MATH 559 - EXERCISES 1 : SOLUTIONS

The Poisson model has mass function

$$f_Y(y;\theta) = \frac{\theta^y \exp\{-\theta\}}{y!} \qquad y = 0, 1, 2, \ldots$$

and zero otherwise, for parameter $\theta > 0$. Two priors to consider in a Bayesian analysis are

(i) $\pi_0(\theta) \equiv Gamma(\alpha_0, \beta_0)$ *for* $\alpha_0, \beta_0 > 0$;

(ii) $\pi_0(\theta)$ *determined by the assumption that* $\phi = \log\theta$ *is* $Normal(\eta_0, \tau_0^2)$ *distributed a priori.*

For values of the hyperparameters $\alpha_0, \beta_0, \eta_0, \tau_0^2$ of your choosing, compute and plot the posterior density $\pi_n(\theta)$ under the two priors for the following data, which constitute a sample of size $n = 50$ are displayed in aggregate form That is, there were

y	0	1	2	3	4	5	6
Count	2	6	7	16	11	6	2

two observations with $y = 0$, six with $y = 1$ and so on.

Solutions: For the data we have $n = 50$ and $s_n = 6+14+48+44+30+12 = 154$. Therefore $\widehat{\theta} = 154/50 = 3.080000$.

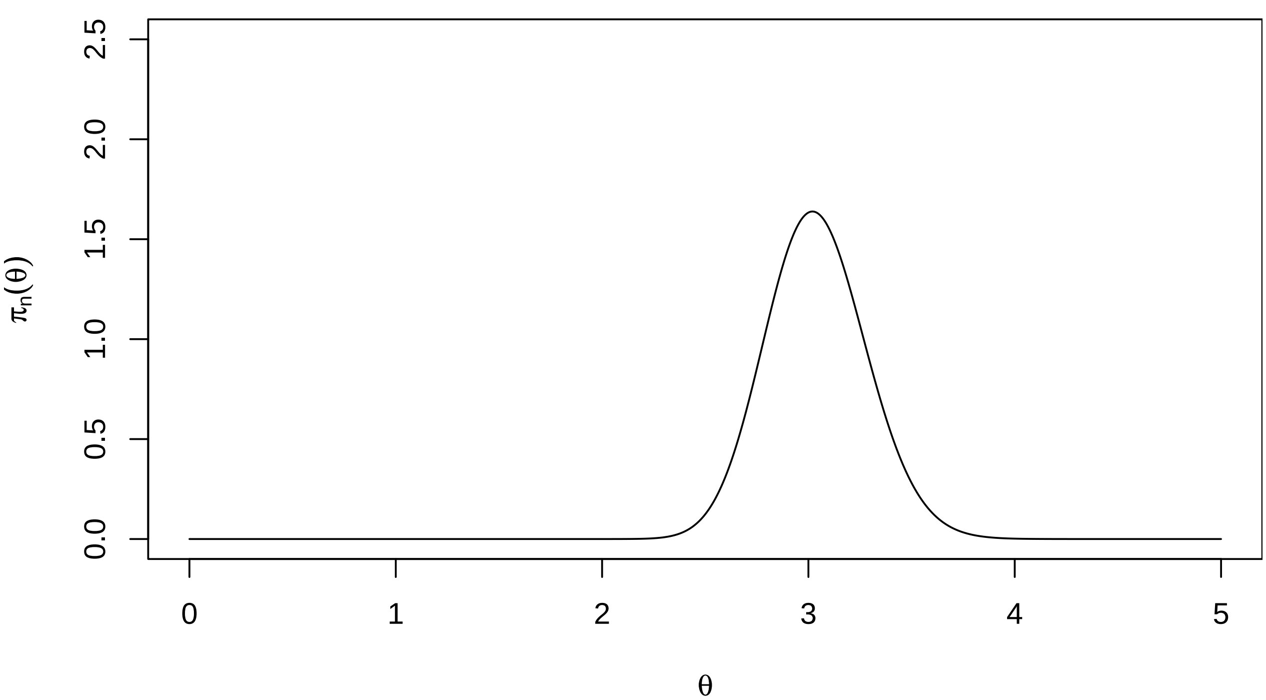
(i) With $\pi_0(\theta) \equiv Gamma(\alpha, \beta)$ for $\alpha, \beta > 0$ we have

$$\pi_n(\theta) \propto \theta^{s_n} \exp\{-n\theta\}\theta^{\alpha-1} \exp\{-\beta\theta\} = \theta^{s_n+\alpha-1} \exp\{-(n+\beta)\theta\}$$

so therefore $\pi_n(\theta) \equiv Gamma(\alpha_n, \beta_n)$ with $\alpha_n = s_n + \alpha$ and $\beta_n = n + \beta$.

We choose $\alpha = \beta = 1$ for illustration:

```
al<-1; be<-1
n<-50
s.n<-6 + 14 + 48 + 44 + 30 + 12
al.n<-s.n+al; be.n<-n+be
xv<-seq(0,5,by=0.001)
yv<-dgamma(xv,al.n,be.n)
par(mar=c(4,4,1,0))
plot(xv,yv,type='l',xlab=expression(theta),ylab=expression(pi[n](theta)),ylim=range(0,2.5))
```



(ii) If $\phi = \log\theta$ is $Normal(\eta, \tau^2)$, then by univariate transformation we have

$$\pi_0(\theta) = \frac{1}{\theta}\left(\frac{1}{2\pi\tau^2}\right)^{1/2} \exp\left\{-\frac{1}{2\tau^2}(\log\theta - \eta)^2\right\}$$

so therefore

$$\pi_n(\theta) \propto \theta^{s_n - 1} \exp\left\{-\frac{1}{2\tau^2}(\log\theta - \eta)^2 - n\theta\right\}.$$

This posterior is not a standard form, so for exact computation, we need to divide by the normalizing constant

$$\int_0^\infty t^{s_n-1} \exp\left\{-\frac{1}{2\tau^2}(\log t - \eta)^2 - nt\right\} \, dt$$

which may be computed using the trapezium rule, or other numerical methods.

For the log-Normal prior, with $\eta = 0$ and $\tau^2 = 10^2$,

```
eta<-0
tau<-10
post.func<-function(xv,ev,tv,nv,sv){

        yv<-dgamma(xv,sv,nv)*dnorm(log(xv),ev,tv)
        return(yv)
}

const<-integrate(post.func,ev=eta,tv=tau,nv=n,sv=s.n,lower=0,upper=10)
const$value

+ [1] 0.03964277

xv<-seq(0,5,by=0.001)
yv<-post.func(xv,eta,tau,n,s.n)/const$value
par(mar=c(4,4,1,0))
plot(xv,yv,type='l',xlab=expression(theta),ylab=expression(pi[n](theta)),ylim=range(0,2.5))
```