MATH 559 - ASSIGNMENT 3

Please submit your solutions by 11.59 pm (EDT) on Wednesday 8th November by uploading a single pdf to myCourses.

1. The $Trinomial(n, \theta_1, \theta_2)$ distribution is a bivariate distribution with pmf

$$p_{Y_1,Y_2}(y_1, y_2; \theta_1, \theta_2) = \frac{n!}{y_1! y_2! (n - y_1 - y_2)!} \theta_1^{y_1} \theta_2^{y_2} (1 - \theta_1 - \theta_2)^{n - y_1 - y_2} \qquad 0 \leq y_1, y_2, y_1 + y_2 \leq n,$$

where $n \geq 1$ is a fixed integer, and parameters $*\theta_1, \theta_2)$ are parameters with parameter space

$$\Theta = \{(\theta_1, \theta_2) : 0 < \theta_1, \theta_2, \theta_1 + \theta_2 < 1\}.$$

   (a) Find the posterior for (θ_1, θ_2) under the conjugate $Dirichlet(\alpha_1, \alpha_2, \alpha_3)$ prior, with pdf

$$\pi_0(\theta_1, \theta_2) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \theta_1^{\alpha_1 - 1} \theta_2^{\alpha_2 - 1} (1 - \theta_1 - \theta_2)^{\alpha_3 - 1}$$

   with support Θ, where $\alpha_1, \alpha_2, \alpha_3 > 0$ are hyperparameters. 3 MARKS

   (b) Plot the joint posterior if $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and $n = 10, y_1 = 3, y_2 = 4$. 4 MARKS

   Hint: `R` *code to plot joint pdfs has been included in past* `knitr` *files.*

   (c) Find the marginal posterior for

$$\phi = \frac{\theta_1}{\theta_1 + \theta_2}$$

   under the prior in (a). 5 MARKS

2. The *Gibbs posterior* for iid data drawn from distribution F_0 using prior $\pi_0^\dagger(\theta)$ is formed by computing the density

$$\pi_n^\dagger(\theta) = \frac{\exp\left\{-\eta \sum_{i=1}^{n} \ell(y_i, \theta)\right\} \pi_0^\dagger(\theta)}{\displaystyle\int \exp\left\{-\eta \sum_{i=1}^{n} \ell(y_i, t)\right\} \pi_0^\dagger(t)\, dt}$$

defined when the denominator is finite, where $\ell(y, \theta)$ is a non-negative function from $\mathcal{Y} \times \Theta$ to $\mathbb{R}^+$, and η is a fixed positive constant. The true value of the parameter, θ_0, is defined by

$$\theta_0 = \arg\min_t \int \ell(y, t)\, dF_0(y)$$

that is, it is the *loss-minimizing* value of the parameter.

   (a) Suppose that $\ell(y, \theta)$ is at least three times differentiable with respect to θ for almost all y (that is, the set of y values for which the function is NOT differential contains probability equal to zero under F_0) at each $\theta \in \Theta$. Suppose that θ_0 lies in an open subset of Θ.

   Describe the behaviour of $\pi_n^\dagger(\theta)$ as $n \longrightarrow \infty$. 4 MARKS

   (b) If $\mathcal{Y} \equiv \Theta \equiv \mathbb{R}$ and $\ell(y, \theta) = |y - \theta|$ show that the Gibbs posterior is equivalent to a standard Bayesian posterior under a particular parametric assumption. 2 MARKS

   (c) If, in fact $F_0(y)$ is an $Exponential(1)$ distribution, describe the behaviour of $\pi_n^\dagger(\theta)$ as $n \longrightarrow \infty$ for the loss function in (b). 2 MARKS