MATH 559 - ASSIGNMENT 1

Please submit your solutions by 11.59 pm (EDT) on Sunday 24th September by uploading a single pdf to myCourses.

For $n \geq 1$, suppose the de Finetti representation for the joint pmf of exchangeable discrete random variables $Y_1, \ldots, Y_n$ is given by

$$p_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \int_\Theta \prod_{i=1}^n p_Y(y_i; \theta)\pi_0(d\theta)$$

where $p_Y(y; \theta)$ is a mass function in y, and θ is a parameter lying in a space $\Theta \subseteq \mathbb{R}^p$, for some (prior) distribution $\pi_0(d\theta)$ defined on Θ, where we may interpret Θ as the smallest set such that

$$\int_\Theta \pi_0(d\theta) = 1.$$

As pointed out by Bernardo & Smith, the *Poisson model* with mass function

$$p_Y(y; \theta) = \frac{\theta^y \exp\{-\theta\}}{y!} \qquad y = 0, 1, 2, \ldots$$

and zero otherwise, for parameter $\theta > 0$, arises by considering observables taking values on the non-negative integers that yield certain summary statistics, or as the limiting case of a discrete selection (multinomial) model.

For this Poisson model:

(a) Find the form of $p_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$ if π_0 is the Gamma density with parameters (α_0, β_0), that is

$$\pi_0(d\theta) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\theta^{\alpha_0 - 1} \exp\{-\beta_0\theta\}d\theta$$

and $\Theta = \mathbb{R}^+ \equiv \{t \in \mathbb{R} : t > 0\}$. 5 MARKS

(b) For the choice of π_0 in (a), compute the implied (marginal) covariance between Y_1 and Y_2.

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(c) Suppose that a *discrete* prior is chosen, where the corresponding mass function takes the form

$$\pi_0(\theta) = \frac{1}{3}\mathbb{1}_{\{1\}}(\theta) + \frac{2}{3}\mathbb{1}_{\{2\}}(\theta)$$

that is, the prior places probability $1/3$ on the value 1, and $2/3$ on the value 2. Compute the implied (marginal) covariance between Y_1 and Y_2 for this prior. 5 MARKS

Hint: in this case, $\Theta \equiv \{1, 2\}$, and the integral in the de Finetti representation reduces to a sum.

(d) For the prior in (a), find the posterior predictive distribution for Y_3 given $Y_1 = y_1$ and $Y_2 = y_2$.

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