment

1. Design sampling schemes to estimate the following integrals by Monte Carlo:

   (a)
   $$I = \int_{-2}^{2} \exp\{x(1-x)/2\}\, dx$$

   (b)
   $$I = \int_{0}^{1}\int_{0}^{1} \frac{\sin(x)}{\log(1+x)} \exp\{-(x+y)\}\, dx\, dy$$

2. The definite integral
   $$\int_{1}^{2} \phi(x)\, dx = \Phi(2) - \Phi(1)$$
   where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard Normal pdf and cdf respectively can be computed using the `pnorm` function in `R`. However, suppose the value of the integral is to be estimated using Monte Carlo methods.

   Compare the variances of the ordinary Monte Carlo estimator ($f(x) \equiv f_0(x) \equiv \phi(x)$) and the optimal Importance Sampling estimator ($f_0(x)$ to be selected).

3. Suppose $f(x) \equiv Beta(5,5)$. Compute the acceptance rate of a rejection sampling algorithm that uses $f_0(x) \equiv Uniform(0,1)$ as the proposal distribution.

4. Consider the density
   $$f(x) = c\exp\{-x^4/2\} \qquad x \in \mathbb{R}$$
   where $c < \infty$ is a constant. Design a rejection sampling scheme to produce samples from $f(x)$.

5. It is often necessary to sample uniformly on a region to compute various integrals numerically.

   (a) Design a method to sample uniformly from the region $\mathcal{D}$ with boundary the unit circle:
   $$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

   (b) Design a method to sample uniformly from the region $\mathcal{R}_d$ where
   $$\mathcal{R}_d = \{(x,y) \in \mathbb{R}^2 : (|x|^d + |y|^d)^{1/d} < 1\}.$$
   where $d > 1$.